Math 504, Homework 1, October 5, 2001

In the exercises below $k$ denotes a field.

1. Think of six interesting questions about the fields $\mathbb{F}_p$ and $\mathbb{Q}(\sqrt{d})$.

2. The field of rational functions in one variable, denoted $k(x)$, consists of all ratios $p/q$ where $p$ and $q$ are polynomials in $x$ having coefficients in $k$, and $q \neq 0$. We add and multiply these in the obvious way. The inverse of a non-zero element $p/q$ is $q/p$. This is the field of rational functions on the affine line over $k$. Likewise, the field $k(x,y)$ of rational functions on the affine plane over $k$ consists of all ratios $p/q$ where $p$ and $q$ are polynomials in the variables $x$ and $y$, and $q \neq 0$. Are the fields $k(x)$ and $k(x,y)$ isomorphic? What does the word “isomorphic” mean in this context?

3. Let $X$ be a set, and $R$ the set of all functions $f : X \to k$. If $f$ and $g$ belong to $R$, how do you suggest we define the sum $f + g$, and the product $fg$? List what you think are the important properties of the sum and product? Is there an element of $R$ that deserves the name zero? Is there an element of $R$ that deserves the name one? If so, say what that element is, and what its properties are that warrant it being given that name?

4. Write $C(X)$ instead of $R$ for the set of all $k$-valued functions on $X$. For each subset $Z$ of $X$, define

$$I(Z) := \{ f \in C(X) \mid f(x) = 0 \text{ for all } x \in Z \}.$$

State all the properties of $I$ that you think are important. For example, how does it behave with respect to the sum and product in $R$? Is there a special name for subsets of $R$ having these properties?

5. Let $Z$ be a subset of $X$. Define $\psi : C(X) \to C(Z)$ by

$$\psi(f) = f|_Z.$$

That is, if $f : X \to k$, $\psi(f)$ is the restriction of $f$ to $Z$. What are the properties of $\psi$ with respect to the addition and multiplication operations in $C(X)$ and $C(Z)$? How do the elements you labelled one and zero behave under $\psi$?

6. Let $R$ be any ring of functions $X \to k$. Associate to each subset $Z$ of $X$ the subset

$$I(Z) := \{ f \in R \mid f|_Z = 0 \}.$$

If $Z' \subset Z$, what is the relation between $I(Z)$ and $I(Z')$? How are $I(Z \cap Z')$ and $I(Z \cup Z')$ related to $I(Z)$ and $I(Z')$?

7. Let $R$ be any ring of functions $X \to k$. Associate to each ideal $I$ in $R$ the subset

$$Z(I) = \{ z \in X \mid f(z) = 0 \text{ for all } f \in I \}.$$
What is the relation between the notions of inclusion, sum and product of ideals, and the notions of inclusion, intersection, and union of subsets of $X$?

8. Let $R$ be any ring of functions $X \rightarrow k$. The previous exercises give functions $I(-)$ and $Z(-)$ between subsets of $X$ and ideals of $R$. What can you say about the compositions $I \circ Z$ and $Z \circ I$?

9. Let $X$ and $Y$ be two sets, and let $\alpha : Y \rightarrow X$ be any function. Define $\psi : C(X) \rightarrow C(Y)$ by

$$\psi(f) = f \circ \alpha;$$

that is, if $y \in Y$, then $\psi(f)(y) = f(\alpha(y))$. What are the properties of $\psi$ with respect to the operations of addition and multiplication in $C(X)$ and $C(Y)$? How do the elements you labelled one and zero behave under $\psi$?

10. Let $\beta : Z \rightarrow Y$ and $\alpha : Y \rightarrow X$ be maps between sets. Let $C(\beta) : C(Y) \rightarrow C(Z)$ and $C(\alpha) : C(X) \rightarrow C(Y)$ be the induced maps, namely $C(\beta)(g) = g \circ \beta$ and $C(\alpha)(f) = f \circ \alpha$. Show that $C(\alpha \beta) = C(\beta) \circ C(\alpha)$. Show that if $\alpha$ is the identity map, then $C(\alpha)$ is also the identity map.
Math 504, Homework 2, October 12, 2001

In the exercises below $k$ denotes a field.

1. State four interesting questions about $\mathbb{C}[x,y,z]/(f)$.

2. State four interesting questions about $\mathbb{C}[x,y,z]/(f,g)$.

3. Show that every non-constant homogeneous polynomial in $\mathbb{C}[x,y]$ factors as a product of linear polynomials. Hence show that the zero locus of a non-constant homogeneous polynomial in $\mathbb{C}[x,y]$ is a union of 1-dimensional subspaces of $\mathbb{C}^2$; that is, a union of complex lines through the origin.

4. Suppose that $f_1, \ldots, f_r$ are homogeneous polynomials in $\mathbb{C}[x_1, \ldots, x_n]$.
   Show that their common zero locus,
   \[
   V(f_1, \ldots, f_r) := \{ p \in \mathbb{C}^n \mid f_1(p) = \cdots = f_r(p) = 0 \},
   \]
   is a union of lines (i.e., complex lines through the origin).

5. If $U$ is a vector space that is the direct sum of various subspaces $U_d, d \geq 0$, and $V$ is a subspace such that $V = \oplus_{d=0}^\infty (V \cap U_d)$, show that
   \[
   U/V \cong \bigoplus_{d=0}^\infty \frac{U_d}{V \cap U_d},
   \]
   and hence that
   \[
   U/V = \bigoplus_{d=0}^\infty \frac{U_d + V}{V}.
   \]

6. Let $I$ be an ideal of $k[x_1, \ldots, x_n]$ that is generated by homogeneous elements. Show that
   \[
   I = \bigoplus_{d=0}^\infty I \cap k[x_1, \ldots, x_n]_d.
   \]

7. Show that $\mathbb{Z}[x]$ is not a principal ideal domain.

8. Show that $k[x,y]$ is not a principal ideal domain.

9. Characterize the irreducible polynomials of degree two in $k[x]$ when $\text{char } k \neq 2$.
   What about when $\text{char } k = 2$?

10. Use the Euclidean algorithm to find the greatest common divisor in $\mathbb{Q}[x]$ of $nx^{n+1} - (n + 1)x^n + 1$ and $x^n - nx + n - 1$. Express the greatest common divisor in the form $af + bg$ where $f$ and $g$ are the two given polynomials and $a$ and $b$ are suitable elements of $\mathbb{Q}[x]$. 
11. Let \( C \) be the curve in \( \mathbb{R}^2 \) cut out by the equation \( y^2 - x^3 = 0 \). Consider \( R = \mathbb{R}[x,y]/(x^3 - y^2) \) as a ring of functions \( C \to \mathbb{R} \). Show that \( R \) is isomorphic to the subring of the polynomial ring \( k[t] \) consisting of those polynomials of the form

\[
\alpha_0 + \alpha_2 t^2 + \cdots + \alpha_n t^n.
\]

12. Continue the previous question. For each point \( p \in C \), let \( m_p \) denote the ideal of \( R \) consisting of those functions that vanish at \( p \). Show that \( \dim_{\mathbb{R}} m_p/m_p^2 = 1 \) if \( p \neq (0,0) \), and that \( \dim_{\mathbb{R}} m_q/m_q^2 = 2 \) when \( q = (0,0) \).

13. Continue the previous question. Decide exactly which ideals \( m_p \) are principal.
Math 504, Homework 3, October 19, 2001

In the exercises below $k$ denotes a field.

1. Let $I$ be a two-sided ideal in a ring $R$. Prove there is a bijection between the set of two-sided ideals in $R$ that contain $I$ and the set of ideals in $R/I$. Under the bijection an ideal $J$ in $R$ corresponds to $J/I$.

Show that

$$R/J \cong \frac{R/I}{J/I}.$$ 

How do the sum and product of ideals correspond under this bijection?

2. Show that a finite domain is a field.

3. Let $R$ be a commutative domain containing a field $k$. Show that $R$ is a field if $\dim_k R < \infty$.

4. Let $R$ and $S$ be rings. Their product $R \times S$ is their cartesian product with component-wise addition and multiplication. This is a ring.

If $I$ and $J$ are ideals in a ring $R$ such that $I+J = R$, show that $R/I \cap J \cong R/I \times R/J$.

5. In a PID show that $gcd(f, g)$ generates the ideal $(f, g)$.

6. Let $R$ be a commutative ring. An ideal $p$ in $R$ is prime if $R/p$ is a domain. This is equivalent to the condition that a product $xy$ can belong to $p$ only if either $x$ or $y$ does. The spectrum of $R$, denoted $\text{Spec } R$, is the set of all prime ideals. Notice that every maximal ideal is prime so $\text{Spec } R$ contains $\text{Max } R$, the set of maximal ideals.

We make $\text{Spec } R$ a topological space by defining the closed sets to be

$$V(I) := \{p \mid p \supset I\},$$

where $I$ runs over all two-sided ideals of $R$. Show this really does make $\text{Spec } R$ a topological space. This is called the Zariski topology.

7. Let $R$ be a PID. Show that $\text{Spec } R = \text{Max } R \cup \{0\}$. Describe the closed subsets of $\text{Spec } R$. In particular, if $k$ is an algebraically closed field, what is $\text{Spec } k[t]$ and what is the topology on it? Think of the example of $k = \mathbb{C}$ and compare this to the usual topology.

8. If $\psi : R \to S$ is a homomorphism between commutative rings show that there is an induced map

$$\psi^\# : \text{Spec } S \to \text{Spec } R,$$

and that this map is continuous. Is this true if we replace $\text{Spec } R$ and $\text{Spec } S$ by $\text{Max } R$ and $\text{Max } S$?
You have just shown that the rule $R \mapsto \text{Spec } R$ and $\psi \mapsto \psi^\sharp$ is a contravariant functor from the category of commutative rings to the category of topological spaces (contravariant because the arrows change direction). Actually you also need to show that $\text{id}_R^\sharp = \text{id}_{\text{Spec } R}$ and $(\psi \phi)^\sharp = \phi^\sharp \psi^\sharp$.

9. View $R = k[x_1, \ldots, x_n]$ as functions $k^n \to k$. Show that there is a natural injective map $k^n \to \text{Max } R \to \text{Spec } R$, so the Zariski topology induces a topology on $k^n$. What are the closed subsets of $k^n$? Show that every polynomial function $f : k^n \to k$, i.e. every $f \in R$, is a continuous map when $k^n$ and $k$ are both given the Zariski topologies. (The Zariski topology on $k$ is obtained from the inclusions $k \to \text{Max } k[t] \to \text{Spec } k[t]$).

10. A boolean ring is a commutative ring in which $x^2 = x$ for every element. If $R$ is boolean show that $\text{Max } R = \text{Spec } R$.

11. Let $R$ be a commutative domain and suppose that every non-zero non-unit is a product of irreducibles. Show $R$ is a UFD if and only if $(x)$ is a prime ideal for all irreducibles in $R$.

In particular, since $R = k[x_1, \ldots, x_n]$ is a UFD, this shows that $R/(f)$ is a domain if and only if $f$ is irreducible.
Math 504, Homework 4, October 26, 2001

In the exercises below $k$ denotes a field.

1. A commutative ring is local if it has a unique maximal ideal. Let $p \in \mathbb{Z}$ be prime. Show that the subring

$$S := \{a/b \mid a, b \in \mathbb{Z}, \ p \text{ does not divide } b\}$$

of $\mathbb{Q}$ is local.

2. Let $p$ be an irreducible element in a PID $R$. Show that the ring

$$S := \{a/b \mid a, b \in R, \ p \text{ does not divide } b\}$$

is local.

3. Let $J$ and $K$ be ideals in $k[x_1, \ldots, x_n]$. Define

$$K : J := \{x \in k[x_1, \ldots, x_n] \mid xJ \subset K\}.$$ 

Show that $V(K : J) = V(K) \setminus V(J)$.

4. In $\mathbb{Z}[x]$ factor into irreducibles $x^n - 1$ for $3 \leq n \leq 10$.

5. Are any two of the following rings isomorphic:

$$\mathbb{Z}/(4), \mathbb{F}_2[x]/(x^2), \mathbb{F}_2[t]/(t^2 - 1), \mathbb{F}_2[y]/(y^2 + y + 1)?$$

Explain. You can sometimes show two rings are not isomorphic by showing that their (lattices of) ideals are different.

6. What is the integral closure of $R = k[x, y]/(y^2 - x^2(x - 1))$ in its field of fractions. Hint: find a subring of $k[t]$ that is isomorphic to $R$. 

Math 504, Homework 5, November 2, 2001

In the exercises below $k$ denotes a field.

1. Let $R$ be a domain with field of fractions $F$. Let $S$ be a subset of $R$ consisting of non-zero elements and suppose that $st \in S$ whenever $s$ and $t$ belong to $S$. Let $S = R[S^{-1}]$ be the subring of $F$ generated by $R$ and the inverses of the elements in $S$. Every element in $S$ can therefore be written in the form $xy^{-1}$ where $x \in R$ and $y \in S$. Show that there is a natural 1-1 correspondence between the prime ideals of $S$ and the prime ideals $p$ of $R$ such that $p \cap S = \phi$. 

2. Continue with the notation of the previous exercise. A previous week’s homework exercise showed that the inclusion map $\psi : R \to R[S^{-1}]$ induces a continuous map

$$\psi^* : \text{Spec } R[S^{-1}] \to \text{Spec } R$$

when the two spectra are given the Zariski topology. Show that this map is a homeomorphism onto the open subset of $\text{Spec } R$ that is the complement of the closed set

$$Z := \{p \in \text{Spec } R \mid p \supset S\}.$$ 

3. Let $L$ be a submodule of a left $R$-module $N$. Show that there is a bijection between the submodules of $N$ that contain $L$ and the submodules of $N/L$. Under the bijection a submodule $M$ lying between $L$ and $N$ corresponds to the submodule $M/L$ of $N/L$. Show that

$$N/M \cong \frac{N/L}{M/L}.$$ 

4. Let $M$ be a left $R$-module. If $I$ is an ideal of $R$ we write

$$IM := \{\sum_{i=1}^n a_i m_i \mid n \geq 0, a_i \in I, m_i \in M\}.$$ 

Show that $IM$ is a submodule of $M$. Show that if $IM = 0$, there is a natural way to make $M$ a left $R/I$-module.

5. Let $S$ be any subring of $R = k[x]$ that is strictly larger than $k$. Show that $R$ is a finitely generated $R$-module. What minimal information about $S$ would allow you to obtain an upper bound on the number of elements needed to generate $R$ as an $S$-module?

6. Hilbert showed that every ideal in a polynomial ring $k[x_1, \ldots, x_n]$ is finitely generated. Find explicit generators for the ideal

$$I := \{f \in \mathbb{R}[x, y, z] \mid f(a, b, c) = 0 \text{ for all } (a, b, c) \in S^2\}.$$ 

where $S^2$ is the sphere

$$S^2 := \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}.$$