In all questions \( k \) denotes a field.

This is a two hour exam. The exam is out of 100 points. That works out at roughly one point every one minute and twelve seconds, or five points every six minutes.

**PART I (40 points)**

State whether each of the following statements is True or False. You get two points for each correct answer and –2 points for each incorrect answer.

In these questions \( R \) is a ring and \( M \) a left \( R \)-module.

1. Let \( K \subseteq L \) be submodules of a left \( R \)-module \( M \). There is a bijection between the set of submodules of \( M \) that contain \( K \) and submodules of \( M/K \) and there is an isomorphism of modules
   \[
   \frac{M/K}{L/K} \cong \frac{M}{L}.
   \]

2. For each \( x \in R \), let \( \rho_x : M \to M \) be defined by \( \rho_x(m) = xm \). Each \( \rho_x \) is a homomorphism of left \( R \)-modules.

3. Let \( Z(R) := \{ z \in R \mid zr = rz \text{ for all } r \in R \} \) denote the center of \( R \). For each \( z \in Z(R) \), let \( \rho_z : M \to M \) be the map in (2) above. Then the map \( \Phi : Z(R) \to \text{End}_R M \) defined by \( \Phi(z) = \rho_z \) is a ring homomorphism.

4. If \( I = \{ x \in R \mid xM = 0 \} \), then \( M \) is a left \( R/I \)-module under the action \( (x + I).m := xm \).

5. If \( I \) is an ideal in a ring \( R \) and \( M \) is a left \( R/I \)-module, then \( M \) is a left \( R \)-module under the action \( xM := [x + I].m \).

6. Let \( N \) be a submodule of \( M \). If there is an \( R \)-module homomorphism \( f : M \to N \) such that \( f(n) = n \) for all \( n \in N \), there is a submodule \( L \subseteq M \) such that \( N \oplus L = M \).

7. Consider the ring
   \[
   R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}
   \]
   acting on \( \mathbb{C}^2 \) by left multiplication and view \( M = \mathbb{C}^2 \) as a left \( R \)-module. Then \( M \) is a semisimple \( R \)-module.

8. Let \( I \) and \( J \) be ideals in a commutative ring \( R \). Then there is an \( R \)-module homomorphism \( \Phi : R \to \text{Hom}_R(R/I, R/J) \) defined by
   \[
   \Phi(x)(r + I) := [xr + J].
   \]
9. Let \( p \) be a prime and let \( R = \mathbb{F}_p[x] \) be the polynomial ring over the field \( \mathbb{F}_p \) having \( p \) elements. There is an isomorphism of rings \( R/(x^p + 1) \cong R/(x^p) \).

10. Let \( p \) be a prime and let \( R = \mathbb{F}_p[x] \) be the polynomial ring over the field \( \mathbb{F}_p \) having \( p \) elements. There is an isomorphism of \( R \)-modules \( R/(x^p + 1) \cong R/(x^p) \).

11. The subring of \( M_2(\mathbb{C}) \) consisting of matrices of the form

\[
\begin{pmatrix}
a & b \\
0 & a
\end{pmatrix}
\]

is isomorphic to the ring \( \mathbb{C}[x]/(x - 1)^2 \).

12. The ring \( \mathbb{Q}[x]/(x^3 + 1) \) is isomorphic to a subring of \( \mathbb{C} \).

13. The ring \( \mathbb{Q}[x]/(x^3 + 2) \) is isomorphic to a subring of \( \mathbb{C} \).

14. A commutative ring with 351 elements can never be a domain.

15. The rings \( \mathbb{Z}[x]/(x^2 - 2x + 1) \) and \( \mathbb{Z} \oplus \mathbb{Z} \) are isomorphic.

16. The rings \( \mathbb{Z}[x]/(x^2 - 2x) \) and \( \mathbb{Z} \oplus \mathbb{Z} \) are isomorphic.

17. There is a ring \( R \) with 64 elements containing \( \mathbb{F}_{16} \), the field with 16 elements.

18. The rings \( \mathbb{Q}[x]/(x^2 - 3) \) and \( \mathbb{Q}[x]/(x^2 - 5) \) are isomorphic.

19. A finite group \( G \) is abelian if and only if all simple \( \mathbb{C}G \)-modules have dimension one.

20. A finite group \( G \) is abelian if and only if all simple \( \mathbb{R}G \)-modules have dimension one.

**PART II** (20 points)

Questions in this section are worth two points each.

Give short (one or two sentence) answers to each of the following questions:

1. State three equivalent conditions for a module \( M \) to be noetherian. One of those conditions can be the definition.

2. State three equivalent conditions for a module \( M \) to be semisimple. One of those conditions can be the definition.

3. State three equivalent conditions for a module \( M \) to be projective. One of those conditions can be the definition.

4. Give an example of a group algebra that is not a semisimple ring. Explain.
5. Let $K$ and $L$ be submodules of a module $M$. Write down a short exact sequence involving $K + L$, $K \cap L$, and $K \oplus L$, saying what the maps are.


7. Define the minimal polynomial of a linear operator $T : k^n \to k^n$.

8. Let $R$ be a principal ideal domain. If $p_1, \ldots, p_t$ are distinct primes and $n_1, \ldots, n_t$ are positive integers, what are the composition factors of $R/(p_1^{n_1} \cdots p_t^{n_t})$?

9. What is the rational canonical form for the linear transformation $T : \mathbb{R}^6 \to \mathbb{R}^6$ having minimal polynomial of $(x^2 - 4)(x^2 + 1)(x^2 + x + 1)$?

10. Complete the following sentence. If $G$ is a finite group and $k$ a field in which $|G| \neq 0$, then every $kG$ module is $\cdots$

PART III (20 points)

Each question is worth 2 points. Complete the following sentences about the structure of modules over a principal ideal domain $R$:

1. Every submodule of a free $R$-module is $\cdots$

2. The torsion submodule of a module $M$ is defined to be $\tau M := \{ \cdots \}$.

3. A finitely generated torsion-free $R$ module is $\cdots$

4. If $M$ is a finitely generated $R$-module, then $M \cong \cdots$.

5. If $p \in R$ is prime, the $p$-primary submodule of $M$ is $M(p) := \{ \cdots \}$.

6. A finitely generated torsion $R$-module is a direct sum of $\cdots$

7. Let $p \in R$ be a prime. The only ideals in $R$ that contain $p^nR$ are $\cdots$

8. If $M$ is a finitely generated $R$-module, then $M \cong \cdots$.

9. The elementary divisors of $M$ are $\cdots$

10. We may also write $M \cong \cdots$ where $\cdots$ are the invariant factors of $M$.

PART IV (20 points)

1. (5 pts) Briefly explain why the problem of finding all 1-dimensional representations of a finite group $G$ reduces to the problem of factoring $x^n - 1$ in $k[x]$.
2. (5 pts) Explain how the theory of modules over a PID is used to obtain the Jordan normal form for a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$.

3. (5 pts) Say all you can about the modules over the ring

$$R = \mathbb{C} \oplus \mathbb{H} \oplus M_2(\mathbb{R}) \oplus M_3(\mathbb{C}) \oplus M_4(\mathbb{H}).$$

4. (5 pts) Briefly tell me what you know about prime elements, and irreducible elements in a commutative domain. What is the relationship between them, and how do they turn up in the theory of principal ideal domains? Also tell me what you know about unique factorization in this context.