12.1.7 Let \( q \) be the natural projection from \( \mathcal{A} \to \mathcal{B}(X/\mathcal{B}) \), defined compositionwise. Then \( \ker q \) clearly contains \( \mathcal{B} \). Take a \( \mathcal{B}_1 \in \mathcal{B} \), assume \( q\mathcal{B}_1 = 0 \), then \( \mathcal{B}_1 \) is the \( \mathcal{B} \)-subspace that has to be in \( \mathcal{B} \), hence \( \mathcal{B} \mathcal{B}_1 \). Since \( q \mathcal{B}_1 \subseteq \ker q \) and \( \mathcal{B}\mathcal{A} / \mathcal{B}\mathcal{B}_1 = \mathcal{B}(\mathcal{A})(\mathcal{B}_1) \). 

12.1.8 Take \( \mathcal{B}_1 \mathcal{B}_2 \), with \( \mathcal{B}_2 \mathcal{B}_1 \). If \( h \in \ker \mathcal{B}_2 \), then \( h\mathcal{B}_2 = 0 \). Consider the ideal \( (h, p) \mathcal{B} \), \( B, B_2 = B_1, \) with \( (h, p) = \mathcal{B}_1 \), hence \( h \), with \( g \neq 0 \).

If \( g \), a unit, then \( q\mathcal{B}_1 = 0 = \mathcal{B}_2 = 0 \). So \( q \mathcal{B}_1 \) unit, hence \( \mathcal{B}_1 \) unit.

And \( h \mathcal{B}_2 (h) = (h, p) \mathcal{B}_2 = 0 \), so \( \ker (\mathcal{B}_2) \subseteq (p) \).

12.1.10 Define \( N_p = \ker (\mathcal{B}_1 \mathcal{B}_2) \). If \( \mathcal{B}_1 \mathcal{B}_2 = 0 \), then \( \exists \mathcal{B}_2, p = 0, p \mathcal{B}_1 = 0 \). Then \( p \mathcal{B}_1 (p + p_1) = 0 \) so \( \mathcal{B}_1 \mathcal{B}_2 \) is a submodule.

Claim: \( N_p = \mathcal{B}_1 \mathcal{B}_2 \). Then \( \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_2 = 0 \) is a submodule.

For each \( \mathcal{B}_1 \mathcal{B}_2 \), define \( S_k = \mathcal{B}_1 \mathcal{B}_2 / \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_2 \). Then \( (\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k) = 0 \) so \( S_k \mathcal{B}_1 \) with \( 1 = \mathcal{B}_1 \mathcal{B}_2 \).

Then \( S_1 \mathcal{B}_1 \mathcal{B}_2 = \mathcal{B}_1, \mathcal{B}_2 \mathcal{B}_2 \). Then \( \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_2 = 0 \) and \( \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_2 \neq 0 \). Now suppose \( \mathcal{B}_1 \mathcal{B}_2 = 0 \). Then \( \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_2 = 0 \), with \( S_1 \) and \( p \mathcal{B}_1 = 0 \). Again define \( S_k = \mathcal{B}_1 \mathcal{B}_2 / \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_2 \). Multiply both sides of \( \mathcal{B}_1 \mathcal{B}_2 = 0 \) with \( S_1 \), we get \( S_1 \mathcal{B}_1 = 0 \), hence \( (\mathcal{B}_1, \mathcal{B}_2) \mathcal{B}_1 = \ker (\mathcal{B}_1) \). But \( (\mathcal{B}_1, \mathcal{B}_2) \mathcal{B}_1 = 0 \), \( \mathcal{B}_1 = 0 \). The sum \( N = \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_2 \mathcal{B}_2 \) is unique and \( N = \mathcal{B}_1 \mathcal{B}_2 \).
12.1.1. Let \( p^k \mid a \), \( p^{k+1} \mid a \). \( p^k M = p^k R/a \sum (p^k + a)/(a) \)

If \( k < n \), \( (a)/(p^k) \subseteq (p^k)/(a) \), so \( (p^k)/(a) = (p^k)/(a) \) and \( p^k M/p^k M \subseteq \)

\( (p^k)/(a) \) \( \subseteq (p^k)/(a) \) so \( (p^k)/(a) = 0 \)

12.1.2. a) \( M/p^k \text{ is a free \( R/(p^k) \)-module} \)

\( \Rightarrow \frac{M}{p^k M} \cong \frac{R}{(p^k)} \cong \frac{p^k M}{p^k M} \)

\( \cong \bigoplus \frac{p^k M/p^k M}{p^k M/p^k M} \)

\( \cong \bigoplus \frac{p^k R/(p^k)}{p^k R/(p^k)} \)

From 12.1.1, \( p^k R/(p^k) \cong \frac{R}{(p^k)} \) if \( p^k R/(p^k) \cong \frac{R}{(p^k)} \)

\( \cong \frac{R}{(p^k)} \) otherwise, hence \( p^k M/p^k M \cong (R/(p^k))^k \)

b) \( M_1 \cong M_2 \Rightarrow F \cong \frac{M_1}{p^k M_1} = \frac{M_2}{p^k M_2} = \frac{M_1}{p^k M_1} \cong \frac{M_2}{p^k M_2} \)

where \( F = R/(p^k) \) is a field \( \Rightarrow M_1 = M_2 \) \( \forall p, k \).

Hence \( M_1 \) and \( M_2 \) have the same set of elementary divisors.

As \# of copies of \( R/(p^k) \) in the direct sum \( M \)

is \( N_{p^k} = N_{p^k} \). And hence the same \# for \( M_1 \) and \( M_2 \).
8.3.2. Take \( a, b \in \mathbb{R} \). Let \( I = (a, b) \), then \( I \) is an ideal of \( \mathbb{R} \), hence \( I = (1) \). And \( \langle c \rangle \) is a least common multiple of \( a \) and \( b \).
\( c | a \Rightarrow a | c \), \( c | b \Rightarrow b | c \). Spec \( c \) is another element of \( \mathbb{R} \) with \( a \) and \( b \) divisible by \( c \), then \( (d) \subset (a) \), \( (d) \subset (b) \) \( \Rightarrow \)
\( (d) \subset (c) = (a, b) \Rightarrow (d) \subset I \).

8.3.4. Let \( I \subset \mathbb{R} \) be an ideal. We will show \( I \) is principle. Assume \( I \) is principle.
Pick \( a, b \in I \), \( (a, b) \in \), hence by assumption \( \exists x \in I \setminus \{0\} \), \( x = a \).
\( a | 1 \Rightarrow a | \frac{1}{x} \Rightarrow x | \frac{1}{a} \). Since \( a - \frac{a}{x} = b \),
\( a(x) < (a, x) \Rightarrow (a) = (x, a) \), \( (a) = I \), hence we can pick \( y \in I \setminus \{0\} \) and define \( (a) = y(1, x, 0, 0, \ldots) \). Continue this process we arrive at a sequence \( (a) \) with \( a_{i+1} | a_i \), \( (a_{i+1}) \supset (a_i) \). Contradict \( \exists I \). So assumption false and \( I \) is principle.

8.3.5(a). Spec \( (a) = (2, 1 + \sqrt{5}) \), then \( 2 = (\sqrt{5} - 1) (2, 1 + \sqrt{5}) \).

and \( 1 + \sqrt{5} = (1 + \sqrt{5}) \left( \frac{3}{2} - 1 + \sqrt{5} \right) \). With all coefficients \( \neq 2 \).

Apply complex norm to both sides we get \( 4 = (1 + \sqrt{5}) \left( 3 - 2 \sqrt{5} \right) \),
\( 2 + 2 \sqrt{5} = 3 + 5 \sqrt{5} \). \( \Rightarrow \alpha + \sqrt{5} \beta = 2 + \sqrt{5} \). \( \Rightarrow a = \alpha, \beta = 0 \).
\( \Rightarrow a = 2 \), but by a similar norm argument, \( (a) \neq (2, 1 + \sqrt{5}) \) so \( (2, 1 + \sqrt{5}) \) is not principle.

b). \( \mathbb{I}^2(2) = (1 + \sqrt{5}) (1 - \sqrt{5}) - 2 \subset \mathbb{I}^2, \) so \( \mathbb{I}^2 \) \( \subset \mathbb{I}^2 \). On the other hand,
\( 2, 2 + (1 + \sqrt{5}) \subset (2, 1 + \sqrt{5}) \) \( \subset \mathbb{I}^2 \) and they generate \( \mathbb{I}^2 \), hence \( (2) = \mathbb{I}^2 \).