MATH 402 FINAL FALL 2005

I will write Z to denote the center of a group, or $Z(H)$ if I have to specify that it is the center of H.

I will write $\mathbb{R}^\times$ and $\mathbb{C}^\times$ to denote the groups of non-zero real and complex numbers under multiplication.

1. True/False

You get 2 points for each correct answer and -2 for each incorrect one. Just write T or F as your answer. My advice is to answer the questions you are sure of first, then do the rest of the exam and finally return to look at those you are not so sure about.

(1) Let G be the group of upper triangular 2×2 matrices and N the subgroup of matrices having determinant one. Then $G/N \cong \mathbb{R}^\times$.
(2) Let G be a finite group and $x \in G$. Let $\phi : \mathbb{Z} \to G$ be the homomorphism $\phi(n) = x^n$. Then $n \in \ker\phi$ if and only if the order of x divides n.
(3) The subgroup of S_8 generated by (31) and (53) and (65) is isomorphic to S_4.
(4) The subgroup of S_8 generated by (25) and (325) and (56) isomorphic to S_4.
(5) The subgroup of S_8 generated by (14) and (15) and (26) is isomorphic to $S_3 \times \mathbb{Z}_2$.
(6) S_6 has cyclic subgroups of orders 8 and 9.
(7) S_6 has abelian subgroups of orders 8 and 9.
(8) The subgroup of S_8 generated by (123456) and (14)(25)(36) is isomorphic to $\mathbb{Z}_6$.
(9) G is abelian if $x^2 = 1$ for all $x \in G$.
(10) There is a surjective homomorphism $S_3 \to \mathbb{Z}_3$.
(11) There is an injective homomorphism $D_7 \to S_7$.
(12) There is a surjective homomorphism $S_7 \to D_7$.
(13) If x is conjugate to y, then x^2 is conjugate to y^2.
(14) If x is conjugate to y, then x^{-1} is conjugate to y^{-1}.
(15) The odd integers together with 0 form a subgroup of $\mathbb{Z}$.
(16) The subgroup of $(\mathbb{Q}, +)$ generated by $\frac{1}{3}$ and $\frac{1}{5}$ contains $\frac{1}{15}$.
(17) The subgroup of $(\mathbb{Q}, +)$ generated by $\frac{1}{3}$ and $\frac{1}{5}$ is generated by $\frac{1}{15}$.
(18) The subgroup of $(\mathbb{Q}, +)$ generated by $\frac{1}{2}$ and $\frac{1}{4}$ is isomorphic to $\mathbb{Z}$.
(19) Every subgroup of $\mathbb{Z}$ except the trivial subgroup is isomorphic to $\mathbb{Z}$.
(20) If $\phi \in \mathrm{Aut}(\mathbb{Z}_{10})$ is defined by $\phi(\bar{1}) = (\bar{7})$, then $\phi^3(l) = \bar{8}$.
(21) There is an element $\phi \in \mathrm{Aut}(\mathbb{Z}_{10}$ such that $\phi(\bar{1}) = l$.
(22) Every group with n elements is isomorphic to a subgroup of S_n.
(23) The automorphism group of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ contains an element of order 2.
(24) The automorphism group of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ contains an element of order 3.
(25) There is a non-abelian group having 47 elements.

(26) There is a non-abelian group having 49 elements.
(27) The automorphism group of $\mathbb{Z}_{17}$ has 16 elements.
(28) For every integer $n \geq 2$ there is an automorphism of $\mathbb{Z}_n$ having order 2.
(29) Let G act on the set X. If $a, b \in G$ move different numbers of elements of X they are not conjugate.
(30) Let G act on the set X. If $a, b \in G$ move the same numbers of elements of X they are conjugate.
(31) An abelian group has no inner automorphisms.
(32) There is an automorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $\phi(3) = -3$.
(33) The elements i and j are conjugate elements in the group of quaternions.
(34) The elements (316) and (241) are conjugate in S_8.
(35) The elements (12)(34) and (13)(24) are conjugate in S_5.
(36) For every group G, the function $\phi : G \to G$ defined by $\phi(x) = x^{-1}$ is a homomorphism.
(37) Let Z denote the center of a finite group G. Then $\alpha(Z) = Z$ for all $\alpha \in \mathrm{Aut}(G)$.
(38) There is a homomorphism $f : S_3 \times \mathbb{Z}_2 \to S_3 \times \mathbb{Z}_2$ such that $f(Z) \not\subset Z$.
(39) If N is the unique subgroup of G isomorphic to S_5, then $\alpha(N) = N$ for all $\alpha \in \mathrm{Aut}(G)$.
(40) The subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of all matrices whose only non-zero entries lie on the diagonal is normal.

2. Complete the sentence

Complete the following sentences. DO NOT waste time by rewriting the part of the sentence that I have already written. Just write the rest of it.

(1) A homomorphism $f : G \to H$ is a function such that ....
(2) The subgroup of G generated by x, y, z is defined to be ...
(3) To show that an automorphism ϕ of a group G is inner one must show that...
(4) To show that elements $x, y \in G$ are NOT conjugate one must show that ...
(5) To show that subgroups A and B of G are conjugate one must show that....
(6) There is a surjective homomorphism $\Phi : G \to \mathrm{Inn}(G)$, the group of inner automorphisms, defined by ...
(7) To show that a subgroup H of G is not normal one must show there are elements .... such that ....
(8) Let $P(X)$ denote the group of all permutations of a set X. If $\phi : G \to P(X)$ is a group homomorphism we can define an action of G on X by $g.x =$ ...
(9) The action of a group G on a set X is transitive if ....
(10) To show that the action of a group G on a set X is not transitive one must show there are elements ... such that ....
(11) The quaternion group Q is not isomorphic to D_4 because ....
(12) The index of a subgroup H in G is defined to be ....
(13) To show that an element $g \in G$ is not in the center of G one must show there is an element .... such that ...
(14) If G acts on X and X is a single G-orbit, then there is a subgroup H in G such that the action of G on X is isomorphic to the action of G on ....; in fact, $H =$...
(15) If $xy \in Z(G)$, then $xy = yx$ because ...
(16) Let G be a non-abelian group with 22 elements and $x \in G$ an element such that $x^2 \neq 1$. The subgroup $\langle x \rangle$ is normal because ...
(17) In Question 16, If $a \notin \langle x \rangle$, then $a^2 \in \langle x \rangle$ because ...
(18) In Question 16, $a^2 x = x a^2$ because ....
(19) In Question 16, $axa^{-1} = x^i$ for some integer i because ...
(20) In Question 16, $axa^{-1} = x^{-1}$ because ...
(21) In Question 16, $G \cong D_{11}$, the dihedral group of order 22 because D_{11} is generated by two elements σ and τ with the relations ....
(22) Let A and B be subgroups of a group G. Suppose that B is normal and that $AB = G$. Then there is an isomorphism between the groups $A/A \cap B$ and G/B given by the function ....
(23) The Class Equation for a finite group G says that ...

3. Short questions and answers

These are one word or one sentence answers. There is no need to show your work.

(1) If $x \in G$ has order $6n$ write down a subgroup of G having $2n$ elements.

(2) The set $\mathbb{Z}$ with the "product" $a * b = a + b - 1$ is a group. What is its identity element ?

(3) The set $\mathbb{Z}$ with the "product" $a * b = a + b - 1$ is a group. What is the inverse of 5?

(4) Where does the permutation $(2613)(4256)(25) \in S_6$ send $2 \in \{1, 2, 3, 4, 5, 6\}$?

(5) What group is the subgroup of S_7 generated by (153) and (726) isomorphic to?

(6) Write $\mathbb{R}^\times$ and $\mathbb{C}^\times$ for the multiplicative groups of non-zero real and complex numbers, respectively. Let $U \subset \mathbb{C}^\times$ be the subgroup consisting of the complex numbers of absolute value one. Write down an isomorphism $\phi : U \times \mathbb{R}^\times \to \mathbb{C}^\times$. You do not need to show it is an isomorphism, just write it down.

(7) Give an example of a group of order 63 that does not have an element of order 9.

(8) Write down two elements in the previous example that have order 21.

(9) Write down an element in the previous example that has order 18 or explain why that group does not have an element of order 18.

(10) Give an example of a subgroup of $\mathbb{C}^\times = (\mathbb{C} - \{0\}, \cdot)$ that is isomorphic to $\mathbb{Z}$.

(11) Give an example of a subgroup of $\mathbb{C}^\times$ that is isomorphic to $\mathbb{Z}_6$.

(12) Give an example of a subgroup of $\mathbb{C}^\times$ that is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}$.

(13) Give an example of an element in the group $(\mathbb{Q}/\mathbb{Z}, +)$ having order 48.

(14) Give an example of a group G such that $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(15) Write down an injective homomorphism $\phi : H \to G \times H$.

(16) Let $\alpha : H \to \operatorname{Aut} N$ be a group homomorphism and $G := N \rtimes_\alpha H$ the semi-direct product. Write down a surjective homomorphism $\psi : G \to H$.

(17) Write down two non-abelian groups having 48 elements.

(18) Write down an injective homomorphism from the dihedral group

$$D_n := \langle a, b \mid a^2 = b^n = 1,\ aba^{-1} = b^{-1} \rangle$$

to the dihedral group

$$D_{mn} := \langle x, y \mid x^2 = y^{mn} = 1,\ xyx^{-1} = y^{-1} \rangle.$$

(19) Write down an automorphism of the cyclic group $\mathbb{Z}_8$ that is not the identity map.

(20) Give an example of a transitive action of the group of quaternions on a set with four elements.

(21) Let $A \subset (\mathbb{Q}, +)$ be the set of elements that can be written as $n/5$ where n is an integer. Let B be the subgroup of $\mathbb{Z}$ consisting of the even integers. Write down an explicit isomorphism $f : A \to B$.

4. Slightly longer questions

(1) Let Z denote the center of a group G and suppose that G/Z is a cyclic group. Show that $Z = G$, i.e., that G is in fact abelian.
(2) Explain why a group of order 51 must have an element of order 3. You can use Lagrange's theorem.
(3) Let H be a subgroup of G such that $[G : H] = n$. Explain why there is a non-trivial homomorphism $\phi : G \to S_n$ having H in its kernel. Hint: Think about G acting on cosets of H.
(4) Show that a subgroup H is normal if and only if every left coset is also a right coset.
(5) If N is a normal subgroup such that $[G : N] = n$ show that $x^n \in N$ for all $x \in G$.
(6) Let p and q be prime numbers with $p > q$. Let G be a group with pq elements. A theorem of Cauchy tells us that G has an element of order q. Let N be the subgroup generated by that element. Show that if N is normal, then G is abelian. Hint: Consider the action of G on N by conjugation.