MATH 402 **GROUP THEORY QUESTIONS** **FALL 2005**

(1) Define a group.
(2) Define a subgroup.
(3) Which cosets of a subgroup H in G are themselves subgroups?
(4) Define a homomorphism.
(5) Define a normal subgroup.
(6) Define the order of an element $x \in G$.
(7) If the order of x is 36, what are the orders of the following elements of G: x^{-1}, x^{-8}, x^{15}, x^{27}? Explain your answers.
(8) Give an example of a group of order 63 that is not cyclic, and an element in it that has order 21.
(9) Give an example of a non-abelian group of order 120 and an element in it that has order 6.
(10) Give an example of a group of order 125 and a subgroup of it that is not cyclic and has order 25.
(11) Suppose $x, y \in G$ are elements of G having orders 2 and 3. Does xy have order 6? Explain your answer.
(12) If $x, y \in G$ show that xy and yx have the same order.
(13) What is the order of the permutation $(167)(12345)$ in the symmetric group S_{10}?
(14) What is the order of $(1567)(134)(12)$ in the symmetric group S_{10}?
(15) Is (36) the same element as (63) in S_8? Explain.
(16) What does the permutation $(654) \in S_8$ do to the element $5 \in \{1, 2, 3, 4, 5, 6, 7, 8\}$?
(17) Define the order of a group.
(18) Explain how the order of an element $x \in G$ is related to the order of a particular subgroup of G.
(19) Define what is meant by "*the subgroup generated by an element $x \in G$*".
(20) Define what is meant by "*the subgroup generated by elements $x, y, z \in G$*".
(21) What are the elements in the subgroup of S_4 generated by (124) and (14)? What is that subgroup isomorphic to?
(22) What are the elements in the subgroup of S_7 generated by (356)? What is that subgroup isomorphic to?
(23) What are the elements in the subgroup of S_7 generated by (6317) and $(73)(16)$? What is that subgroup isomorphic to?
(24) What are the elements in the subgroup of S_7 generated by (53) and (26)? What is that subgroup isomorphic to?
(25) Show that the kernel of a homomorphism is a normal subgroup.
(26) If N is a normal subgroup of G, show that $xN = Nx$ for all $x \in G$.

(27) Let N be a normal subgroup of G. Define G/N; what are its elements, and what is the law of composition on it?
(28) Let N be a normal subgroup in G. What is the identity element in G/N, and what is the inverse of an element in G/N?
(29) In proving that G/N is a group where do we first use the fact that N is normal? What would go wrong if we tried to show that G/N is a group when N is a subgroup that is not normal?
(30) Let N be a normal subgroup, define $\phi : G \to G/N$ by $\phi(x) = xN$. Show this is a group homomorphism. What is $\ker \phi$?
(31) List all subgroups of $\mathbb{Z}$.
(32) State Lagrange's Theorem.
(33) State and prove Lagrange's Theorem.
(34) Let H be a subgroup of a group G. If $x, y \in G$ show that either $xH \cap yH = \phi$ or $xH = yH$.
(35) Show that a coset xH has the same number of elements as H.
(36) Let G be a group of order n. Does Lagrange's theorem tell you that the order of every element in G must divide n? Justify your answer.
(37) Let G be any group and $x \in G$ an element of order n. Define a homomorphism $\phi : \mathbb{Z} \to G$ whose image is $\langle x \rangle$. Apply the First Isomorphism Theorem to this situation, and explain why it shows that all cyclic groups of order n are isomorphic.
(38) Show that two left cosets of a subgroup H in G are either identical or have empty intersection.
(39) Give an example of the automorphism of the integers other than the identity map.
(40) Write down an isomorphism $f : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$.
(41) Write down an isomorphism $f : (\mathbb{R}_{>0}, \cdot) \to (\mathbb{R}, +)$.
(42) Write down three *different* isomorphisms $f : (\mathbb{R}_{>0}, \cdot) \to (\mathbb{R}, +)$.
(43) Define a new law of composition on $\mathbb{Z}$ as follows:

$$a * b := a + b + 2$$

where the operations on the right-hand sides of this definition are the usual operations in $\mathbb{Z}$.
  (a) Show that is associative.
  (b) Show there is an identity for this law of composition.
  (c) What is the inverse of $a \in \mathbb{Z}$ for this law of composition.
  (d) Find an isomorphism $f : (\mathbb{Z}, +) \to (\mathbb{Z}, *)$.
(44) Show that the intersection of two normal subgroups of G is a normal subgroup of G.
(45) The center of a group is $\{x \in G \mid xy = yx \text{ for all } y \in G\}$. We usually denote it by Z or $Z(G)$ (from the German word *zentrum*). Show that the center of G is a normal subgroup of G.
(46) If $n \geq 3$, show that $Z(S_n) = \{1\}$, i.e., consists only of the identity element.

(47) The cartesian product $A \times B$ is made into a group by defining a product
$$(a, b).(c, d) :=???$$
for $a, c \in A$ and $b, d \in B$.
(48) In particular, $A \times A$ is a group if A is. If x is an element of A having order 6, what is the inverse of $(x, x^2) \in A \times A$?
(49) If x is an element of A having order 36, what is the inverse of $(x^9, x^{30}) \in A \times A$?
(50) If x is an element of A having order 36, what is the order of $(x^6, x^{15}) \in A \times A$? Explain.
(51) Write down an isomorphism $\phi : \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$.
(52) If a and b are positive integers whose greatest common divisor is 1 show that $\mathbb{Z}_{ab}$ is isomorphic to $\mathbb{Z}_a \times \mathbb{Z}_b$.
(53) Write down an automorphism of the cyclic group $\mathbb{Z}_8$ that is not the identity map.
(54) Is the automorphism in the last question an inner automorphism? Explain your answer.
(55) State the First Isomorphism Theorem.
(56) State and prove the First Isomorphism Theorem.
(57) State the Second Isomorphism Theorem.
(58) State and prove the Second Isomorphism Theorem.
(59) State the Third Isomorphism Theorem.
(60) State and prove the Third Isomorphism Theorem.
(61) Prove that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the only groups of order 4 up to isomorphism.
(62) Prove that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
(63) Prove that $\mathbb{Z}_6$ and S_3 are the only groups of order 6 up to isomorphism.
(64) Complete the following sentence: *"Elements $x, y \in G$ are conjugate if ..."*.
(65) What must you prove in order to show that elements $x, y \in G$ are NOT conjugate?
(66) Show that (316) and (241) are conjugate in S_8.
(67) Suppose that $x, y \in S_n$. Show that x are not conjugate if they move different numbers of elements in $\{1, 2, \dots, n\}$.
(68) Define the symmetric group S_n.
(69) Write down an explicit injective homomorphism $S_4 \to S_7$.
(70) Let p be a prime and n a positive integer. Prove that a group of order p^n has an element of order p.
(71) Let p be a prime number. Give an example of a group of order p^6 that has an element of order p^3 but no element of order p^4.
(72) What is the quaternion group? Write down its multiplication table.
(73) Let Q be the quaternion group. Is the map $\phi : Q \to Q$ given by $\phi(x) = x^2$ a homomorphism?
(74) Are all subgroups of the quaternion group normal?
(75) Why is the only homomorphism $\phi : S_6 \to \mathbb{Z}_{77}$ the trivial one, $\phi(g) = 0$ for all $g \in S_6$?
(76) Is a subgroup of a subgroup a subgroup? Explain.
(77) Is a normal subgroup of a normal subgroup a normal subgroup? Explain.

(78) Give an example of a group G and subgroups A and B of orders 4 and 6 respectively such that $A \cap B$ has two elements. Give reasons.

(79) Give an example of a group G and subgroups A and B of orders 12 and 20 respectively such that $A \cap B$ has two elements.

(80) Give an example of a group G and subgroups A and B of orders 12 and 20 respectively such that $A \cap B$ has four elements.

(81) Give an example of a group G and subgroups A and B of orders 12 and 20 respectively such that $A \cap B \cong \mathbb{Z}_4$.

(82) Give an example of a group G and subgroups A and B of orders 12 and 20 respectively such that $A \cap B \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(83) Let $\phi : G \to H$ be an isomorphism. Show that $\phi^{-1} : H \to G$ is an isomorphism.

(84) Show that a composition of homomorphisms is a homomorphism.

(85) Show that a composition of isomorphisms is an isomorphism.

(86) Let x and y be different elements of a group G, neither of which is the identity, and suppose that

$$x^2 = y^2 = (xy)^2 = 1.$$

Show that the subgroup they generate is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (HINT: First determine all the elements in $\langle x, y \rangle$.)

(87) Let G be the cyclic group of order 12. How many subgroups of G have order 3? Explain.

(88) Let G be the cyclic group of order 12 and H a subgroup of order 3. Write down the cosets of H. To do this you first need to set up some notation for the elements of G so make sure you do that.

(89) Let G be a group and H a subgroup. Let $\mathbb{V}$ be the set of all right cosets aH of H in G, and let S denote the group of permutations of $\mathbb{V}$, i.e., S is the set of all bijective maps from $\mathbb{V}$ to $\mathbb{V}$ made into a group by defining the product of two bijections to be their composition; thus, if $\mathbb{V}$ has n elements S is isomorphic to the symmetric group S_n.

Show that the function $\phi : G \to S$ defined by

$$\phi(x)(C) := xC$$

is a homomorphism. Here we use the notation $xC := \{xc \mid c \in C\}$; check that gC is a right coset if C is. You must show that $\phi(xy) = \phi(x) \circ \phi(y)$. What is the kernel of ϕ?

(90) Show that a homomorphism ϕ is injective if and only if $\ker \phi = \{1\}$.

(91) Classify the groups of order 6. Give a proof.

(92) Let A and B be normal subgroups of G such that $A \cap B = \{1\}$. Show that
  (a) $ab = ba$ for all $a \in A$ and $b \in B$, and that
  (b) $AB \cong A \times B$.

(93) Let A and B be subgroups of G such that $AB = BA$. Show that AB is a subgroup of G.

(94) Give an example of subgroups A and B of S_3 such that $AB \neq BA$.

(95) Give an example of subgroups A and B of S_3 such that AB is not a subgroup of S_3. Explain why AB is not a subgroup.
(96) Let A and B be subgroups of a finite group G and suppose that $[G : A]$ and $[G : B]$ are relatively prime. Show that $G = AB$. What happens if G is infinite?
(97) In the symmetric group S_4 find two different elements that commute with each other and are conjugate to one another.
(98) In the symmetric group S_4 find two different elements that do not commute with each other and are conjugate to one another.
(99) In the symmetric group S_4 find two non-identity elements that commute with each other and are not conjugate to one another.
(100) In the symmetric group S_4 find two elements of order two that are not conjugate to one another.
(101) In the symmetric group S_4 find two elements that neither commute with each other and are not conjugate to one another.
(102) Write down all groups of order ≤ 11.
(103) Suppose that the group $(\mathbb{Q}, +)$ is isomorphic to a product of groups, say $\mathbb{Q} \cong A \times B$. Show that either A or B is the trivial group.
(104) If $\phi : G \to H$ is a bijective group homomorphism (i.e., an isomorphism) show that ϕ^{-1} is an isomorphism.
(105) Is the subgroup of S_5 generated by (31) and (15) isomorphic to S_3? Explain your answer.
(106) Is the subgroup of S_5 generated by (25) and (325) isomorphic to S_3? Explain your answer.
(107) Is the subgroup of S_5 generated by (14) and (235) isomorphic to S_3? Explain your answer.
(108) Is the subgroup of S_5 generated by (12)(34) and (34) isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$? Explain your answer.
(109) Is the subgroup of S_5 generated by (1234) and (34) isomorphic to $\mathbb{Z}_8$? Explain your answer.
(110) Is the subgroup of S_5 generated by (1234) and (13)(24) isomorphic to $\mathbb{Z}_4$? Explain your answer.
(111) Is the subgroup of S_5 generated by (124) and (35) isomorphic to $\mathbb{Z}_6$? Explain your answer.
(112) Find all homomorphisms from $\mathbb{Z}_7$ to S_5. Explain.
(113) Find the smallest n such that S_n contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Give reasons.
(114) Define an automorphism of a group G and show that the set $\mathrm{Aut}(G)$ of all automorphisms is a group.
(115) Let Z denote the center of a group G. Show that $\alpha(Z) = Z$ for all $\alpha \in \mathrm{Aut}(G)$.
(116) Define an inner automorphism of a group G and show that the set $\mathrm{Inn}(G)$ of all inner automorphisms is a normal subgroup of $\mathrm{Aut}(G)$.
(117) Show that $\mathrm{Inn}(G) \cong G/Z(G)$, where $Z(G)$ denotes the center of G. (Hint: Use the First Isomorphism Theorem.)

(118) Let H be a normal subgroup of G and let

$$A := \{\alpha \in \mathrm{Aut}(G) \mid \alpha(H) = H\}.$$

Show that A is a subgroup of $\mathrm{Aut}(G)$ and that there is a homomorphism $\Psi : A \to \mathrm{Aut}(G/H)$.

(119) Use the previous problem to show that there is a homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(G/Z)$.

(120) Suppose that G/Z is abelian. Show that $\mathrm{Inn}(G)$ is contained in the kernel of the homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(G/Z)$ in the previous problem.

(121) If $xy \in Z(G)$, show that $xy = yx$.

(122) If $\alpha \in \mathrm{Aut}(G)$, show that $H := \{a \in G \mid \alpha(a)a^{-1} \in Z(G)\}$ is a normal subgroup of G.

(123) Define an action of a group G on a set X.

(124) Show that the set $P(X)$ of all permutations of a set X is a group.

(125) Suppose $\phi : G \to P(X)$ is a group homomorphism. Write down an action of G on X that depends on ϕ.

(126) Suppose that G acts on X. Write down a group homomorphism $\phi : G \to P(X)$ that depends on that action. Show ϕ is a group homomorphism..

(127) Let G be a finite group with n elements. Show that G is isomorphic to a subgroup of the symmetric group S_n.

(128) Let Z denote the center of a group G and suppose that G/Z is a cyclic group. Show that $Z = G$, i.e., that G is in fact abelian.

(129) Give an example of a group G such that $G/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(130) If A and B are normal subgroups of G such that $A \cap B = \{1\}$ show that $ab = ba$ for all $a \in A$ and $b \in B$. Hint: consider $aba^{-1}b^{-1}$.

(131) Let H be a normal subgroup of G having order two. Show that H is contained in the center of G. Give an example to show that this does not hold if "two" is replaced by "three".

(132) Are the following subgroups H of $\mathrm{GL}_n(\mathbb{R})$ normal:

  (a) H consists of all matrices whose only non-zero entries lie on the diagonal.
  (b) H consists of all matrices whose only non-zero entries lie on the diagonal and all those diagonal entries are the same.
  (c) H consists of all those matrices that have zeroes below the diagonal.

(133) Write $\mathbb{R}^\times$ and $\mathbb{C}^\times$ for the multiplicative groups of non-zero real and complex numbers, respectively. Let $U \subset \mathbb{C}^\times$ be the subgroup consisting of the complex numbers of absolute value one. Define an isomorphism $\phi : U \times \mathbb{R}^\times \to \mathbb{C}^\times$.

(134) Let $\mathbb{R}$ be the additive group of real numbers. Define a surjective homomorphism $\phi : \mathbb{R} \to U$ and determine its kernel.

(135) Fix a set X and write $\mathcal{P}(X)$ for the set of all subsets of X. This is called the power set of X. Define a "product" on $\mathcal{P}(X)$ by the rule

$$A \bullet B := A \cap B.$$

This operation does *not* make $\mathcal{P}(X)$ a group but

(a) show that $\bullet$ is associative;
(b) show there is an identity, i.e., there is an element $I \in \mathcal{P}(X)$ such that $I \bullet A = A \bullet I = A$ for all $A \in \mathcal{P}(X)$;
(c) explain why inverses do not exist.

(136) [Not for the exam, but a cute exercise.] Fix a set X and write $\mathcal{P}(X)$ for the set of all subsets of X. This is called the power set of X. Define a "product" on $\mathcal{P}(X)$ by the rule

$$A \bullet B := A \cup B - A \cap B.$$

This operation makes $\mathcal{P}(X)$ a group:
(a) show that $\bullet$ is associative;
(b) show there is an identity, i.e., there is an element $I \in \mathcal{P}(X)$ such that $I \bullet A = A \bullet I = A$ for all $A \in \mathcal{P}(X)$;
(c) show that inverses exist.
(d) If $|X| = 4$, what is $\mathcal{P}(X)$?

(137) What is the center of the group of upper triangular 3×3 matrices with entries belonging to $\mathbb{R}$ and 1s on the diagonal?

(138) Let G be the group of upper triangular 2×2 matrices and let N be the subgroup of those matrices having determinant one. Show that $G/N \cong \mathbb{R}^\times$.

(139) Let G be any group and $x \in G$. Let $\phi : \mathbb{Z} \to G$ be the map $\phi(n) = x^n$. Show ϕ is a homomorphism. (In fact, we define x^n just so this is true!). Describe the kernel of ϕ in terms of the order of x. What does the First Isomorphism Theorem say in this context.

(140) If I is a subgroup of $\mathbb{Z}$ show that $I = d\mathbb{Z}$ for some $d \in \mathbb{Z}$. Prove this.

(141) Let H be a subgroup of G. Do not assume G is finite in this exercise. Show there is a bijection between the sets of right and left cosets of H in G so we can define the index $[G : H]$ in terms of right or left cosets. [Hint: there is an obvious map ϕ from left to right cosets, namely $\phi(aH) = Ha$; does this give a bijection? If not, what goes wrong in trying to prove it is a bijection?]

(142) Show that a subgroup H is normal if and only if every left coset is also a right coset.

(143) If N is a normal subgroup such that $[G : N] = n$ show that $x^n \in N$ for all $x \in G$.

(144) Let S be any subset of G and define $H := \{ginG \mid gs = sg \text{ for all } s \in S\}$. Show that H is a subgroup of G.

(145) Let p be a prime number dividing $|G|$. Let $x \in G$ and write $H := \{g \in G \; gx = xg\}$ and $C := \{gxg^{-1} \mid g \in G\}$. Show that p divides either $|H|$ or $|C|$. HInt: use the conjugation action of G on itself.

(146) Suppose $x \in G$ has order n. Show G has an element of order m for every positive number m that divides n.

(147) Let G be a finite abelian group whose order is divisible by the prime number p. Show G has an element of order p. Hint: induction on $|G|$; if $g \in G$ then p divides the order of $\langle g \rangle$ or $G/\langle g \rangle$.

(148) Let G be a finite group whose order is divisible by the prime number p. Show G has an element of order p. Hint: if G is abelian use the previous exercise, so you may assume G is not abelian and pick an element x that is not in the center of G. Use Exercise 145 and the Class equation. Also use induction on $|G|$.

(149) Let C be a conjugacy class in G. Prove that $D := \{x^{-1} \mid x \in C\}$ is a conjugacy class.

(150) Let G be a group of order n and let m be an integer relatively prime to n. Show that if $x^m = y^m$, then $x = y$. Hence show that for each $z \in G$ there is a unique $x \in G$ such that $x^m = z$.

(151) State the Class Equation for a finite group G.

(152) Show G is abelian if $x^2 = 1$ for all $x \in G$.

(153) If p is an odd prime show that a group of order $2p$ is either $\mathbb{Z}_{2p}$ or the dihedral group of order $2p$.