PROPOSITIONS ABOUT GROUP HOMOMORPHISMS

Definition. Suppose that G and G' are groups. A map $\varphi : G \to G'$ is called a homomorphism if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.

In the following propositions, we will always assume that G, G', etc. are groups. We let e, e' denote the identity elements of those groups.

1. If $\varphi : G \to G'$ is a homomorphism, then $\varphi(e) = e'$ and, for all $a \in G$, $\varphi(a^{-1}) = \varphi(a)^{-1}$. More generally, for any $a \in G$ and any $n \in \mathbf{Z}$, $\varphi(a^n) = \varphi(a)^n$.

2. If $\varphi : G \to G'$ and $\varphi' : G' \to G''$ are group homomorphism, then so is $\phi' \circ \phi : G \to G''$.

3. Suppose that $\varphi : G \to G'$ is a homomorphism. If $a \in G$ has finite order, then $\varphi(a)$ has finite order. If the order of a is m, then the order of $\varphi(a)$ divides m.

4. Suppose that $\varphi : G \to G'$ is a homomorphism. Let

$$\varphi(G) \;=\; \{\varphi(a) \mid a \in G\}$$

which is called the image of G under φ, or just the image of φ. Then $\varphi(G)$ is a subgroup of G'.

5. Suppose that $\varphi : G \to G'$ is a homomorphism. Define

$$Ker\ \varphi \;=\; \{a \in G \mid \varphi(a) = e'\ \}.$$

Then $K = Ker\ \varphi$ is a normal subgroup of G. It is called the *"kernel"* of the homomorphism φ. If $a, b \in G$, then $\varphi(a) = \varphi(b)$ if and only if $aK = bK$.

6. Suppose that $\varphi : G \to G'$ is a homomorphism. Then φ is injective if and only if $Ker\ \varphi = \{e\}$. If that is the case, then G is isomorphic to the subgroup $\varphi(G)$ of G'.

7. (First Homomorphism Theorem.) Suppose that $\varphi : G \to G'$ is a surjective homomorphism. Let $K = Ker\ \varphi$. Define $\psi : G/K \to G'$ by

$$\psi(aK) = \varphi(a).$$

Then ψ is an isomorphism of G/K to G'. If φ is not assumed to be surjective, then ψ is an isomorphism of G/K to the subgroup $\varphi(G)$ of G'.

8. (Correspondence Theorem.) Suppose that $\varphi : G \to G'$ is a surjective homomorphism and that K denotes the kernel of φ. Then there is a one-to-one correspondence between the following two sets:

the set of subgroups H of G which contain K and *the set of subgroups H' of G'.*

This correspondence is defined as follows.

$$H' = \varphi(H) = \{\varphi(h) \mid h \in H\}, \qquad H = \varphi^{-1}(H') = \{h \in G \mid \varphi(h) \in H'\}.$$

We then have an isomorphism $H/K \cong H'$. Furthermore, under this correspondence, H is a normal subgroup of G if and only if H' is a normal subgroup of G'.

9. (Third Homomorphism Theorem.) Under the assumptions of proposition 7, assume that N is a normal subgroup of G which contains K and that N' is the subgroup of G' corresponding to N. (That is, $N' = \varphi(N)$.) Then $G/N \cong G'/N'$.

10. (Second Homomorphism Theorem.) Suppose that H is a subgroup of G and that N is a normal subgroup of G. Then *(i)* HN is a subgroup of G, *(ii)* N is a normal subgroup of HN, *(iii)* $H \cap N$ is a normal subgroup of H, and *(iv)* $(HN)/N \cong H/(H \cap N)$.