THE LEFT AND RIGHT COSET DECOMPOSITIONS

We assume that G is a group and H is a subgroup of G.

Definition: Suppose that $a \in G$. The set $aH = \{ah \mid h \in H\}$ is called the *left coset* of H for a. The set $Ha = \{ha \mid h \in H\}$ is called the *right coset* of H for a.

Basic Properties:

1. If $h \in H$, then $hH = Hh = H$. Thus, H is both a left coset and a right coset for H.

2. If $a \in G$, then there is a bijection between H and aH. Thus, every left coset of H in G has the same cardinality as H. The same statements are true for the right cosets of H in G.

3. Suppose that $a \in G$. If $b \in aH$, then $bH = aH$. Similarly, if $b \in Ha$, then $Hb = Ha$.

4. If two left cosets of H in G intersect, then they coincide. If two right cosets of H in G intersect, then they coincide.

5. Every element of G belongs to exactly one left coset of H in G. Every element of G belongs to exactly one right coset of H in G. Thus, G is the disjoint union of the distinct left cosets of H in G. Also, G is the disjoint union of the distinct right cosets of H in G.

6. The set of left cosets of H in G (denoted by G/H) has the same cardinality as the set of right cosets of H (denoted by $H\backslash G$). If these sets are finite, their cardinality is denoted by $[G : H]$.

7. (The order-index equation) If G is finite, then $|G| = [G : H]\ |H|$.

8. If $a, b \in G$, we will write $a \equiv_L b \pmod{H}$ if $a^{-1}b \in H$. We refer to this relation on G as *"left congruence modulo H"*. It is an equivalence relation on G. The equivalence classes are precisely the left cosets of H in G. If $a \in G$, then the equivalence class for a under left congruence modulo H is the left coset aH.

9. If $a, b \in G$, we write $a \equiv_R b \pmod{H}$ if $ab^{-1} \in H$. We refer to this relation on G as *"right congruence modulo H"*. Similar statements to those in (8) are valid.

10. Suppose that $a \in G$. Then $aH = Ha$ if and only if $aHa^{-1} = H$.

Definition: A subgroup H of G is said to be a *"normal"* subgroup of G if $aHa^{-1} = H$ for *all* $a \in G$.

11. Here are some sufficient conditions for a subgroup H of G to be a normal subgroup of G:

(i): G is abelian and H is any subgroup. *(ii)*: G is any group and $[G : H] = 1$ or 2.

(iii): $H \subset Z(G)$. *(iv)*: For all $a, b \in G$, we have $aba^{-1}b^{-1} \in H$.