Math 402: Group Theory

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CHAPTER 0

In the beginning...

Every area of human enquiry requires its own particular language. That language is built up over a long period of time. It evolves in order to express the ideas that arise in the area. New words are created as new concepts develop. These languages are often impenetrable to outsiders. Usually there are good reasons for this. Mathematics is no exception in this regard.

Two fundamental concepts in mathematics are the notions of *sets* and *functions*. In order to state the definitions and results in any branch of mathematics we use the language of *set theory*. We use the English language too. In Spain though, they use the Spanish language together with the language of sets. It might be an enjoyable exercise for you to look at a mathematics book written in Spanish and observe that there is a lot in it you can understand without speaking Spanish simply because Spanish and English mathematicians use the same set-theoretic language, or notation.

Notation. We will use the standard notations of set theory. The notation is important. It is designed so we can say things briefly and precisely. It is part of the language of mathematics. You must learn to use it, and use it correctly. You must not, for example, confuse the symbols $\in$ and $\subset$; the first is used for elements, the second for subsets; it is ok to say that $2 \in \mathbb{Z}$, but not ok to say $\{2\} \in \mathbb{Z}$; the first of these reads "2 *is an element of* $\mathbb{Z}$, and the second of these reads "*the set consisting of* 2 *is an element of* $\mathbb{Z}$; it is ok to write $\{2\} \subset \mathbb{Z}$ because this reads "*the set consisting of* 2 *is a subset of* $\mathbb{Z}$.

It is common practice to use upper case letters for sets and lower case letters for elements. Of course, we can not always follow this practice—for example, the subsets $E = \{\text{even integers}\}$ and $O = \{\text{odd integers}\}$ of the integers are themselves elements of the two element set $\{O, E\}$.

1. The language of sets

1.1. A set is a collection of things. Not necessarily mathematical things: there is the set of US presidents, past and present, the set of people who are alive, the set of dogs, the set of single women with red hair that own a house in Paris, the set of letters in the Greek alphabet, the set of popsicles, the set of New Zealand citizens, and so on. Mathematical examples include the set of whole numbers (called integers), the set of prime numbers, the set

of even numbers, the set of odd numbers, the set of squares, the set of lines in the plane $\mathbb{R}^2$, the set of pairs of distinct points in 3-space, and so on.

Usually we use an upper case letter to denote a set and when specifying the set we use curly parentheses. For example, we could write

$$\begin{aligned} A &= \{\text{living mothers}\} \\ P &= \{\text{prime numbers}\} \\ E &= \{\text{even numbers}\} \\ O &= \{\text{odd numbers}\}. \end{aligned}$$

Particularly important sets have special names and are denoted by special symbols. For example,

$$\begin{aligned} \mathbb{N} &= \{\text{the natural numbers}\} = \{0,1,2,3,\dots\} \\ \mathbb{Z} &= \{\text{integers}\} = \{\text{whole numbers}\} \\ &= \{\dots, -2, -1, 0, 1, 2, \dots\} \\ \mathbb{Q} &= \{\text{rational numbers}\} = \{\text{fractions}\} \\ &= \{\tfrac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0\} \\ \mathbb{R} &= \{\text{real numbers}\} \\ \mathbb{C} &= \{\text{complex numbers}\} \\ \mathbb{R}^2 &= \{\text{the points in the plane}\}. \end{aligned}$$

We also improvise on these standard notations. Some examples are

$$\begin{aligned} 2\mathbb{N} &= \{\text{the even natural numbers}\} = \{0,2,4,6,\dots\} \\ \mathbb{Z}_{\leq 0} &= \{\text{the non-positive integers}\} = \{\dots, -3, -2, -1, 0\dots\} \\ 3\mathbb{Z} &= \{\text{all integer multiples of 3}\} = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\} \\ \mathbb{R}_{\geq 0} &= \{\text{the non-negative real numbers}\} = \{r \in \mathbb{R} \mid r \geq 0\}, \end{aligned}$$

Of particular importance is the empty set, denoted ϕ, that has no elements at all. It might seem a little odd to talk about the empty set and to have a special symbol for it, but think of the parallel with the symbol we use for zero, 0. It is quite interesting to read about the history of zero on Wikipedia. Check it out!

1.2. Elements. The things that belong to a set are called its elements. For example, the prime number 37 is an element of the set P above. The point $(3, -2)$ belongs to the set $\mathbb{R}^2$. The number π is an element the set $\mathbb{R}$.

We usually use lower case letters for the elements of a set. When x is an element of a set X, we write $x \in X$, and read this as *x is an element of X* or *x belongs to X* or *X contains x* or, simply, *x is in X*.

Before we can talk about a particular set we need a precise description of it. One possibility is to list its elements—for example, we can define the set V consisting of the vowels by writing $V = \{a, e, i, o, u, y\}$. If a set is finite, i.e., it does not have an infinite number of elements in it, it might be

possible to describe the set by explicitly writing out a list of all its members. The set of people now living on planet earth is finite, but we are not able to list all the elements in it. However, if a set is infinite, or even finite but extremely large, this is not possible, so we must find some short way to describe the set precisely.

A set is completely determined by its elements. Two sets are *equal* if and only if they have the same elements.

1.3. The symbol meaning "such that". We already mentioned the set of rational numbers $\mathbb{Q}$. You already know what fractions are but let's define them using set notation:

$$\mathbb{Q} = \left\{ \tfrac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\, b \neq 0 \right\}. \tag{0-1}$$

The vertical symbol | is read as "such that". Thus, the mathematical sentence (0-1) should be read as follows: $\mathbb{Q}$ is the set of all numbers $\frac{a}{b}$ such that a and b are integers and b is not zero.

Another common notation for "such that" is the colon. Using the colon the above sentence would be

$$\mathbb{Q} = \left\{ \tfrac{a}{b} \;:\; a, b \in \mathbb{Z},\, b \neq 0 \right\}. \tag{0-2}$$

I prefer the symbol | to the symbol : because it is more visible.

By using the symbol for such that we can give more succinct definitions of sets. For example, if $a \in \mathbb{Z}$, we write $a\mathbb{Z}$ for the set of integer-multiples of a and can write this succinctly as

$$a\mathbb{Z} := \{ax \mid x \in \mathbb{Z}\} = \{\cdots, -2a, -a, 0, a, 2a, \cdots\}.$$

In a similar way we introduce the notation

$$a - b\mathbb{Z} = \{a - bx \mid x \in \mathbb{Z}\}. \quad \text{and} \quad a + b\mathbb{Z} = \{a + bx \mid x \in \mathbb{Z}\}.$$

More explicitly, for example, $1 + 3\mathbb{Z} = \{\cdots, -5, -2, 1, 4, 7, \cdots\}$.

1.4. Cardinality. We denote the number of elements in a set X by $|X|$. We call the number of elements in a set its cardinality. For example, the cardinality of the empty set is zero; it is the only set whose cardinality is zero. Two sets are said to have the same cardinality if they have the same number of elements.

The cardinality of infinite sets is a subtle matter but we can't say more about this until we discuss the notion of injective, surjective, and bijective, functions. We will do this shortly and say more about cardinality once we have those notions in hand.

1.5. Subsets and containment. We say that a set X is contained in a set Y if every element of X is an element of Y. More formally, we say X is a subset of Y if it is contained in Y and write

$$X \subset Y$$

to denote this situation. Thus the symbol "$\subset$" is read as *is a subset of* or *is contained in*. For example,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

If X is any set, then $X \subset X$ and $\phi \subset X$.

It is quite common to prove the equality of two sets X and Y by proving that $X \subset Y$ and $Y \subset X$. In other words, $X = Y$ if and only if $X \subset Y$ and $Y \subset X$.

1.6. Intersection and union. Two important operations on sets are intersection and union. They bear some resemblance to addition and multiplication of numbers.

The intersection of sets X and Y consists of the elements that are in both X and Y. It is denotedby $X \cap Y$. Using the examples above, we have, for example,

$$P \cap E = \{2\}$$

because 2 is the only even prime number. You may know that there are infinitely many primes, so $P \cap O$ is an infinite set, i.e., it has infinitely many elements. Notice we wrote $P \cap E = \{2\}$ *not* $P \cap E = 2$. There is an important difference—the intersection of two sets is a set, and the number 2 is different from the set whose only element is 2.

The union of sets X and Y, which we denote by $X \cup Y$, consists of the elements that are in either X or Y. For example, $\{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}$. Likewise,

$$O \cup E = \mathbb{Z}$$

because every number is either even or odd. Notice that

$$O \cap E = \phi$$

because there are no numbers that are both even and odd.

If X is a subset of Y it is clear that $X \cap Y = X$ and $X \cup Y = Y$.

1.7. OR versus EXCLUSIVE OR. Sometimes students are confused by the way mathematicians use the word "or". In everyday language we tend to use the word "or" in a context where something is one thing or the other but can not be both. An example will help. We might say, *"my leg hurts"* and the listener might respond with the question *"does your left leg or your right leg hurt"* assuming that only one of your legs hurts and you will answer by stating which of your two legs hurts. However, the pedantic mathematician might answer *"yes"*, meaning that it is true that *at least one of his legs hurts.*

In mathematics when we say that either P or Q is true we mean that one of the following is true:

- P is true but Q is false;
- Q is true but P is false;
- both P and Q are true.

Thus, if someone asks *"is 729 even or odd"*, the pedantic mathematician will answer *"yes"*. Frustrating for others, but that's the way it is.

To distinguish the mathematician's notion of *OR* from the general public's idea mathematicians introduce the notion of what is called *"exclusive or"*. We sometimes write XOR to mean *"exclusive or"*. Thus, one would answer yes to the question *"is P XOR Q true"* only if

- P is true but Q is false;
- Q is true but P is false;

I haven't explained that very well...read about it on the web.

1.8. Properties of intersection and union. You already know the basic properties of the arithmetic operations $+$ and $\times$. For example, there are the associative rules,

$$a + (b + c) = (a + b) + c \qquad \text{and} \qquad a \times (b \times c) = (a \times b) \times c$$

which implies that the expressions $a + b + c$ and $a \times b \times c$ are unambiguous. You also know that

$$a + b = b + a \qquad \text{and} \qquad a \times b = b \times a.$$

Slightly more sophisticated is the distributive rule

$$a \times (b + c) = a \times b + a \times c$$

which involves both operations, addition and multiplication. And zero has two special properties

$$0 \times a = 0 \qquad \text{and} \qquad 0 + a = a$$

for all numbers a.

There are analogous properties for the set operations $\cup$ and $\cap$. For any sets X, Y, and Z,

$$\begin{aligned}
X \cup Y &= Y \cup X \\
X \cap Y &= Y \cap X \\
X \cup (Y \cup Z) &= (X \cup Y) \cup Z \\
X \cap (Y \cap Z) &= (X \cap Y) \cap Z \\
X \cap (Y \cup Z) &= (X \cap Y) \cup (X \cap Z) \\
X \cup (Y \cap Z) &= (X \cup Y) \cap (X \cup Z) \\
\phi \cap X &= \phi \\
\phi \cup X &= X.
\end{aligned}$$

Mathematicians like it when there are similarities like this between the arithmetic operations $+$ and $\times$ and the set operations $\cup$ and $\cap$. Of course, there are some significant differences too. For example, $X \cap X = X \cup X = X$. One other difference is that there are *two* distributive laws for $\cap$ and $\cup$ but only one distributive law for $+$ and $\times$ ($+$ does not distribute across $\times$).

All these properties are easy to check. The only ones that might require some care are the distributive laws. You should try to prove them yourself. The strategy to use is that I mentioned earlier for showing two sets are equal: show each is a subset of the other.

There is also a similarity between $\subset$ and $\leq$. It is a good exercise for you to write down some of the similarities.

1.9. Disjoint union. The union of two subsets B and C of a set A is denoted by $B \cup C$. If $B \cap C = \phi$, we say that B and C are disjoint. We sometimes write $B \sqcup C$ to denote the union of two disjoint sets, and call it the disjoint union of B and C. For example, $\mathbb{Z}$ is the disjoint union of the even and the odd numbers. More generally, if n is a positive integer, then $\mathbb{Z}$ is the disjoint union of the n subsets,

$$i + n\mathbb{Z} := \{a \in \mathbb{Z} \mid a \text{ leaves a remainder of } i \text{ when divided by } n\}$$

as i runs through the numbers 0 to $n-1$. Of course, you know this already but a rigorous proof requires some thought (see Theorem 11.1).

Notice that $n\mathbb{Z}$ consists of the numbers divisible by n, which is exactly $\{nb \mid b \in \mathbb{Z}\}$; that is why we write $n\mathbb{Z}$ for this set—it is all multiples of n. And the notation $i+n\mathbb{Z}$ is explained by the fact that $i+n\mathbb{Z} = \{i+nb \mid b \in \mathbb{Z}\}$.

Of course, we could define $i + n\mathbb{Z}$ for any integer i to be $\{i + nb \mid b \in \mathbb{Z}\}$. We will do this later, but you need to be warned that $i+n\mathbb{Z} = j+n\mathbb{Z}$ if $i-j$ is divisible by n. For example, the set of odd numbers is $1 + 2\mathbb{Z} = 3 + 2\mathbb{Z} = -57 + 2\mathbb{Z}, \ldots$.

1.10. Set difference. If $A \subset B$ we define

$$B - A := \{b \in B \mid b \notin A\}.$$

We also use this notation when A and B are subsets of a set C, even though B need not be contained in A. For example, if $A = \{0, 1, 2, 3, 4\} \subset \mathbb{Z}$ and E is the set of even integers, then $A - E = \{1, 3\}$.

1.11. Products. The Cartesian product of sets X and Y is the set

$$X \times Y := \{(x, y) \mid x \in X,\, y \in Y\};$$

that is, $X \times Y$ consists of all ordered pairs (x, y) in which x belongs to X and y belongs to Y. The Cartesian product $Y \times X$ consists of all ordered pairs (y, x) in which x belongs to X and y belongs to Y. Note that $X \times Y \neq Y \times X$ unless $X = Y$,

Notice this: if X and Y are finite, we have the formula

$$|X \times Y| = |X| \times |Y|.$$

That's why we use the word "product"; the number of elements in the product of a sets is the product of the number of elements in each set. In fact, it is interesting to pause and think that thousands of years ago, before man had much mathematics, he must have had a notion of Cartesian product: if we have 3 families and each family needs two spears we need a total of six

spears. I've expressed that poorly, but when you think about it the notion of multiplication must surely have arisen *after* the notion (intuitive and not explicitly expressed) of Cartesian product.

You should also wonder why we use the word Cartesian. Any ideas?

2. Functions

2.1. A function $f : X \to Y$ from a set X to a set Y is a *rule* that assigns to each element in x an element $f(x)$ in Y. We also call a function a map or mapping.

A deceptively important, deceptive because apparently trivial, function is the identity function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) = x$ for all $x \in X$. There are many identity functions, one for each set.

2.2. Composition of functions. We can compose two functions: If $f : X \to Y$ and $g : Y \to Z$, we define $g \circ f : X \to Z$ to be the function

$$(g \circ f)(x) := g(f(x)).$$

We call $g \circ f$ the composition of g and f. Somtimes we omit $\circ$ and just write gf.

2.2.1. *Composition of functions is associative.* Composition is associative, meaning that

$$(hg)f = h(gf)$$

whenever these compositions make sense; we usually write hgf for this. Do *not* read this sentence only once and pass on, think about why it is true. Can you prove it? Associativity of composition of functions is every bit as important as the fact that multiplication of numbers is associative. Look at the proof of Lemma 0.2 and isolate the step in which associativity of composition is used.

Associativity is a deceptively simple property. Too often beginners use it in situations where it does not apply. One of my favorite examples of its misuse, if one can use the word *"favorite"* to choose among terrible sins, occurs in elementary calculus. Certainly, students in an elementary calculus course would agree that subtraction is not an associative operation because $(a-b)-c \neq (a-b)-c$ in general. Perhaps with a little more thought such students might agree that division is not an associative operation. Despite that, I have seen many students write the following

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{-h}{x(x+h)}}{h}$$

but the last expression is ambiguous because $-$ is not associative. For example,

$$\frac{\left(\frac{1}{2}\right)}{2} \neq \frac{1}{\left(\frac{2}{2}\right)}.$$

2.2.2. *The identity function.* Identity functions have the following important property: if $f : X \to Y$ is any function, then

$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f.$$

In this regard the identity function behaves like the number 1. Perhaps a better analogy is with matrices: there are many identity matrices, one of size $n \times n$ for each $n \in \mathbb{N}$, and the result of the product of a matrix A with the appropriate sized identity matrix is A.

If $f : X \to X$ we often write f^2 rather than ff or $f \circ f$. Likewise we write f^3 for fff and so on. It is convenient to define f^0 to be id_X. We then have the wonderful formula

$$f^m f^n = f^{m+n}$$

for all $m, n \in \mathbb{N}$.

2.3. Injective, surjective, and bijective, functions. We now meet some particularly important classes of functions.

Let $f : X \to Y$. We say that f is

- injective, or one-to-one or 1-1, if $f(x) \neq f(x')$ whenever $x \neq x'$;[1]
- surjective, or onto, if each $y \in Y$ is equal to $f(x)$ for some $x \in X$;
- bijective if it is both injective and surjective.

An injective, surjective, or bijective function is called an injection, surjection, or bijection for short.

To check your understanding of these notions prove the following result.

LEMMA 0.1. *Let $g : X \to Y$ and $f : Y \to Z$ be functions.*

(1) *If $f \circ g$ is injective so is g.*
(2) *If $f \circ g$ is surjective so is f.*
(3) *$f \circ g$ is injective if f and g are.*
(4) *$f \circ g$ is surjective if f and g are.*

You should also decide whether the converse of each of these statements is true or not.

2.3.1. *The range of a function.* The range of a function $f : X \to Y$, which we denote by $\mathcal{R}(f)$, is the set of all values it takes, i.e.,

$$\mathcal{R}(f) := \{f(x) \mid x \in X\}.$$

I like this definition because it is short but some prefer the equivalent definition

$$\mathcal{R}(f) := \{y \in Y \mid y = f(x) \text{ for some } \ x \in X\}.$$

[1]Many people think *"one-to-one"* is a lousy name. I agree. It would be better to say such a function is *"different-to-different.* After all, that is its defining property: f is injective if it sends different elements of X to different elements of Y.

2.3.2. Consider the function

$$f : \{\text{men}\} \to \{\text{women}\}, \qquad f(x) := \text{the mother of } x.$$

Is f well-defined, injective, surjective, bijective? What is the range of f, Give reasons, in standard English prose, for your answers.

2.3.3. The sine function $\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one because $\sin \pi = \sin 2\pi$. It is not onto because its range is $[-1, 1] = \{y \mid -1 \leq y \leq 1\}$. However, if we consider sin as a function from $\mathbb{R}$ to $[-1, 1]$ it is becomes onto, though it is still not one-to-one. If we consider sin as a function from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1, 1]$ it is becomes bijective. That is why we define the inverse sine function $\sin^{-1}$ as a function from $[-1, 1]$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

2.3.4. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not one-to-one because $f(2) = f(-2)$ for example, and is not onto because its range is $\mathbb{R}_{\geq 0} := \{y \in \mathbb{R} \mid ly \geq 0\}$. However, the function

$$f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \qquad f(x) = x^2,$$

is both one-to-one and onto.

2.3.5. *Exercise.* Let X be a finite set and $f : X \to X$ a function. Show that f is injective if and only if it is surjective.

2.3.6. *Domain and codomain.* If f is a function from X to Y we call X the domain and Y the codomain of f. I don't use these words very often myself but you can read this paragraph as part of your general cultural education. The examples in sections 2.3.3 and 2.3.4 show that for a give formula $f(x)$ the question of whether f is injective or surjective depends in a delicate way on the choice of domain and codomain.

If we want to be very precise, we define a function as a triple (f, X, Y) consisting of a *domain* X, a *codomain* Y, and a subset Γ_f of $X \times Y$ such that if $x \in X$, there is a unique y in Y such that $(x, y) \in \Gamma_f$. We then write $f(x)$ for the unique element in Y such that $(x, f(x))$ is in Γ_f. We call Γ_f the graph of the function.

Thus, when we just give a formula $f(x)$ to define a function we are not really giving a complete definition. We must also state the domain and codomain to give a complete definition of a function.

LEMMA 0.2. *The following properties of a function $f : X \to Y$ are equivalent:*

(1) *f is bijective;*
(2) *for each y in Y there is a unique $x \in X$ such that $f(x) = y$;*
(3) *there is a unique function $g : Y \to X$ such that $fg = id_Y$ and $gf = \mathrm{id}_X$.*

Proof. (1) $\Rightarrow$ (2) Let $y \in Y$. Since f is surjective, $y = f(x)$ for *some* $x \in X$. Since f is injective there can be only one $x \in X$ such that $f(x) = y$.

(2) $\Rightarrow$ (3) Define $g : Y \to X$ by declaring that $g(y)$ is the unique x in X such that $f(x) = y$. If $y \in Y$, then $fg(y)$ is y because $g(y)$ is defined just so it has the property that $f(g(y)) = y$. Thus $fg = \mathrm{id}_Y$. If $x \in X$, and $y = f(x)$,

then $g(y)$ is defined to be x; i.e., $x = g(y) = gf(x)$. Thus $gf = \mathrm{id}_X$. Thus the function g we have just defined has the claimed properties.

Suppose $g' : Y \to X$ also had the property that $fg' = id_Y$ and $g'f = \mathrm{id}_X$. Let $y \in Y$. Then $y = \mathrm{id}_Y(y)$ so

$$g(y) = g\big(\mathrm{id}_Y(y)\big) = g(fg'(y)) = (gf)(g'(y)) = \mathrm{id}_X\big(g'(y)\big) = g'(y).$$

Thus $g(y) = g'(y)$ for all $y \in Y$ and we conclude that $g = g'$. (Notice we used the associative law for composition of functions in our calculation.)

(3) $\Rightarrow$ (1) Since $fg = \mathrm{id}_Y$ and id_Y is surjective, f is surjective. Since $gf = \mathrm{id}_X$ and id_X is injective, f is injective. □

2.4. The inverse of a function, if it exists. The function g in part (3) of Lemma 0.2 is called the inverse of f and is denoted by

$$f^{-1}.$$

It may be defined by declaring that $f^{-1}(y) = x$ provided that $f(x) = y$. We then have

$$f^{-1} \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}_Y .$$

These formulas provide the defining property of f^{-1} provided it exists.

If f is a bijective function from a set X to itself and $n \in \mathbb{N}$, we define f^{-n} to be

$$(f^{-1})^n = \underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{n \text{ times}}.$$

We then have the wonderful formula

$$f^m f^n = f^{m+n}$$

for all $m, n \in \mathbb{Z}$.

2.4.1. *An example.* The inverse of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$, is the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{2}x$ because

$$fg(x) = 2g(x) = 2(\frac{1}{2}x) = x \qquad \text{and} \qquad gf(x) = g(2x) = \frac{1}{2}(2x) = x.$$

2.4.2. *Warning: A function need not have an inverse!* For example, the function $h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = 2n$, does not have an inverse. Why?

2.4.3. *Important:* Notice I defined *the* inverse of a function (if it exists) not *an* inverse. The result in the next lemma explains why I used *the* and not *an*.

LEMMA 0.3. *A function can have at most one inverse.*

Proof. Let $f : X \to Y$. Suppose $g_1 : Y \to X$ and $g_2 : Y \to X$ are such that

$$fg_1 = fg_2 = \mathrm{id}_Y \qquad \text{and} \qquad g_1 f = g_2 f = \mathrm{id}_X .$$

If $y \in Y$, then $y = \mathrm{id}_Y(y) = fg_1(y)$ so

$$g_2(y) = g_2\big(fg_1(y)\big) = \big(g_2(fg_1)\big)(y) = \big((g_2 f)g_1\big)(y) = (\mathrm{id}_X\, g_1)(y) = g_1(y).$$

Since $g_1(y) = g_2(y)$ for all $y \in Y$, $g_1 = g_2$. □

Because f can have at most one inverse we denote that inverse by f^{-1} whenever it exists.

2.4.4. *One-sided inverses.* Let $f : X \to Y$. It is possible for there to be a function $g : Y \to X$ such that $fg = \mathrm{id}_Y$ but $gf \neq \mathrm{id}_X$. Likewise, there might be a function $h : Y \to X$ such that $hf = \mathrm{id}_X$ but $fh \neq \mathrm{id}_Y$. For example, let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and let $f : \mathbb{N} \to \mathbb{N}$ be the function $f(n) = n + 1$. Let $g : \mathbb{N} \to \mathbb{N}$ be the function

$$g(n) = \begin{cases} n & \text{if } n = 0 \\ n - 1 & \text{if } n \geq 1. \end{cases}$$

Then $gf = \mathrm{id}_{\mathbb{N}}$ but $fg \neq \mathrm{id}_{\mathbb{N}}$ because $fg(0) = 1$. Thus g is a left inverse to f but not a right inverse. Similarly, f is a right inverse to g but not a left inverse.

However, if $f : X \to Y$ has a left inverse and a right inverse, then the left inverse must equal the right inverse so f has an inverse. To see this, suppose there are functions $g : Y \to X$ and $h : Y \to X$ such that $fg = \mathrm{id}_Y$ and $hf = \mathrm{id}_X$. Then

$$g = \mathrm{id}_X \circ g = (hf) \circ g = (hf)g = h(fg) = h \circ \mathrm{id}_Y = h.$$

2.4.5. *Warning about notation:* Do *not* confuse the function f^{-1} with the function $1/f$. In fact, we have not even defined a function called $1/f$ so there can't possibly be any confusion. Still, you might imagine a situation in which it is logical to use the label $1/f$ for a certain function related to f; indeed, there are such situations but it need not be the case that $1/f$ has anything to do with f^{-1}.

2.4.6. *Exercise.* Think of a situation in which one might reasonably label a function $1/f$ yet $1/f$ need have nothing to do with f^{-1}.

2.5. More about cardinality. Sets X and Y have the same cardinality if there is a bijection $f : X \to Y$. For example the set of integers has the same cardinality as the set of even integers, *even though* $2\mathbb{Z}$ *is a proper subset of* $\mathbb{Z}$. The function $f : \mathbb{Z} \to 2\mathbb{Z}$, $f(n) := 2n$, is a bijection.

Show that the property of having the same cardinality is an equivalence relation. The equivalence classes are called *cardinals*. It is convenient to actually say that $|X|$ denotes the equivalence class that X belongs to. We write $|X| \leq |Y|$ if there is an injective function $f : X \to Y$. If $|X| \leq |Y|$ and $|Y| \leq |X|$ is $|X| = |Y|$?

A set having the same cardinality as $\mathbb{Z}$ is said to be countable.

Two important examples, both for illustrating the idea of cardinality, and for their historical impact are these: (1) $\mathbb{Z}$ and $\mathbb{Q}$ have the same cardinality; (2) $\mathbb{Z}$ and $\mathbb{R}$ have different cardinalities. The latter example shows there are different kinds of infinity. Indeed, we say that

Show for finite sets X and Y that there is a bijection (see below) $f : X \to Y$ if and only if $|X| = |Y|$.

Infinite cardinals. Different cardinals.

One of the most famous arguments in mathematics is Cantor's diagonal argument to show that $|\mathbb{Q}| = |\mathbb{Z}|$. This is famous not just because it proves an important fact but famous because no one had even thought of asking the question before Cantor. Indeed, before Cantor there was not even a mathematical framework in which one could rigorously ask the question whether $\mathbb{Z}$ and $\mathbb{Q}$ have the same number of elements.

To appreciate this you might ask yourself this: if there are injective functions $f : X \to Y$ and $g : Y \to X$, does it make sense to say that X and Y have the same number of elements? Is $|X| = |Y|$?

Is $|\mathbb{R}| = |\mathbb{Z}|$. If not, is there a set S with $|\mathbb{Z}| < |S| < |\mathbb{R}|$?

You can read about these topics on the web.

2.5.1. *Exercise.* Let's see whether you have understood some of the above. First we introduce the notation

$$\mathrm{Map}(X, Y) := \{\text{all functions } f : X \to Y\}.$$

If $|X| = |X'|$ and $|Y| = |Y'|$ prove that

$$|\,\mathrm{Map}(X, Y)| = |\,\mathrm{Map}(X', Y')|.$$

Prove this by choosing explicit bijections $a : X \to X'$ and $b : Y \to Y'$ and using those to define an explicit bijection

$$\Phi : \mathrm{Map}(X, Y) = \mathrm{Map}(X', Y').$$

Let X, Y, and Z, be any sets; show there is a (natural) bijection

$$\Psi : \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

Please note that we are *not* assuming that the sets X, Y, or Z, are finite so you can't prove this just by counting elements!

3. Writing Mathematics

3.1. In a mathematics book you will see the words

Lemma, Proposition, Theorem, Corollary.

Each of these is followed by a precise statement of a result/fact, and that is followed by a proof of the statement. Or it should be! These four words indicate, to some extent, the status of the result. A theorem is very important and a proposition important. A lemma is usually a result proved in preparation for proving a theorem or proposition. Some lemmas that have a short and simple proof acquire a particular importance because they encapsulate simple observations that are used over and over again. For example, we frequently make use of the observation that a function from a finite set to itself is injective if and only if it is surjective. A corollary is a useful consequence of a theorem, usually with a much simpler proof than the theorem. A deep theorem might have many corollaries worth stating.

A good topic for discussion when you next kick back with some other math geeks is why we use such labels. Why do we bother to organize the results is such a way? Why do we care about proofs anyway? And, what

constitutes a proof? Why are some proofs so hard to understand? If you find a particular proof difficult to understand it is often a good idea to try writing it out in a different way. For example, you might try to rearrange the order of the different paragraphs or steps in it. Why should we bother understanding the proofs in this course? The flippant answer to the last question is because your grade depends on it.

This will be the first or second course in which you encounter

the axiomatic method.

Although the axiomatic method is the rock on which Euclid's books are built, it is only over the past 200 years that mathematicians have adopted this formal and precise way of presenting their subject as *the right way to present the material*.

The foundation of the axiomatic method consists of

Definitions.

We make precise definitions of the objects and precise definitions of their properties.

In a group theory course the primary objects are groups and rings; there are many secondary objects too that require precise definitions: subgroups, normal subgroups, quotients, ideals, ... It is all rather frightening at first. But it is through making precise definitions that we place our subject on a solid foundation; and I mean rock-solid. There can be no debate about what a group is once the definition has been made.

In a linear algebra course the primary objects are matrices, vector spaces, linear transformations, bases, and notions such as dimension, invertibility, and so on.

It is having rock-solid definitions and rock-solid proofs that distinguishes mathematics from all other lines of human inquiry. There is a certainty in mathematics that you will not see in physics, chemistry, biology, economics, psychology, sociology, astrology, and so on. All sciences strive to mathematicize, to place themselves on a solid footing. Man seeks certainty.

Having precise definitions, and making precise statements in Theorems and their proofs, can give mathematics a dry appearance. However, the examples and applications are what give the subject life; an analogy with biology can help; the examples, for example the various groups that we know, the groups that nature has given us, are the living creatures that we study; they all have their special characteristics and features; there are patterns, they can occur in families, there are the cyclic groups, the symmetric groups, the simple groups, and so on. We study these creatures. The theorems and so on are reports on the results of our studies: for example, every symmetric group S_n for $n \geq 5$ contains a simple subgroup with exactly half as many elements as itself; if H is a subgroup of a group G, then the number of elements in H divides the number of elements in G; and so on.

3.2. English. These notes are written in a mixture of standard english prose and mathematical notation. So are mathematics books. That is how to express the ideas of mathematics: in prose and notation, woven together into *grammatically correct sentences.* You must do the same in this course. It is the only way to express your ideas clearly. When you write out a solution to a problem your goal is to convey certain ideas to the reader. It is your responsibility to do that clearly and unambigously. It is not the reader's responsibility to struggle to discern your meaning. You must be the reader's guide and friend, making his or her life easy.

You must begin every sentence with an upper-case letter. You must end each sentence with a period. A sentence has a subject, a verb, and an object. Sentences should not be unduly long. Follow the rules of English grammar. These rules are introduced in third grade and by sixth grade most children have learned them. You are now at a university and must write at the university level.

I have sometimes heard the complaint that this is a mathematics course, not an english course. But, whatever the field of human inquiry, the only way to convey its ideas is through language, and that language is always a mixture of everyday prose and the technical terminology and notation of the particular field.

So, write, and write well.

3.3. Proofs. You must write a lot of proofs in this course. A proof is an argument designed to convince your reader that something is true, or false.

A proof is not a list of equations that you have used to persuade yourself that *you* understand why something is true. A proof is an argument to persuade *someone else* that something is true. You must not expect the reader to assemble your list of equations into an argument by inserting appropriate phrases and punctuation. It is your job to do that. Your equations and calculations are typically the work you do prior to assembling an argument. A good argument will *use* your equations but will weave them into an argument.

Think of the analogy of cooking. A proof is something like a recipe. It is not just a list of ingredients. Certainly one needs to know the ingredients, but one must also know how to combine them and in what order and in what proportions. It is no good giving a list of ingredients and then in the narrative part of the recipe mentioning only some of them. You leave the reader baffled—was I supposed to add sugar or not? Likewise, in a proof, you should say everything that is necessary and nothing that is superfluous. Like cooking, it is a difficult skill. It takes years to develop and you can spend a lifetime honing that skill. Developing that skill is an important part of this course. Practice, experience, and judgment, are required to do this well.

Your proof should use phrases like "If..., then ...", and words like "because", "since", "therefore", "so", "since", "but", and so on.

3.4. Definitions. I will ask you to state a large number of definitions in the midterm and finals exams. Definitions are as important as proofs. I can do no better than to quote Giuseppe Peano:

> *There is no need to prove every theorem in class, but let us at least have precise concepts and correct definitions. Rigor does not consist in proving everything. It consists in saying what is true and not saying what is not true.*

Definitions are important historical landmarks. They generally emerge over an extended period of time as mathematicians come to isolate the important, essential, and fruitful concepts. The important features of a mathematical object have names. For example, important features of a group include whether it is *finite*, *abelian*, *cyclic*, of prime *order*, *simple*, and so on? Other important features of a group involve its *subgroups*, *conjugacy classes*, and so on. To compare two groups we use *homomorphisms*, their *kernels* and *images*, *isomorphisms*, and so on. There is no avoiding the fact that you must learn these and other definitions. There are lots of them and it looks daunting at first. As with any new vocabulary, whether English or foreign, using the new words is the way to embed them in your mind. Use them in conversation with others. Doing many exercises will help those definitions to take root in your mind.

CHAPTER 1

Integers

We begin with elementary properties of the integers, the set

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}.$$

We will not prove anything new or surprising but we will examine carefully familiar properties of the integers and see what fundamental principles make them true.

1. The natural numbers

Children experience a degree of bewilderment when they first encounter negative numbers. Before that a child's notion of number is confined to the set of natural numbers, the set

$$\mathbb{N} := \{1, 2, 3, \ldots\}.$$

Before learning about the algebraic operations + and × a child understands the order relation on $\mathbb{N}$—a child is aware that two is larger than one, three larger than two, and so on.

That order relation is extended to the integers: if x and y are integers we say that x is less than y, and write $x < y$, if $x + n = y$ for some $n \in \mathbb{N}$. We write $x \leq y$ if $x = y$ or $x < y$.

Even before they can speak, children are intuitively aware of the well-ordering principle:

every non-empty subset of $\mathbb{N}$ *has a smallest element.*

(Just ask a child to choose among several boxes of chocolate.) Although the well-ordering principle can be proved from a more fundamental set of axioms, we will take it on faith. No doubt, you have used the well-ordering principle for many years, and believe it, so the leap of faith is small. However, just because you take something for granted does not mean that we should not look afresh at it with a critical eye.

Actually, if you think about it, young children, and even monkeys and other animals, are aware of the following fact which is equivalent to the well-ordering principle:

every non-empty subset of $\mathbb{N}$ *has a largest element.*

Just ask a child to choose from a collection of different piles of jelly beans: if you say *"you can choose one pile"* the child will unerringly zero in on the pile with the most jelly beans. Likewise with monkeys and bananas.

Why is this equivalent to the well-orderining principle?

Fractions. For now we will restrict our attention to the integers, but let's keep in the back of our mind the rational numbers, or fractions, consisting of such things as $0, -\frac{15}{34}, 7\frac{1}{4}, \ldots$. More formally, we define the rational numbers to be

$$\mathbb{Q} = \{\tfrac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0\}.$$

Notice that a subset of the non-negative rational numbers need not have a smallest element: for example, $\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$ has no smallest element.

2. Division and remainders

Let a and b be integers. We say that b divides a, or that a is a multiple of b, if there is an integer c such that $a = bc$. We write $b|a$ to mean "b divides a", and $b \not| a$ to mean "b does not divide a.

Basic Facts about divisibility. You should know and be able to use the following facts. We will use them frequently with no further comment.

(1) every integer divides 0;
(2) 1 divides every integer;
(3) if $a|b$ and $x \in \mathbb{Z}$, then $a|bx$;
(4) if $a|b$ and $b|c$, then $a|c$;
(5) if $a|b$ and $a|c$, then $a|bx + cy$ for all $x, y \in \mathbb{Z}$;
(6) if $a|b$ and $b|a$, then either $a = b$ or $a = -b$.

You should be able to prove all the above. You might just read out loud items (3)–(6) to see if you can translate the symbols into fluent sentences.

Generally speaking, when we try to divide one integer by another, we end up with a remainder; for example, trying to divide 89 by 12, we find that

$$89 = 12 \times 7 + 5.$$

We call 89 the dividend, 12 the divisor, 7 the quotient, and 5 the remainder.

THEOREM 1.1 (The division algorithm). *Let $a, b \in \mathbb{Z}$ with $b > 0$. Then there exist unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $0 \leq r < b$.*

Some comments before the proof. We are dividing a by b to obtain a quotient q and a remainder r. There is no requirement that a be positive, although we do assume that b is. Thus, q might be negative, but r is not negative. The point of the theorem is that *q and r are unique!* If we did not require r to lie between 0 and $b - 1$, there would be no uniqueness.

The key step in the proof is *the existence of r* which we will prove as a consequence of the well-ordering principle.

With this preamble, let's prove the theorem.

Proof. [Theorem 1.1] Let

$$a - b\mathbb{Z} := \{a - bt \mid t \in \mathbb{Z}\}.$$

Because b is positive, if t is sufficiently negative $a - bq$ is positive; thus $a - b\mathbb{Z}$ contains non-negative integers; by the well-ordering principle $a - b\mathbb{Z}$ has a smallest element; we call that r and let q be the integer such that $a - bq = r$.

If $r \geq b$, then $r-b \geq 0$ and $r-b = a-b(q-1)$, so $r-b$ is a non-negative element of $a - b\mathbb{Z}$ and is smaller than r; that contradicts our choice of r so we conclude that $r < b$. Hence r and q exist as claimed, and it remains to prove their uniqueness.

Suppose $a = bq' + r'$ and $0 \leq r' < b$. Then $r - r' = b(q - q')$ and $|r - r'| < b$; but the only integer that has absolute value strictly less than b and is a multiple of b is zero, so $r - r' = 0$; thus $r = r'$ and $q = q'$, proving uniqueness of q and r. □

COROLLARY 1.2. *If n is a positive integer, then $\mathbb{Z}$ is the disjoint union of the sets $r + n\mathbb{Z}$, $0 \leq r \leq n-1$.*

Here is a cute application of the division algorithm.

EXAMPLE 1.3. If a is an integer that is divisible neither by 2 nor 3, then 24 divides $a^2 - 1$. We use the "rabbit out of the hat" method of proof. By the division algorithm, we can write $a = 6b + r$ with $0 \leq r < 5$. If r were 0, 2, or 4, then a would be even, hence divisible by 2; but we assumed this was not the case. If r were 3, then 3 would divide a contrary to our hypothesis. The only remaining possibility is that $r = 1$ or $r = 5$.

Thus we can write $a = 6n \pm 1$. Therefore

$$a^2 - 1 = (6n+1)^2 - 1 = 36n^2 \pm 12n = 12n(3n \pm 1).$$

If n is even, then 24 divides $12n$, and therefore divides $a^2 - 1$. If n is odd, then $3n \pm 1$ is even, so again 24 divides $a^2 - 1$. ◊

3. The Euclidean Algorithm

3.1. Greatest common divisor. The notion of greatest common divisor appears repeatedly in this course. If a and b are non-zero integers, their greatest common divisor is the largest integer that divides both a and b. It is denoted (a, b) or $\gcd(a, b)$. E.g., $(12, -12) = 12$.

You should be asking yourself *"do a and b actually have a greatest common divisor?"* They do and we give the argument in the next paragraph.

Consider the set of integers that divide both a and b. This set contains 1 so is non-empty. However, it cannot contain any number bigger than $|a|$, the absolute value of a, so it must have a biggest member. (Actually, writing out the details of this argument carefully, you will see we have used the Well Ordering Principle again: if X denotes the set of positive integers which divide both a and b, then $X' = \{|a| - x \mid x \in X\}$ is a non-empty set of non-negative integers, so has a smallest element, say $x' = |a| - x$, and it is easy to check that x must be the largest member of X, so x is the gcd).

3.2. Finding the greatest common divisor. The previous section shows that the greatest common divisor of a and b exists, but how do we go about finding it? We find it by using the Euclidean Algorithm which consists of repeatedly using the division algorithm. We begin with integers a and b,

$b > 0$, and obtain sequences of integers $q_0, q_1, q_2, \ldots$ and $r_0, r_1, r_2, \ldots$ such that

$$\begin{aligned} a &= bq_0 + r_0 && \text{and } 0 \le r_0 < b, \\ b &= r_0q_1 + r_1 && \text{and } 0 \le r_1 < r_0, \\ r_0 &= r_1q_2 + r_2 && \text{and } 0 \le r_2 < r_1, \\ &\vdots \\ r_{t-2} &= r_{t-1}q_t + r_t && \text{and } 0 \le r_t < r_{t-1}, \\ r_{t-1} &= r_tq_{t+1}. \end{aligned}$$

You keep dividing the latest remainder into the previous one; the proceedure stops because the remainders keep getting smaller

$$b > r_0 > r_1 > \cdots \ge 0.$$

When one hits a remainder of zero, as indicated above, then r_t is the gcd of a and b. Before proving this we illustrate the idea.

EXAMPLE 1.4. Find $(1547, 560)$.

$$\begin{aligned} 1547 &= 560 \times 2 + 427, \\ 560 &= 427 \times 1 + 133, \\ 427 &= 133 \times 3 + 28, \\ 133 &= 28 \times 4 + 21, \\ 28 &= 21 \times 1 + 7, \\ 21 &= 7 \times 3. \end{aligned}$$

So $(1547, 560) = 7$. ◊

PROPOSITION 1.5. *The last remainder in the Euclidean Algorithm gives the gcd.*

Proof. We adopt the notation set up above. Write $d = \gcd(a, b)$. Since $d|a$ and $d|b$, $d|r_0 = a - bq_0$. Hence $d|r_1 = b - r_0q_1$; et cetera. Eventually, we see that $d|r_t$. Since $r_t > 0$, $d \le r_t$. But also, $r_t|r_{t-1}$, so $r_t|r_{t-2} = r_{t-1}q_t + r_t$, and $r_t|r_{t-3} = r_{t-2}q_{t-1} + r_{t-2}$ et cetera. Eventually, we see that $r_t|b$ and $r_t|a$. Therefore, r_t is the gcd. □

THEOREM 1.6. *If $d = (a, b)$, then there exist $u, v \in \mathbb{Z}$ such that*

$$d = au + bv.$$

Proof. Just use the sequence of equalities in the Euclidean Algorithm from the bottom up to express r_t in terms of earlier remainders:

$$\begin{aligned} r_t &= r_{t-2} - r_{t-1}q_t \\ &= r_{t-2} - (r_{t-3} - r_{t-2}q_{t-1})q_t \\ &\vdots \qquad \vdots \end{aligned}$$

The next example makes the procedure clear. □

EXAMPLE 1.7. Reconsider Example 1.4. We will use the calculations in that exercise to show there are integers u and v such that $7 = 1547u + 560v$. We get

$$\begin{aligned} 7 &= 28 - 1 \times 21 \\ &= 28 - (133 - 28 \times 4) = 28 \times 5 - 133 \\ &= (427 - 133 \times 3) \times 5 - 133 = 427 \times 5 - 133 \times 16 \\ &= 427 \times 5 - (560 - 427) \times 16 = 427 \times 21 - 560 \times 16 \\ &= (1547 - 560 \times 2) \times 21 - 560 \times 16 \\ &= (1547 \times 21 - 560 \times 58. \end{aligned}$$

COROLLARY 1.8. *If $a|bc$ and $(a, b) = 1$, then $a|c$.*

Mention/look ahead to the Chinese Remainder Theorem.

THEOREM 1.9 (The Fundamental Theorem of Arithmetic). *Every integer is a product of prime numbers in a unique way.*

3.2.1. *The greatest common divisor of several elements.* It should be obvious that we can define the greatest common divisor of any set of integers as the largest integer that divides all of them. We do that. You should check that

$$\gcd\big(a, \gcd(b, c)\big) = \gcd\big(\gcd(a, b), c\big) = \gcd(a, b, c),$$

and so on for larger sets of integers.

3.3. The least common multiple.

CHAPTER 2

Groups

1. Permutations

A permutation of a set X is a bijective function $\sigma : X \to X$. For example, the identity function id_X is a permutation of X. We call it the trivial permutation.

In this section we examine individual permutations and the set of all permutations on X.

1.1. The set of permutations of a set X. Let X be any set.

The set of permutations of X is our first example of a group. If we combine our first result about permutations, Proposition 2.1 below, with the observation that id_X is a permutation, and the fact that composition of functions is associative, we have, in effect, verified that the set of permutations of X is a group.

So, all results about permutations in this section are, in fact, results about permutation groups or, as they are usually called when X is finite, the *symmetric groups*. Symmetric groups are of great importance.

PROPOSITION 2.1. *Let f and g be permutations of X. Then*

(1) *fg is a permutation of X;*
(2) *f^{-1} is a permutation of X.*

Proof. These are facts about bijections (see section 2.3 in Chapter 0). A composition of bijections is bijective. A bijection has an inverse and that inverse is itself a bijection. □

1.1.1. *The number of permutations of a set.* There are infinitely many permutations of an infinite set. For now we will mostly be concerned with permutations of a finite set.

If n is a positive integer, then n factorial is the number

$$n! := n(n-1)(n-2)\cdots 2.1,$$

the product of the integers between 1 and n. For example,

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120,$$
$$6! = 720, \quad 7! = 5040, \quad 8! = 40320, \quad \text{and so on.}$$

It is convenient to define

$$0! = 1.$$

LEMMA 2.2. *There are $n!$ permutations of a set having n elements.*

Proof. Let's assume $X = \{1, 2, \ldots, n\}$. In defining a bijective function $f : X \to X$, there are n choices for $f(1)$ then, once $f(1)$ has been selected, $n-1$ choices for $f(2)$, then $n-2$ choices for $f(3)$, and so on, giving a total of $n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1 = n!$ choices for f. □

1.1.2. *Convention and warning.* If σ and τ are permutations, the notation $\sigma\tau$ means first do τ, then do σ. After all, since σ and τ are functions what could be more sensible than using the convention we are already familiar with: fg means first perform the function g then the function f. However, not all books adopt this convention. For example, P.M. Cohn's three volume text *Algebra* uses the opposite convention. In all other respects I love these books.

1.2. Cycle notation for permutations.

1.2.1. *Remarks on notation.* We need good notation for permutations. Sometimes it is easy to do that: for example, the function $f(n) = n + 5$ is a permutation of the set $\mathbb{Z}$. For permutations of finite sets, the case of most interest to us, we can rarely use such simple formulas.

Before dealing with permutations let's consider some notations for integers. The notation 10010 for the number ten thousand and ten immediately gives us a sense of its *size*—it's around ten thousand. It is also apparent that 10010 is divisible by 10, hence by the primes 2 and 5. But the notation 10010 does not reveal any arithmetic properties of 10010 other than its divisiblity by 2 and 5. In contrast, if we write 10010 as $2 \times 5 \times 7 \times 11 \times 13$ we immediately see its arithmetic properties.

I don't need to say anything about Roman numerals like MDCCLIV. The fact that this notation was discarded long ago, except for some specialized and ceremonial uses, is sufficient testament to its inadequacies.

1.2.2. *Analogy with prime numbers.* We will adopt a notation for permutations that is analogous to writing an integer as a product of prime numbers. The analogue of a prime number is a cyclic permutation, or cycle, for short. In analogy with the fact that every integer can be written as a product of prime numbers in a unique way we will show that every permutation can be written as a product of disjoint cycles in a unique way. We will now explain these terms, first somewhat informally.

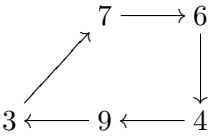
1.2.3. *Cycles.* Let's start with an example. Let σ be the permutation of $\{1, \ldots, 9\}$ defined by

$$\sigma(1) = 1,\ \sigma(2) = 5, \sigma(3) = 7,\ \sigma(4) = 9,\ \sigma(5) = 2,$$
$$\sigma(6) = 4,\ \sigma(7) = 6, \sigma(8) = 8,\ \sigma(9) = 3.$$

The expression for σ as a product of disjoint cycles will be written

$$\sigma = (2\,5)(3\,7\,6\,4\,9).$$

The meaning is this. First, σ is the product/composition of two simpler permutations/functions, (2 5) and (3 7 6 4 9). The permutation denoted (2 5) is the function that sends 2 to 5 and 5 to 2 and fixes the integers 1, 3, 4, 6, 7, 8, 9. The permutation denoted (3 7 6 4 9) is the function that sends 3 to 7, 7 to 6, 6 to 4, 4 to 9, 9 to 3, and fixes the integers 1, 2, 5, 8. We call the factors (2 5) and (3 7 6 4 9) *cycles*. Each factor cycles around some of the numbers in $\{1, \ldots, 9\}$. The following picture to illustrate this:

$(2\,5) =$ $(3\,7\,6\,4\,9) =$ 

Each arrow indicates the action of σ. For example, $7 \to 6$ means $\sigma(7) = 6$. The notation (2 5)(3 7 6 4 9) immediately reveals the behavior of σ, just as writing 10010 as $2 \times 5 \times 7 \times 11 \times 13$ immediately reveals its fundamental arithmetic properties. The notation (2 5)(3 7 6 4 9) is efficient because we omit the numbers i for which $\sigma(i) = i$. There could be no shorter notation than (2 5 7) for the permutation τ defined by $\tau(2) = 5$, $\tau(5) = 7$, $\tau(7) = 2$, and $\tau(i) = i$ for all other i.

In some books the permutation $\tau = (2\,5\,7)$ will be written as

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,5\,3\,4\,7\,6\,2\,8\,9 \end{pmatrix}. \tag{2-1}$$

The general principle is that we write $\tau(i)$ below i, i.e.,

$$\tau = \begin{pmatrix} 1 & 2 & \ldots & 9 \\ \tau(1) & \tau(2) & \ldots & \tau(9) \end{pmatrix}.$$

This notation is cumbersome. You must examine it carefully to uncover the essential properties of τ. And it is a lot of writing compared to (2 5 7)!

Likewise,

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,5\,7\,8\,2\,4\,6\,9\,3 \end{pmatrix}.$$

is a cumbersome and obscure notation for the permutation (2 5)(3 7 6 4 8 9 3).

1.2.4. *Warning: Different notations for the same cycle.* Be warned that the permutations (3 7 6 4 9) and (6 4 9 3 7) are the same!

1.2.5. *Disjoint cycles.* The cycles (2 5) and (3 7 6 4 9) are said to be *disjoint* because none of the terms in (2 5) appears in (3 7 6 4 9).[1] Likewise, the cycles (1 3 8), (2 4), and (5 6 7) are disjoint. On the other hand the cycles (2 5) and (2 7 6 4 9) are not disjoint because 2 appears in each of them.

The permutation $\theta := (1\,2\,3)(3\,4)(4\,5\,2\,6)(7\,1\,5\,3)$ is a product of four cycles but they are not disjoint. Our convention for composition of functions is that fgh means first apply the function h, then g, then f. Hence θ is the

[1]Recall that two subsets, A and B say, of a set X are disjoint if $A \cap B = \phi$. We say three subsets A, B, and C, are disjoint if $A \cap B = B \cap C = A \cap C = \phi$. And so on.

function first apply the permutation $\delta = (7\,1\,5\,3)$, then $\gamma = (4\,5\,2\,6)$, then $\beta = (3\,4)$, then $\alpha = (1\,2\,3)$. So, the effect of θ is as follows:

$$1 \xrightarrow{\delta} 5 \xrightarrow{\gamma} 2 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 3$$

$$3 \xrightarrow{\delta} 7 \xrightarrow{\gamma} 7 \xrightarrow{\beta} 7 \xrightarrow{\alpha} 7$$

$$7 \xrightarrow{\delta} 1 \xrightarrow{\gamma} 1 \xrightarrow{\beta} 1 \xrightarrow{\alpha} 2$$

$$2 \xrightarrow{\delta} 2 \xrightarrow{\gamma} 6 \xrightarrow{\beta} 6 \xrightarrow{\alpha} 6$$

$$6 \xrightarrow{\delta} 6 \xrightarrow{\gamma} 4 \xrightarrow{\beta} 3 \xrightarrow{\alpha} 1$$

$$4 \xrightarrow{\delta} 4 \xrightarrow{\gamma} 5 \xrightarrow{\beta} 5 \xrightarrow{\alpha} 5$$

$$5 \xrightarrow{\delta} 3 \xrightarrow{\gamma} 3 \xrightarrow{\beta} 4 \xrightarrow{\alpha} 4.$$

Hence $\theta = (1\,3\,7\,2\,6)(4\,5)$. In particular, θ is a product of disjoint cycles. Perhaps you can already see that every permutation of a finite set is a product of disjoint cycles. Although this might be intuitively obvious to you we will give a rigorous proof, i.e., an explanation, of this fact in Theorem 2.5 below.

An important property of disjoint cycles is that they commute with one another: for example, $(1\,3\,7\,2\,6)(4\,5) = (4\,5)(1\,3\,7\,2\,6)$. It is best for you to think about this yourself and why this is rather than have someone explain it to you. As another example, observe that the six different products of the three disjoint cycles $(1\,3)$, $(2\,4\,7)$, and $(5\,6)$, are equal to each other.

1.3. The orbits of a permutation. It is useful to have a graphic, dynamic mental image of a permutation. To encourage this we say *"σ acts on X"* and speak of *"the action of σ on X."* The word *"action"* has a dynamic feel to it.

Let $x \in X$. The orbit of x under the action of σ is defined to be the set

$$O_x := \{\sigma^n(x) \mid n \in \mathbb{Z}\} = \{\ldots, \sigma^{-2}(x), \sigma^{-1}(x), x, \sigma(x), \sigma^2(x), \ldots\}.$$

We call O_x a σ-orbit. The size of the orbit is the number of elements in it.

The association of the word *orbit* with the movement of planets further encourages us to have a dynamic picture of a permutation. Permutations of X move the elements of X around.

LEMMA 2.3. *Let O_x and O_y be the σ-orbits of x and y. Then either $O_x = O_y$ or $O_x \cap O_y = \phi$.*

Proof. Suppose $O_x \cap O_y \neq \phi$. Then $\sigma^m(x) = \sigma^n(y)$ for some integers m and n. It follows that $\sigma^k(x) = \sigma^{k+n-m}(y)$ and $\sigma^\ell(y) = \sigma^{\ell+m-n}(x)$ for all integers k and ℓ. Hence $O_x \subset O_y$ and $O_y \subset O_x$. Thus $O_x = O_y$. □

If $A_1, A_2, \ldots$ are subsets of a set X such that $X = A_1 \cup A_2 \cup \cdots$ and $A_i \cap A_j = \phi$ whenever $i \neq j$ we say that X is the disjoint union of $A_1, A_2, \ldots$ and write $X = A_1 \sqcup A_2 \sqcup \cdots$ to denote this fact.

PROPOSITION 2.4. *Let σ be a permutation of X. Then X is the disjoint union of its σ-orbits.*

Proof. Since x belongs to the orbit O_x, X is the union of all the σ-orbits. By Lemma 2.3, different orbits are disjoint. Hence if $A_1, A_2, \dots$ are the distinct orbits, then $X = A_1 \sqcup A_2 \sqcup \cdots$. □

Now assume σ is a permutation of a finite set X. Then every σ-orbit is finite and there are only finitely many σ-orbits. Thus $X = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k$ where $A_1, A_2, \dots, A_k$ are the different σ-orbits. Suppose

$$A_1 = \{a_{11}, a_{12}, \dots, a_{1r_1}\}$$

and

$$\sigma(a_{11}) = a_{12}, \quad \sigma(a_{12}) = a_{13}, \quad \dots, \quad \sigma(a_{1\,r_1-1}) = a_{1r_1}, \quad \sigma(a_{1r_1}) = a_{11}.$$

Then the action of the cycle $(a_{11}\, a_{12}\, \dots\, a_{1r_1})$ on A_1 is the same as the action of σ. Likewise, if

$$A_2 = \{a_{21}, a_{22}, \dots, a_{2r_2}\}$$

and

$$\sigma(a_{21}) = a_{22}, \quad \sigma(a_{22}) = a_{23}, \quad \dots, \quad \sigma(a_{2\,r_2-1}) = a_{2r_2}, \quad \sigma(a_{2r_2}) = a_{21}$$

then the action of the cycle $(a_{21}\, a_{22}\, \dots\, a_{2r_2})$ on A_2 is the same as the action of σ. And so on. It follows that

$$\sigma = (a_{11}\, a_{12}\, \dots\, a_{1r_1})(a_{21}\, a_{22}\, \dots\, a_{2r_2}) \cdots (a_{k1}\, a_{k2}\, \dots\, a_{kr_k}).$$

Since the different orbits are disjoint we have proved the next result.

THEOREM 2.5. *Every permutation can be written as a product of disjoint cycles in a unique way.*

Strictly speaking, we only proved this in the case when X is finite but the same idea and a little more notation will prove the same result for any set.

1.4. Cycles. You probably noticed that I have not given a precise definition of a cycle yet, woeful sinner that I am.

Suppose σ is a permutation of a set X. Suppose too that $\sigma \neq \mathrm{id}_X$. We call σ a cycle if it has at exactly one orbit of size > 1. If that orbit has size r we call σ an r-cycle if the size of that orbit is r.

We also declare that id_X is a cycle, the unique 1-cycle. We sometimes call it the trivial cycle.

Two non-trivial cycles σ and τ are disjoint if their non-trivial orbits are disjoint.

The inverse of a cycle is a cycle. For example, $(3\,5\,2\,6\,1)^{-1} = (1\,6\,2\,5\,3)$.

A 2-cycle is called a transposition.

PROPOSITION 2.6. *Every permutation is a product of transpositions.*

Proof. Every cycle is a product of transpositions because, for example, $(1\,2\,\ldots\,m-1\,m) = (1\,m)(1\,m-1)\cdots(1\,4)(1\,3)(1\,2)$. But every permutation is a product of cycles, so the result follows. □

There is no uniqueness to the factorization of a permutation as a product of transpositions. For example,

$$(1\,2\,3\,4) = (1\,2)(2\,3)(3\,4) = (1\,4)(1\,3)(1\,2).$$

1.5. The symmetric groups. Let n be a positive integer. The set of all permutations of the set $\{1, 2, \ldots, n\}$ is called the n^{th} symmetric group, or just the *symmetric group* if n is clear from the context. It is often denoted by S_n or Σ_n.

The number of elements in S_n is $n!$.

Thus S_1 has a single element, the identity permutation $\mathrm{id}_{\{1\}}$.

The second symmetric group S_2 has two elements, $\mathrm{id}_{\{1,2\}}$ and (12).

The third symmetric group, S_3, has six (=3!) elements, namely

$$1, \quad (1\,2), \quad (2\,3), \quad (1\,3), \quad (1\,2\,3), \quad (3\,2\,1).$$

The multiplication table for S_3 is:

•	1	(12)	(13)	(23)	(123)	(321)
1	1	(12)	(13)	(23)	(123)	(321)
(12)	(12)	1	(321)	(123)	(23)	(13)
(13)	(13)	(123)	1	(321)	(12)	(23)
(23)	(23)	(321)	(123)	1	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(321)	1
(321)	(321)	(23)	(12)	(13)	1	(123)

The entry in the row labelled (12) and the column labelled (123) is $(12)(123) = (23)$. And so on.

2. Groups

2.1. Definition and first observations. In the previous section we encountered a group, the symmetric group S_n. Keep that example in mind when reading the next definition—why does S_n satisfy axioms (1)–(3) in the definition?

Definition 2.7. A group is a non-empty set G together with a map $G \times G \to G$ that we usually denote by $(x, y) \mapsto xy$, satisfying the following properties:

(1) $(xy)z = x(yz)$ for all $x, y, z \in G$;
(2) there is an element $e \in G$ called the identity with the property that
$$ex = xe = x \qquad \text{for all } x \in G;$$
(3) for each $x \in G$ there is an element x^{-1} in G such that $xx^{-1} = x^{-1}x = e$.

We say the group is abelian if $xy = yx$ for all $x, y \in G$.[2] $\Diamond$

We call the map $G \times G \to G$, $(x, y) \mapsto xy$, the group operation when we want to be a little vague but in particular situations we often call it multiplication, or addition, or composition.

2.1.1. *The symmetric group is a group.* Let's check that S_n, the set of permutations of the set $\{1, 2, \ldots, n\}$, endowed with the binary operation *"composition of functions"* satisfies the axioms of Definition 2.7. A composition of bijections is a bijection so "composition of functions" is indeed a map $S_n \times S_n \to S_n$. We also note that S_n is non-empty because it contains the identity function $\mathrm{id} : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Composition of functions is associative so the associative axiom is satisfied. There is an identity element, namely the identity map. Finally, the inverse of a bijection is a bijection so every element in S_n has an inverse. Thus S_n is a group.

It is OK if you skip immediately to the examples in the next two sections and return to the next result after looking at some of those. The next lemma is an important, albeit elementary, result and you might find it easier to appreciate its proof after you have some examples in mind.

LEMMA 2.8. *Let G be a group. Then*

(1) *G has exactly one identity element;*
(2) *each $x \in G$ has a unique inverse;*
(3) *you can cancel in a group: if $x, y, z \in G$ and $xy = xz$, then $y = z$. Likewise, if $yx = zx$, then $y = z$.*

Proof. (1) If e and e' both satisfy condition (2) in Definition 2.7 they are equal: we have $e = ee' = e'$, the first "=" because e' satisfies (2) and the second "=" because e satisfies (2).

(2) if $x \in G$ and $xy = yx = e$ and $xz = zx = e$, then $y = ye = y(xz) = (yx)z = ez = z$. The proof used condition (1), the associativity of multiplication.

(3) If $xy = xz$, then

$$y = ey = (x^{-1}x)y = x^{-1}(xy) = x^{-1}(xz) = (x^{-1}x)z = ez = z.$$

A similar string of equalities with x^{-1} on the right instead of the left proves the other cancellation rule. $\square$

2.2. Some infinite groups. You already know lots of groups! Let's now mention some of them. Although most will be familiar look at each one in light of the axioms for a group and check that all three of the axioms are satisfied.

[2]Named after Niels Henrik Abel, 1802-29.

2.2.1. *The group of integers.* The first group you meet as a child is the group of integers $\mathbb{Z}$ with the group operation being addition. We denote the result of the group operation by $x + y$ rather than xy. To check that $\mathbb{Z}$ really is a group first observe that it is non-empty and that its addition is associative, $x + (y + z) = (x + y) + z$. The identity is zero because $0 + x = x + 0 = x$ and the inverse of x is $-x$.

Note that $\mathbb{Z}$ is *not* a group under multiplication: although multiplication is associative, $x(yz) = (xy)z$, and there is an identity, namely 1 ($1.x = x.1 = x$), inverses do not exist—e.g., neither 2 nor 0 has an inverse.

2.2.2. *The group of non-zero real numbers under multiplication.* The non-zero real numbers with their ordinary multiplication form a group, an abelian group because $xy = yx$. The identity element is 1, and the inverse of x is $1/x$. There are two common notations for the set of non-zero real numbers, $\mathbb{R} - \{0\}$ and $\mathbb{R}^\times$, and we often write $(\mathbb{R}^\times, \cdot)$ or $(\mathbb{R} - \{0\}, \cdot)$ for it—the first position denotes the set G and the second position denotes the operation. Following that convention, the previous example is $(\mathbb{Z}, +)$ and the next example is $(\mathbb{R}, +)$.

2.2.3. *The group of positive real numbers under multiplication.* The set of positive real numbers with their ordinary multiplication form a group. We denote this group by $(\mathbb{R}_{>0}, \times)$.

2.2.4. *The group of real numbers under addition.* The real numbers under addition form a group that we denote by $(\mathbb{R}, +)$. The identity in $(\mathbb{R}, +)$ is 0 and the inverse of x is $-x$.

2.2.5. *Rational numbers.* Similar examples with the rational numbers in place of the reals are $(\mathbb{Q} - \{0\}, \cdot)$, $\mathbb{Q}_{>0}, \times)$, and $(\mathbb{Q}, +)$.

2.2.6. *The general linear groups.* The set of invertible $n \times n$ matrices with real entries is denoted by $\mathrm{GL}(n, \mathbb{R})$. It is a group under matrix multiplication: the product of two $n \times n$ invertible matrices is an invertible matrix of size $n \times n$; the identity matrix is the identity; the inverse of A is exactly the matrix you already know as its inverse; multiplication of matrices is associative. We call $\mathrm{GL}(n, \mathbb{R})$ the general linear group (of size n over $\mathbb{R}$). If $n \geq 2$, $\mathrm{GL}(n, \mathbb{R})$ is not abelian.

There are other general linear groups such that $\mathrm{GL}(n, \mathbb{Z})$, the group of $n \times n$ invertible matrices whose entries, and the entries of its inverse, are integers.

2.2.7. *The special linear groups.* We denote by $\mathrm{SL}(n, \mathbb{R})$ the set of $n \times n$ matrices having determinant 1. The identity matrix belongs to $\mathrm{SL}(n, \mathbb{R})$. Since $\det(AB) = (\det A)(\det B)$, a product of two matrices in $\mathrm{SL}(n, \mathbb{R})$ is in $\mathrm{SL}(n, \mathbb{R})$. Since $\det(A^{-1}) = (\det A)^{-1}$, the inverse of a matrix in $\mathrm{SL}(n, \mathbb{R})$ is in $\mathrm{SL}(n, \mathbb{R})$. We call $\mathrm{SL}(n, \mathbb{R})$ the special linear group.

2.3. Some finite groups. You have already met one of the most important classes of finite groups, the symmetric groups S_n. Symmetric groups

are quite complicated. Indeed, they have been intensely studied for well over a century now and although an enormous amount is known about them there is still much we don't know about them. You could devote a lifetime to their study. Many people have done just that. But let's start here with some simple examples, some of which will be familiar.

2.3.1. *The trivial group.* The simplest group of all consists of only one element, $G := \{e\}$ with $ee = e$. We call it the trivial group.

2.3.2. *The group* $\{\pm 1\}$. A familiar group with two elements is the set

$$\mu_2 := \{1, -1\}$$

with the usual multiplication.

2.3.3. *The multiplication table of a group.* At primary school you probably wrote out some multiplication tables.

The order of a group is the number of its elements. We write $|G|$ for the number of elements in G. A group is said to be *finite* if it has only finitely many elements.

For a finite group it is conceivable that we could write out its entire multiplication table. We do this for some small groups below. We adopt the following convention: the rows and columns are labeled by the elements of the group; the entry in row a and column b is the product ab; the entry in row b and column a is ba; we use the same order for the labeling of the rows and columns.

You should be aware of some "patterns" that in the multiplication table.

If G is abelian, there is a symmetry of the table about the diagonal line of slope -1. That symmetry expresses the fact that $ab = ba$.

The identity must appear exactly once in each row and column because given x there is an element y such that $xy = e = yx$. The identity e appears on the diagonal exactly when $x^2 = e$, i.e., when an element is its own inverse.

In fact, *every* element of the group must appear in each row and column; for example, the entries in the row labelled by x are $\{xg \mid g \in G\}$, and given $y \in G$ there is a g such that $xg = y$, namely $g = x^{-1}y$, so y appears in the row labelled x.

2.3.4. *Positive and negative numbers.* Here is a group whose elements are themselves sets. Let P and N denote the sets of positive and negative real numbers respectively. We define the group operation, which we call multiplication, by declaring that The group operation says that

$\times$	P	N
P	P	N
N	N	P

positive $\times$ positive = positive negative $\times$ negative = positive
positive $\times$ negative = negative negative $\times$ positive = negative.

2.3.5. *The parity group* $\mathbf{P}$. The elements in this group are themselves sets. Let E denote the set of even integers and O the set of odd integers. Young children know that

$$\text{odd} + \text{odd} = \text{even}, \text{even}+\text{odd}=\text{odd}=\text{odd}+\text{even}, \text{even}+\text{even}=\text{even}.$$

In other words, they know the addition table for the parity group $\mathbf{P} := \{E, O\}$: Thus E is the identity in $\mathbf{P}$ and O is its own inverse.

$+$	E	O
E	E	O
O	O	E

2.3.6. *The group* $\mathbb{F}$. This group is abelian and its addition table is defined to be Thus $\mathbb{F} = \{0, 1\}$ and the addition is as defined above.

$+$	0	1
0	0	1
1	1	0

2.3.7. *Remark.* The last four examples are essentially the same. This idea will be made precise when we introduce the notion of isomorphism. We will then be able to say that the last four groups are *isomorphic* to one another where the word isomorphism has a precise meaning.

2.3.8. *Exercise: a group with three elements.* There is a group with three elements. There is essentially only one group with three elements. When you begin to write out its multiplication table the axioms force your hand. It is a very good exercise to see why this is. For example, write 1 for the identity element and let x be some other element in your putative three-element group. Now xx is either 1, x, or the third element in it. Consider all three possibilities—perhaps you can exclude one of them, or even two of them. Perhaps you should consider xxx.

2.3.9. *The* 4^{th} *roots of unity.* Let i denote a square root of -1. Then the set of complex numbers

$$\mu_4 := \{1, -1, i, -i\}$$

is a group under multiplication.

$\cdot$	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

2.3.10. *Exclusive OR.* Fix a set X. It might help if you think of an explicit X such as the set of integers. Let G be the set of all finite subsets of X including the empty set. If A and B are in G, i.e.,, subsets of X, we define

$$A \oplus B := \{\text{elements that are in either } A \text{ or } B \text{ but not both}\}.$$

The identity element is the empty set ϕ. Every element is its own inverse because $A \oplus A = \phi$. You can draw a Venn diagram involving three sets in order to convince yourself that the associative law $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ holds.

If $X = \{3\}$, the "multiplication table" for the group is

$\oplus$	ϕ	$\{3\}$
ϕ	ϕ	$\{3\}$
$\{3\}$	$\{3\}$	ϕ

If $X = \{3, 8\}$, the "multiplication table" for the group is

$\oplus$	ϕ	$\{3\}$	$\{8\}$	$\{3,8\}$
ϕ	ϕ	$\{3\}$	$\{8\}$	$\{3,8\}$
$\{3\}$	$\{3\}$	ϕ	$\{3,8\}$	$\{8\}$
$\{8\}$	$\{8\}$	$\{3,8\}$	ϕ	$\{3\}$
$\{3,8\}$	$\{3,8\}$	$\{8\}$	$\{3\}$	ϕ

2.3.11. *Another group with four elements,* $\mathbb{F}^2$. The set

$$\mathbb{F}^2 := \{00, 01, 10, 11\}$$

can be made into a group:

$+$	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

2.3.12. *The group generated by the reflections in the x- and y-axes.* The set consisting of the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ ST = TS = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a group under matrix multiplication. Note a product of two elements in G is still in G. Each element is its own inverse. We know that multiplication of *any* matrices is associative so the associative law holds in G. Each of these matrices represents a linear transformation in the plane: S is the reflection in the x-axis, T is the reflection in the y-axis, and ST is rotation by 180 degrees. This group is "essentially the same" as the previous example as you can see by comparing the multiplication/addition tables:

$\cdot$	I	S	T	ST
I	1	S	T	ST
S	S	I	ST	T
T	T	ST	I	S
ST	ST	T	S	I

2.4. Multiplicative versus additive notation. In the above examples we used various notations for the group operations. The only hard and fast rule is that we *never* use the symbol + for the group operation in a non-abelian group. That would be too confusing because in all our prior experience + has been a commutative operation whether adding numbers, or matrices, or functions, etc. Thus if we speak about a group G with no real knowledge of its nature we will *always* write xy for the result of the group operation on the elements x and y. We then say that we are using *multiplicative notation* for the group operation.

For example, we used multiplicative notation when we defined a group. And, when we define *isomorphism* below we will again use multiplicative notation.

2.4.1. *Exponent Notation.* If x is an element of a group G and n a positive integer we write x^n for the product of x with itself n times. For example, $x^2 = xx$, $x^3 = xxx$, and so on. The associative law ensures that x^3, for example, is unambiguous (because $x(xx) = (xx)x$). We also declare that $x^0 = 1$, the identity element in G, If n is a positive integer, we write x^n for $(x^{-1})^n$.

It is a good exercise to check that $x^m x^n = x^{m+n}$ for all integers m and n, and that $(x^m)^n = x^{mn}$.

For a familiar group like $(\mathbb{Z}, +)$ it would be very confusing to use any symbol other than + for the addition in $\mathbb{Z}$. However, the exponent notation above is not sensible to use in a group where we use + for the group operation. It makes more sense to write nx for $x + x + \cdots + x$, the n-fold sum of x with itself. Now the "exponent rules" become $(n + m)x = nx + mx$.

2.4.2. *Notation for the identity.* In the definition of a group we wrote e for its identity element. It is common to use the symbol 1 to denote the identity in a group where we are using multiplicative notation. It is also common to write 0 for the identity element in a group where we are using + for the group operation. However, when we do this you must not confuse 1 with the real number 1. We are simply overworking the symbol 1 by using it to denote two different things. Of course, you already have some experience in doing that with matrices. The $m \times n$ matrices form a group under addition and we denote the identity in this group, the $m \times n$ zero matrix, by 0. Thus, in matrix algebra the symbol 0 is very overworked. For example, you might even write $0 \times 0 = 0$ where the first 0 is the 2×3 zero

matrix, the second 0 is the 3×4 zero matrix,and the 0 on the right is the 2×4 zero matrix.

2.5. Complex roots of unity. For each integer $n \geq 1$ we define

$$\mu_n := \{z \in \mathbb{C} \mid z^n = 1\}.$$

This is a group under multiplication: it contains 1 which serves as the identity in μ_n; if z is an n^{th} root of unity, so is z^{-1}; if $z^n = w^n = 1$, then $(zw)^n = 1$; and, of course, the multiplication is associative. We call μ_n the group of n^{th} roots of unity. We can explicitly list its elements

$$\mu_n := \{e^{2\pi im/n} \mid 0 \leq m \leq n-1\}.$$

Notice that $|\mu_n| = n$.

3. Isomorphisms

3.1. A preparatory result. Our next result is an important result about group homomorphisms and their kernels but we don't want to introduce that terminology yet so we will state the result without any fancy language. In any case, we will use parts of it frequently once we begin our discussion of isomorphisms.

We will use the notation e_G for the identity element in a group G when we need to distinguish it from the identity element in a group H.

LEMMA 2.9. *Let $f : G \to H$ be a function between two groups such that*

$$f(xy) = f(x)f(y)$$

for all $x, y \in G$. Define $K := \{g \in G \mid f(g) = e_H\}$. Then

1. *$e_G \in K$;*
2. *$f(x^{-1}) = f(x)^{-1}$ for all $x \in G$;*
3. *the multiplication in G makes K a group;*
4. *f is injective if and only if $K = \{e_G\}$.*

Proof. (1) The hypothesis on f implies that

$$e_H f(e_G) = f(e_G) = f(e_G e_G) = f(e_G)f(e_G).$$

However, we can cancel in H so $e_H = f(e_G)$.

(2) Let $x \in G$. Then $f(x)f(x^{-1}) = f(xx^{-1}) = f(e_G) = e_H$. Similarly, $f(x^{-1})f(x) = f(x^{-1}x) = f(e_G)$. Thus $f(x^{-1}) = f(x)^{-1}$.

(3) The following five observations show that K is a group.

(a) By (1), $e_G \in K$, so K is not empty.

(b) If $x, y \in K$, then $f(xy) = f(x)f(y) = e_H e_H = e_H$ so $xy \in K$. Thus, when restricted to elements of K, the multiplication in G gives a composition $K \times K \to K$.

(c) That composition is associative because it is associative for all elements in G.

(d) If $x \in K$, then $xe_G = x = e_G x$ so K contains an identity element, namely e_G

(e) If $g \in K$, then $f(g^{-1}) = f(g)^{-1} = e_H^{-1} = e_H$ so $g^{-1} \in K$. Hence every element in K has an inverse in K.

(4) Certainly, if f is injective $K = \{e_G\}$. To prove the converse suppose that $K = \{e_G\}$. If $f(x) = f(y)$, then

$$f(xy^{-1}) = f(x)f(y^{-1}) = f(y)f(y^{-1}) = f(yy^{-1}) = f(e_G) = e_H$$

so $xy^{-1} \in K = \{e_G\}$, i.e., $x = y$. Hence f is injective. □

3.2. When are two groups the same? The notion of *isomorphism* formalizes the notion of sameness for groups. Two groups are "essentially the same" if and only if they are isomorphic (see below).

Consider the groups $\mu_2 = \{1, -1\}$, $\mathbf{P} = (\{E, O\}, +)$, and $\mathbb{F} = (\{0, 1\}, +)$.

The three sets μ_2, $\mathbf{P}$, and $\mathbb{F}$, are different because their elements are different. Thus if someone asks *"are any two of these groups the same"* we are bound to reply *"no, their elements are the different."* However, if we focus on the group operations and compare their multiplication tables they are essentially the "same". To see this write their multiplication tables side-by-side and compare them.

It is the *algebraic structure* of a group that is important rather than the individual nature of its elements. Group theory is not about how we label the elements of a group. It is about the behavior of the group operation, the algebraic features of the group.

Definition 2.10. Let G and H be groups. A bijection $f : G \to H$ is called an isomorphism if

$$f(xy) = f(x)f(y)$$

for all $x, y \in G$. In this case we say G and H are isomorphic groups and use the notation $G \cong H$ to denote this fact. ◊

Every group is isomorphic to itself: the identity map $\mathrm{id}_G : G \to G$ is an isomorphism.

Proposition 2.11. *Let G and H be groups and suppose $f : G \to H$ is an isomorphism. Then*

(1) *$f(e_G) = e_H$;*
(2) *$f(x^{-1}) = f(x)^{-1}$ for all $x \in G$;*
(3) *$f^{-1} : H \to G$ is an isomorphism;*
(4) *if $g : H \to K$ is an isomorphism, then gf is an isomorphism.*

Proof. Parts (1) and (2) were already proved in Lemma 2.9.

(3) Let $a, b \in H$. Then there are elements $x, y \in G$ such that $f(x) = a$ and $f(y) = b$. By definition of f^{-1}, $f^{-1}(a) = x$ and $f^{-1}(b) = y$. Because f is an isomorphism, $f(xy) = f(x)f(y) = ab$ so

$$f^{-1}(ab) = xy = f^{-1}(a)f^{-1}(b)$$

which shows that f^{-1} is an isomorphism.

(4) Since f and g are bijective so is gf. If $x, y \in G$, then

$$gf(xy) = g(f(x)f(y)) = gf(x)gf(y)$$

so gf is an isomorphism. □

The point is this. A group as a set *with additional structure*. That additional structure is the product rule which satisfies the conditions in the definition of a group. The requirement that $f(xy) = f(x)f(y)$ is saying, in effect, that the bijection f is "compatible" with the additional structure, namely the products.

3.2.1. *Remark.* We used multiplicative notation when we defined "isomorphism" and when we stated and proved Proposition 2.11. However, often we will have isomorphisms between groups G and H in which the group laws are written additively. In that case the condition $f(xy) = f(x)f(y)$ must be replaced by $f(x + y) = f(x) + f(y)$. Here's a simple example of that.

3.2.2. *An example:* $\mathbb{Z} \cong \mathbb{Z}d$. Let d be a non-zero integer. We write $\mathbb{Z}d$ for the set of all multiples of d; i.e., $\mathbb{Z}d = \{nd \mid n \in \mathbb{Z}\}$. Then $(\mathbb{Z}d, +)$ is a group: it is non-empty; a sum of two multiples of d is a multiple of d so ordinary addition is a binary operation $\mathbb{Z}d \times \mathbb{Z}d \to \mathbb{Z}d$; addition is certainly associative; 0 is a multiple of d so is in $\mathbb{Z}d$ and is the identity in $\mathbb{Z}d$; and every element in $\mathbb{Z}d$ has an inverse in $\mathbb{Z}d$, namely its negative. Now we know $(\mathbb{Z}d, +)$ is a group, I claim that the function

$$f : \mathbb{Z} \to \mathbb{Z}d, \qquad f(n) := nd,$$

is an isomorphism. Certainly it is bijective; its inverse being the function that sends an element $x \in \mathbb{Z}d$ to x/d. Furthermore,

$$f(m + n) = (m + n)d = md + nd = f(m) + f(n)$$

so f is an isomorphism, and we may therefore write $\mathbb{Z} \cong \mathbb{Z}d$.

There was nothing special about d in what we just did. If r is any non-zero real number that set of all *integer* multiples of r is a group under addition and is isomorphic to $\mathbb{Z}$. Fill in the details if this is not immediately obvious,

3.2.3. *Remark.* Sometimes we have an isomorphism between groups where the group operation is written additively in one and multiplicatively in the other. In that case one has to change the condition $f(xy) = f(x)f(y)$ in the appropriate way. We will now give a simple example of this phenomenon. This example can be considered a warm-up for the much more important example that appears in section 3.3 below.

3.2.4. *Doubly infinite geometric progressions.* Let $a \in \mathbb{R} - \{-1, 0, 1\}$. Then

$$\mathbf{A} := \{a^n \mid n \in \mathbb{Z}\}$$

is a doubly infinite geometric progression: it consists of the numbers

$$\ldots a^{-2},\ a^{-1},\ 1,\ a,\ a^2, \ldots.$$

The set $\mathbf{A}$ is a group under multiplication. Of course, 1 is the identity, the inverse of a^n is a^{-n}, and multiplication is associative. I claim that the function

$$f : \mathbb{Z} \to \mathbf{A}, \qquad f(n) := a^n,$$

is an isomorphism of groups. Because a is not -1, 0, or -1, all the powers of a are different. Hence f is injective. It is obviously surjective, and therefore bijective. Since

$$f(m+n) = a^{m+n} = a^m a^n = f(m)f(n)$$

for all $m, n \in \mathbb{Z}$, f is an isomorphism. Notice that f^{-1}, which is also an isomorphism, satisfies $f^{-1}(xy) = f^{-1}(x) + f^{-1}(y)$.

3.2.5. *Groups with four elements.* We have seen several groups having four elements. Are they all isomorphic to one another or not? For example, the group $\mathbb{F}^2$ in section 2.3.11 is isomorphic to the group in section 2.3.12 with an isomorphism $f : \mathbb{F}^2 \to \{I, S, T, ST\}$ being given by the function

$$f(00) := I, \qquad f(01) := S, \qquad f(10) := T, \qquad f(11) := ST.$$

Certainly this f is bijective, but a little more care is required to see that

$$f(x+y) = f(x)f(y)$$

for all $x, y \in \mathbb{F}^2$. Although the group operation in $\mathbb{F}^2$ is denoted by $+$ and the operation in the other group is written multiplicatively this has no bearing on the question of isomorphism. Both groups are abelian but it is natural to use $+$ for one and $\times$ for the other.

The above f is not the only isomorphism from $\mathbb{F}^2$ to $\{I, S, T, ST\}$. The function $g : \mathbb{F}^2 \to \{I, S, T, ST\}$ given by

$$g(00) := I, \qquad g(01) := T, \qquad g(10) := ST, \qquad g(11) := S.$$

There are two more isomorphisms from $\mathbb{F}^2$ to $\{I, S, T, ST\}$. Can you find them?

Is $\mathbb{F}^2$ is isomorphic to μ_4? The answer is *no* but how do we see this? The best way to see this is to look for some algebraic feature that one of the groups has but the other does not have. For example, every element x in $\mathbb{F}^2$ has the property that $x + x$ is equal to the identity, but in μ_2 we have $i^2 \neq 1$. (The fact that the group operations are written differently has no relevance.) Alternatively, there is an element ξ in μ_4 such that every element in μ_4 is a power of ξ, i.e., $\mu_4 = \{\xi, \xi^2, \xi^3, \xi^4\}$. But there is no element x in $\mathbb{F}^2$ such that $\{x, x+x, x+x+x, x+x+x+x\}$ is equal to $\mathbb{F}^2$.

Of course, we need to turn these observations into more rigorous proofs, but that is easy—the harder part is to make the observations in the last paragraph.

We use a proof by contradiction to prover that $\mathbb{F}^2 \ncong \mu_4$. Suppose to the contrary that $f : \mathbb{F}^2 \to \mu_4$ is an isomorphism. Then for all $x \in \mathbb{F}^2$ we would have $f(x+x) = f(00) = 1$ (remember that an isomorphism sends the identity to the identity) and therefore $f(x)f(x) = 1$. But f is surjective so

we would have $\xi^2 = 1$ for all $\xi \in \mu_4$ and that is not the case. Hence no such f can exist.

Thus we have two different, meaning *non-isomorphic*, groups of order 4. Are there any others? No. If G is a group with 4 elements it is isomorphic to $\mathbb{F}^2$ or μ_4. It is a good exercise to try proving that.

PROPOSITION 2.12. *Let p be a positive prime number. Then there is only one group with p elements, i.e., if G and H are groups having p elements, then $G \cong H$.*

Proof. Let G be any group with p elements. Let e denote the identity element in G and fix an element x in G that is not e.

Let μ_p be the group of complex p^{th} roots of unity. Write $\xi = e^{2\pi i/p}$. Then $\mu_p = \{1, \xi, \xi^2, \ldots, \xi^{p-1}\}$. Define $f : \mu_p \to G$ by

$$f(\xi^k) = x^k.$$

Then $f(\xi^k\xi^\ell) = f(\xi^{k+\ell}) = x^{k+\ell} = x^k x^\ell = f(\xi^k)f(\xi^\ell)$.

Let $K = \{\xi^n \mid f(\xi^n) = e$. Then K is a subgroup of μ_p by Lemma 2.9. Suppose K contains some element other than 1, say ξ^n with $1 \le n \le p-1$. Since $\gcd(n, p) = 1$, there are integers a and b such that $an+bp = 1$. Because K is a group it contains

$$(\xi^n)^a = \xi^{na} = \xi^{1-bp} = \xi.$$

But that is absurd because $f(\xi) = x \neq e$. We conclude that K must equal $\{1\}$. It now follows from Lemma 2.9 that f is injective. But G and μ_p have the same number of elements so f is bijective and therefore an isomorphism.

We have shown that $G \cong \mu_p$. If H is another group with p elements then $H \cong \mu_p$ too. Hence $H \cong G$. □

3.3. Logarithms. The discovery that the groups $(\mathbb{R}_{>0}, \times)$ and $(\mathbb{R}, +)$ are isomorphic had a large impact on science. There are many isomorphisms between. Fix a real number $b > 1$. The functions

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}_{>0} \quad f(x) := b^x \\ g &: \mathbb{R}_{>0} \to \mathbb{R} \quad g(x) = \log_b x \end{aligned}$$

are mutually inverse isomorphisms.

The historical significance of this isomorphism lies in the fact that it simplified the task of calculation. For example, if one has to multiply two positive real numbers x and y, one uses the fact that

$$xy = fg(xy) = f\big(g(x) + g(y)\big) = f(\log_b x + \log_b y)$$

Large tables giving the values of $f(x)$ and $g(x)$ were published so one could simply look up the values of $g(x)$ and $g(y)$, add them, then look up the value of $f\big(g(x) + g(y)\big)$. Similarly, if one wanted to compute x^r, one used the fact that

$$x^r = fg(x^r) = f(\log(x^r)) = f(r \log x).$$

The discovery of the method of logarithms, i.e., the discovery of the isomorphisms f and g, is usually attributed to the Scotsman John Napier, the 8^{th} Laird of Merchiston (1550-1617). The method was first revealed to the public with the publication of Napier's book *Mirifici Logarithmorum Canonis Descriptio* in 1614. It was written in Latin, the language of science at the time, and contained 57 pages of explanation and 90 pages of tables of logarithms. Napier did not use a base as we now understand it, but his logarithms were, up to a scaling factor, effectively to base $1/e$.

Henry Briggs (1561-1630), Professor of Mathematics at Gresham College, London, from 1596 to 1619, and from 1619 Savilian professor of geometry at Oxford, visited Scotland in 1615 and 1617 seeking Napier's permission to publish a table of common logarithms, i.e., logarithms to the base 10. The first installment of Briggs' table of common logarithms, containing a brief account of logarithms and a long table of the logarithms of all integers below 1000 to 8 decimal places, was published in 1617 under the title *Logarithmorum Chilias Prima*.

In 1624, Briggs published *Arithmetica Logarithmica*, containing the logarithms of all integers from 1 to 20,000 and from 90,000 to 100,000 to fourteen places of decimals, together with an introduction in which the theory and use of logarithms are fully developed. The interval from 20,000 to 90,000 was filled by Adriaan Vlacq, a Dutch mathematician, but in his table, which appeared in 1628, the logarithms were given to only ten places of decimals. Vlacq's table was later found to contain 603 errors, but considering that the table was the result of original hand-calculation, and contained more than 2,100,000 digits, the number of errors is remarkably small. An edition of Vlacq's work, containing many corrections, was issued at Leipzig in 1794 under the title *Thesaurus Logarithmorum Completus* by Jurij Vega.

The great astronomer, Johannes Kepler (1571-1630) was an enthusiastic supporter of Napier's work and in 1624 published a clear explanation of how they worked. Napier had not done that and many had been reluctant to use Napier's logarithms prior to Kepler's explanation. At that time the main task of astronomers was to produce tables of astronomical data, in large part as an aid to astrology. The production of these tables involved huge amounts of computation so Kepler's enthusiasm is understandable.

Napier formed the word logarithm to mean a number that indicates a ratio: the Greek word logos meant proportion, and arithmos meant number. Napier chose that because the difference of two logarithms determines the ratio of the numbers they represent, so that an arithmetic series of logarithms corresponds to a geometric series of numbers. The term antilogarithm was introduced in the late 17th century and persisted in collections of tables until they fell into disuse around the 1970s.

Jost Bürgi (1552-1632), a Swiss clockmaker and maker of astronomical instruments, invented logarithms independently of John Napier. There is evidence that Bürgi invented the method of logarithms as early as 1588, six years before Napier began work on the same idea. By delaying the

publication of his work to 1620, and even then publishing only after repeated requests from Johannes Kepler, Bürgi lost his claim for priority.[3]

3.4. The circle group. Let $U(1)$ denote the set of complex numbers of length 1. If z and w are complex numbers, then $|zw| = |z||w|$ so the usual multiplication of complex numbers is an associative binary operation $U(1) \times U(1) \to U(1)$. The number 1 has length one so belongs to $U(1)$ and is its identity element. If z is a non-zero complex number, then $|z^{-1}| = |z|^{-1}$ so every element in $U(1)$ has an inverse in $U(1)$. Hence $U(1)$ is a group.

We call $U(1)$ the circle group because if you think of the complex numbers as points in the euclidean plane $\mathbb{R}^2$ the complex numbers of length one form a circle of radius one centered at 0.

3.5. Rotations in the plane. Given an angle θ we write T_θ for the linear transformation of the plane $\mathbb{R}^2$ that rotates a point by an angle of θ radians in the counterclockwise direction about the origin. Notice that $T_\theta = T_{\theta+2n\pi}$ for all $n \in \mathbb{Z}$, so each rotation can be labelled in infinitely many different ways.

It is obvious that $T_\theta T_\psi = T_{\theta+\psi}$ and that rotation in the counterclockwise direction by an angle of $-\theta$ is the same as rotation in the *clockwise* direction by an angle of θ so T_θ has an inverse, namely $T_{-\theta}$. it follows at once that *the set of all rotations is a group.* We call it the rotation group in $\mathbb{R}^2$ and denote it by SO(2). Actually, in keeping with its importance, it goes by a grander name, the special orthogonal group. As the notation suggests there are special orthogonal groups $\mathrm{SO}(n)$ for all integers $n \geq 0$ but for now we confine our attention to SO(2). The identity element is T_0, the rotation through zero degrees. In fact, $T_{2n\pi} = \mathrm{id}_{\mathbb{R}^2}$ for all $n \in \mathbb{Z}$, and $T_{(2n+1)\pi} = -\mathrm{id}_{\mathbb{R}^2}$ for all $n \in \mathbb{Z}$.

[3]In 1615, Kepler's mother, Katharina Kepler, was accused of witchcraft by a prostitute. European witch hunting was at its peak during Kepler's career, and was supported by all levels of society, including secular officials and intellectuals in universities. Kepler spent several years making legal appeals and hiding his mother from legal authorities seeking to torture her into confessing to witchcraft. Examining an accused witch *ad torturam* was a standard court procedure during this era. In 1620, under court order, Kepler's mother was kidnapped in the middle of the night from her daughter's home and taken to prison. Kepler spent the next year appealing to the Duke of Württemberg to prevent his imprisoned mother from being examined *ad torturam*. However, on September 28, 1621 Frau Kepler was taken from her prison cell into the torture room, shown the instruments of torture and ordered to confess. She replied "Do with me what you want. Even if you were to pull one vein after another out of my body, I would have nothing to admit," and said the Lord's Prayer. She was taken back to prison and freed on October 4 upon order of the duke, who ruled that her refusal to confess under threat of torture proved her innocence. He also ordered her accusers to pay the cost of her trial and imprisonment. After having spent most of the last seven years under the legal threat of imminent torture, Katharina Kepler died on April 13, still being threatened with violence from those who insisted she was a witch.

Let's use column vectors $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to label the points in the plane $\mathbb{R}^2$ and write A_θ for the unique 2×2 matrix such that $T_\theta(\underline{x}) = A_\theta \underline{x}$ for all $\underline{x} \in \mathbb{R}^2$. If we write our matrix with respect to the ordered basis

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the first column of A_θ is given by $T_\theta(\underline{e}_1)$ and the second column of A_θ is given by $T_\theta(\underline{e}_2)$. Elementary trigonometry (draw the diagrams and check!) then gives

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2-2}$$

There are many interesting aspects to this:

- if you have forgotten your formulas for $\sin(\theta + \psi)$ and $\cos(\theta + \psi)$ you can recover them by using the fact that $A_\theta A_\psi = A_{\theta+\psi}$ and computing the product on the left;
- the determinant of A_θ is 1 because $\cos^2\theta + \sin^2\theta = 1$;
- $A_\theta^{-1} = A_{-\theta}$ and using the formula for the inverse of a 2×2 matrix in you can recover the formulae for $\sin(-\theta)$ and $\cos(-\theta)$ if you have forgotten them.

3.6. The isomorphism between $U(1)$ and $\mathrm{SO}(2)$. Define $f : \mathrm{SO}(2) \to U(1)$ by

$$f(T_\theta) := e^{i\theta}.$$

This *is* a well-defined function: although $T_\theta = T_{\theta+2n\pi}$, $f(T_\theta) = f(T_{\theta+2n\pi})$ because $e^{2n\pi i} = 1$. Furthermore, f is bijective: it is surjective because if $z = a + ib \in U(1) - \{\pm i\}$, then $z = e^{i\theta}$ where

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

and $i = e^{i\pi/2}$ and $-i = e^{-i\pi/2}$; it is injective because if $f(T_\theta) = f(T_\psi)$, then $e^{i\theta} = e^{i\psi}$ which implies $e^{i(\theta-\psi)} = 1$ and $\theta - \psi = 2n\pi$ for some integer n, whence $T_\theta = T_\psi$.

Finally, since

$$f(T_\theta)f(T_\psi) = e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)} = f(T_{\theta+\psi}) = f(T_\theta T_\psi)$$

we conclude that f is an isomorphism.

3.7. More about isomorphisms. Equivalence relation

Suppose G and H are isomorphic groups. Then

(1) $|G| = |H|$;
(2) G is abelian if and only if H is abelian;
(3)

3.7.1. *What does isomorphism mean?* Two groups are isomorphic if and only if all their "essential" group-theoretic, or algebraic, features are the same. The word *essential* is vague, but here is an example of a difference that is not essential: let $\mu_2 = \{1, -1\}$ and let $\mathbf{P} = \{E, O\}$ be the parity group. The groups μ_2 and $\mathbf{P}$ are *different* because we use different labels for their elements and different symbols for the group operation ($\cdot$ for μ_2 and addition $+$ for $\mathbf{P}$). However, in terms of their group-theoretic properties they are the same: they each have two elements and one other element that is its own inverse.

There are some obvious questions you might ask. For example, are any two groups with the same numbers of elements isomorphic? No; for example, $\mathbb{Z}_6$ and S_3 have six elements but are not isomorphic because one is abelian and the other is not. We have already seen that μ_4 and $\mathbb{F}^2$ both have order 4 but are not isomorphic because every element in $\mathbb{F}^2$ is its own inverse but μ_4 does not have that property.

3.7.2. *Joke.* An exam question describes two groups and then asks *"Are these two groups isomorphic?"* The student answers *"The first one is but the second one isn't."* :)

4. Subgroups

4.1. Definition. Let G be a group. A subset $H \subset G$ is a subgroup of G if the multiplication on G makes H a group.

Note that G itself is a subgroup of G and so is $\{e\}$. These are the boring subgroups.

To show that a subset H of G is a subgroup we must show H satisfies axioms (1), (2), and (3), of Definition 2.7. Even before checking those we must check condition (0), that xy is in H whenever x and y are. If H passes that test we say H is closed under multiplication.

Oh, we must check H is non-empty; let's assume it is.

Condition (1), associativity of multiplication, will be satisfied by elements $x, y, z \in H$ because it is already satisfied for all $x, y, z \in G$. So we do not need to check condition (1).

Condition (2) says H must have an identity. Could H have an identity that is different from e, the identity of G? If $e' \in H$ is such that $e'x = x$ for even one element $x \in H$, then $e'x = ex$; but we can cancel in G so $e' = e$. So, the identity in H has to be the identity in G. Thus H must contain e.

To see if condition (3) holds for H, the uniqueness of inverses tells us that x^{-1} must belong to H whenever x does. All this proves the next result.

PROPOSITION 2.13. *A subset H of G is a subgroup if and only*

(1) *H contains the identity of G, and*
(2) *xy belongs to H whenever x and y do, and*
(3) *x^{-1} belongs to H whenever x does.*

A shorter characterization of subgroups is given by the next result.

PROPOSITION 2.14. *Let H be a non-empty subset of a group G. Then H is a subgroup if and only xy^{-1} belongs to H whenever x and y do.*

Proof. ($\Rightarrow$) This is trivial.

($\Leftarrow$) By hypothesis, H is non-empty so contains an element, h say. It also contains hh^{-1}, the identity. Thus, if $h \in H$, so is $eh^{-1} = h^{-1}$, so H contains the inverse of every element in it. Finally, if $u, v \in H$, then $v^{-1} \in H$ so H contains $u(v^{-1})^{-1} = uv$. Hence H is a subgroup. □

Let d be an integer. Because the difference of two multiples of d is a multiple of d Proposition 2.14 tells us that $d\mathbb{Z} := \{dn \mid n \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.

PROPOSITION 2.15. *The subgroups of $\mathbb{Z}$ are the subsets $d\mathbb{Z}$, $d \geq 0$.*

Proof. Let H be a subgroup of $\mathbb{Z}$. Since $\{0\} = 0\mathbb{Z}$ we may assume $H \neq \{0\}$.

If h is in H so is $-h$. Hence there is a smallest positive integer in H. Let d be that integer. Because H is a subgroup, every multiple of d belongs to H. Thus $d\mathbb{Z} \subset H$.

Now let h be any element in H. Then $h = qd + r$ for some integers q and r with $0 \leq r < d$. Since $r = h - qd$, r belongs to H. By choice of d it follows that $r = 0$, whence $h \in d\mathbb{Z}$. Thus $d\mathbb{Z} = H$. □

4.1.1. *Exercise:* If H is a subgroup of G and $f : G \to G'$ an isomorphism show that $f(H)$ is a subgroup of G' and that the restriction of f to H is an isomorphism from H to $f(H)$.

PROPOSITION 2.16. *If H and K are subgroups of G so is $H \cap K$.*

Proof. Certainly $H \cap K$ is non-empty because both H and K contain the identity element. Suppose that x and y belong to $H \cap K$. Then $xy^{-1} \in H$ because H is a subgroup and $xy^{-1} \in K$ because K is a subgroup. Thus $xy^{-1} \in H \cap K$ and it now follows from the previous result that $H \cap K$ is a subgroup of G. □

The next result explains, in part, the importance of the symmetric groups—they are ubiquitous.

THEOREM 2.17 (Cayley's Theorem). *If G is a group with n elements, then G is isomorphic to a subgroup of the symmetric group S_n.*

Proof. The idea of the proof is simple: if $x \in G$, then the function $\lambda_x : G \to G$ defined by $\lambda_x(y) := xy$ is a permutation of G. Write P for the group of permutations of G. Then P is a group isomorphic to S_n so it is enough to show that G is isomorphic to a subgroup of P.

Define $f : G \to \mathsf{P}$ by $f(x) := \lambda_x$. First we check that λ_x really is in P: if $y \neq y'$, then $xy \neq xy'$ so $\lambda_x(y) \neq \lambda_x(y')$ which implies that λ_x is injective; if $g \in G$, then $g = \lambda_x(x^{-1}g)$ so λ_x is surjective; thus λ_x is a permutation of G. It is clear that $f(x)f(y)$ is the function that sends g to $\lambda_x\lambda_y(g) = xyg = \lambda_{xy}(g)$ so $f(x)f(y) = f(xy)$.

Certainly $f(G)$ is a subgroup of P because $f(1) = \mathrm{id}_G$; $f(x)f(y) = f(xy)$; and $f(x)^{-1} = f(x^{-1})$.

It is clear that f is injective because if $x \neq y$, $f(x)(1) = x$ whereas $f(y(1) = y$ so $f(x) \neq f(y)$. Thus, considered as map from G to $f(G)$, which is a subgroup of P, f is bijective and hence an isomorphism. □

4.2. The center of G. The center of G is the subgroup

$$Z(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

In German, the word for *center* is *zentrum*. That's why we use the notation $Z(G)$. Germany was a hotbed of group theory in the late 1800s.

The center of G is a subgroup of G, a particularly important one.

In fact, it is a special case of the following more general result which we will prove after introducing a generalization of the center. The centralizer of a subset $\mathcal{S}$ of G is defined to be

$$C_G(\mathcal{S}) := \{z \in G \mid zg = gz \text{ for all } g \in \mathcal{S}\}.$$

In other words, $C_G(\mathcal{S})$ consists of the elements in G that commute with all elements in $\mathcal{S}$. Hence $Z(G) = C_G(G)$.

LEMMA 2.18. *If $\mathcal{S}$ is a subset of G, then $C_G(\mathcal{S})$ is a subgroup of G.*

Proof. Certainly the identity of G commutes with all elements in $\mathcal{S}$—it commutes with all elements in G! If x and y commute with all elements in $\mathcal{S}$ so do x^{-1} and xy. To see this let $g \in \mathcal{S}$. The cancellation law implies that $xg = gx$ if and only if $x^{-1}(xg)x^{-1} = x^{-1}(gx)x^{-1}$, i.e., if and only if $gx^{-1} = x^{-1}g$, so $x^{-1} \in C_G(\mathcal{S})$. Furthermore, $xyg = xgy = gxy$ so $xy \in C_G(\mathcal{S})$. □

4.3. The subgroup generated by a subset of G. If S is a subset of G we introduce the notation

$$\langle S\rangle := \{\text{the smallest subgroup of } G \text{ containing } S\}$$

and call $\langle S\rangle$ the subgroup of G generated by S. For the definition to make sense we must check there *is* a smallest such subgroup: if H and K are subgroups containing S, then $H \cap K$ is a subgroup containing S, so one sees that

$$\langle S\rangle = \text{the intersection of all the subgroups of } G \text{ that contain } S.$$

Notice that Proposition **??**.2.6 says that the symmetric group S_n is generated by transpositions. However, one can be efficient and generate it with just $n-1$ transpositions. Show that $S_n = \langle (1\,2), (2\,3), \ldots, (n-1\,n)\rangle$.

LEMMA 2.19. *If $x \in G$, then $\langle x\rangle = \{x^n \mid n \in \mathbb{Z}\}$.*

Proof. Certainly $\{x^n \mid n \in \mathbb{Z}\}$ is a group: it contains the identity, x^0, and contains inverses, $(x^n)^{-1} = x^{-n}$, and is closed under products, $x^m x^n = x^{m+n}$. Hence $\{x^n \mid n \in \mathbb{Z}\}$ is a subgroup of G that contains x.

Let H be a subgroup of G containing x. Then H contains x^n for all $n > 0$ because it is closed under multiplication; H contains inverses so contains x^{-1}, and products of x^{-1} with itself, so contains x^{-n} for all $n > 0$; H contains the identity so contains x^0 too. Thus, H contains $\{x^n \mid n \in \mathbb{Z}\}$. Hence $\{x^n \mid n \in \mathbb{Z}\}$ is the smallest subgroup of G containing x. □

4.3.1. *Exercise.* If $a_1, \ldots, a_n \in \mathbb{Z}$ show that $\langle a_1, \ldots, a_n \rangle = d\mathbb{Z}$ where d is the greatest common divisor of $a_1, \ldots, a_n$.

4.3.2. *A non-example.* Let H be the subset of the group $G = (\mathbb{Q} - \{0\}, \cdot)$ consisting of the negative integers and 1. This fails to be a subgroup because the product of two elements in H need not belong to H; but it satisfies all the other axioms.

4.4. Two subgroups of G. Suppose H and K are subgroups of G. We have seen that $H \cap K$ is also a subgroup of G. It is the *largest* subgroup of G contained in both H and K.

4.4.1. *The abelian case.* There is an analogy with least common multiples. If a and b are non-zero integers, then $\mathbb{Z}a \cap \mathbb{Z}b = \mathbb{Z}c$ where c is the least common multiple of a and b. For example, $20\mathbb{Z} \cap 12\mathbb{Z} = 60\mathbb{Z}$.

It is natural to ask if we can do something similar with greatest common divisors. We can. The *smallest*subgroup that contains $20\mathbb{Z}$ and $12\mathbb{Z}$ is $4\mathbb{Z}$ and $4 = \gcd\{12, 20\}$. If A and B are subsets of the integers we sometimes write

$$A + B := \{a + b \mid a \in A \text{ and } b \in B\}.$$

For example, $20\mathbb{Z} + 12\mathbb{Z} = 4\mathbb{Z}$. If A and B are *subgroups* of $\mathbb{Z}$, then $A + B$ is a subgroup of $\mathbb{Z}$ and is the smallest subgroup that contains both A and B.

If a and b are non-zero integers, then $\mathbb{Z}a + \mathbb{Z}b = \mathbb{Z}d$ where $d = \gcd\{a, b\}$ and $\mathbb{Z}d$ is the smallest subgroup of $\mathbb{Z}$ that contains both $\mathbb{Z}a$ and $\mathbb{Z}b$.

We extend this notation in an obvious way.

Proposition 2.20. *Let H and K be subgroups of an abelian group $(G, +)$. Define*

$$H + K := \{x + y \mid x \in H \text{ and } y \in K\}.$$

Then $H + K$ is a subgroup of G and is the smallest subgroup that contains both H and K.

Proof. □

In other words, $H + K$ is the subgroup of G generated by $H \cup K$. We also write $H + K = \langle H, K \rangle$.

4.4.2. *The non-abelian case.* If G is not abelian a little more care is required. First we will write G multiplicatively and we define

$$AB := \{ab \,|a \in A \text{ and } b \in B\}$$

whenever A and B are subsets of G.

PROPOSITION 2.21. *Let H and K be subgroups of a group G. If $HK = KH$, then HK is a subgroup of G and is the smallest subgroup that contains both H and K.*

Proof. □

You might want to look ahead to Theorem 2.33 for a result that is somewhat in the same spirit as Proposition 2.21.

5. Cosets and Lagrange's Theorem

5.1. Let H be a subgroup of G. For each element $x \in G$, we define

$$xH := \{xh \mid h \in H\}.$$

We call the sets xH the right cosets of H in G.

There is a similar notion of left cosets.

Because $e \in H$, $x \in xH$, so G is the union of the right cosets.

LEMMA 2.22. *If H is a subgroup of G, then G is the disjoint union of the distinct right cosets xH.*

Proof. If $xH \cap yH \neq \phi$, then $xa = yb$ for some $a, b \in H$. If $c \in H$, then $xc = xaa^{-1}c = yba^{-1}c$ which is in yH because $a, b, c \in H$. Hence $xH \subset yH$. Similarly, $yH \subset xH$; thus $xH = yH$. □

It follows from the disjointness result that xH is the *unique* right coset of H that contains x.

LEMMA 2.23. *If H is a subgroup of G, then $|xH| = |H|$.*

Proof. There is a map $H \to xH$, $h \mapsto xh$; this map is surjective and is injective because if $xh = xh'$, then $h = h'$; hence xH has the same number of elements as H. □

PROPOSITION 2.24 (Lagrange's Theorem). *Let H be a subgroup of a finite group G. Then $|H|$ divides $|G|$, and $|G|/|H|$ is the number of right cosets of H in G.*

Proof. Since G is a disjoint union of sets xH, each of which has the same number of elements as H,

$$|G| = n|H|$$

where n is the number of right cosets of H in G. □

Of course, $|G|/|H|$ is also the number of left cosets of H in G.

5.1.1. *Warning:* Although the number of left cosets of H is the same as the number of right cosets of H, in general a left coset is not a right coset. For example, if H is the subgroup $\{1,(12)\}$ of S_3, then

$$(13)H = \{(13),(13)(12)\} = \{(13),(123)\}$$

but

$$H(13) = \{(13),(12)(13)\} = \{(13),(132)\} \neq (13)H.$$

Let H be a subgroup of a group G. The index of H in G is

$$[G:H] := \text{the number of cosets of } H \text{ in } G.$$

The proof of Lagrange's theorem tells us the following.

COROLLARY 2.25. *If H be a subgroup of a group G, then*

$$|G| = |H| \cdot [G:H].$$

Proof. This is because G is the disjoint union of the cosets of H, each of which has $|H|$ elements, and there are $[G:H]$ different cosets. □

5.2. Coset notation. We extend the notation xH for a coset of a subgroup by writing

$$xS := \{xs \mid s \in S\}$$

for *any* subset $S \subset G$. Likewise,

$$Sx := \{sx \mid s \in S\}.$$

More generally, if T is another subset of G we write

$$ST := \{st \mid s \in S, t \in T\}.$$

This notation behaves in a nice way. For example, $x(yS) = (xy)S$ and $(xS)y = x(Sy)$, so we simply write xyS and xSy for these subsets. Similarly, $x(ST) = (xS)T$, and so on.

6. Cyclic groups

6.1. Definition. A group G is cyclic if it is generated by one element, i.e., if $G = \langle x \rangle$ for some x. We then call x a generator of G and say that x generates G.

Equivalently, G is cyclic if it equals $\{x^n \mid n \in \mathbb{Z}\}$ for some $x \in G$. Note we do not insist that x^n and x^m are different if m and n are different.

The group of n^{th} roots of unity,

$$\mu_n := \{z \in \mathbb{C} \mid z^n = 1\},$$

is cyclic. For example, $e^{2\pi i/n}$ is a generator for μ_n. In fact, if r is any integer relatively prime to n, $e^{2r\pi i/n}$ is a generator for μ_n.

The group of integers $(\mathbb{Z},+)$ is cyclic. Clearly, $\mathbb{Z} = \langle 1 \rangle$ and $\mathbb{Z} = \langle -1 \rangle$.

A cyclic group is abelian because $x^m x^n = x^{m+n} = x^{n+m} = x^n x^m$.

Every group contains a cyclic subgroup, namely $\langle x \rangle$ for any $x \in G$.

6.2. The order of an element. The order of an element $x \in G$ is the smallest positive integer n such that $x^n = 1$. If there is no such n we say that x has infinite order.

PROPOSITION 2.26. *The order of x is equal to the number of elements in $\langle x \rangle$.*

Proof. Suppose first that x has infinite order. If $\langle x \rangle$ was finite there would be integers $m \neq n$ such that $x^m = x^n$. But then $x^{m-n} = x^{n-m} = 1$ so x would have finite order. This contradiction shows the result is true when x has infinite order.

Now suppose x has finite order, n say. We will show that

$$\langle x \rangle = \{1, x, x^2, \ldots, x^{n-1}\}$$

and that these n elements are distinct from one another. If m is any integer, then $m = nq + r$ for some $q, r \in \mathbb{Z}$ such that $0 \leq r \leq n-1$. It follows that

$$x^m = x^{na+r} = x^{na}x^r = (x^n)^a x^r = x^r.$$

Hence $\langle x \rangle = \{1, x, x^2, \ldots, x^{n-1}\}$.

If $x^i = x^j$ for some integers $0 \leq i < j \leq n-1$, then $x^{i-j} = x^{j-i} = 1$, so $x^{|i-j|} = 1$; but $0 < |i-j| < n$, contradicting the hypothesis that x has order n. □

PROPOSITION 2.27. *Let x be an element in a finite group G. Then*

(1) *x has finite order;*
(2) *the order of x divides $|G|$;*
(3) *$x^{|G|} = e$.*

Proof. (1) Since $\langle x \rangle$ is finite the order of x is finite.

(2) The order of x is the number of elements in $\langle x \rangle$, and Lagrange's Theorem tells us that this number divides $|G|$.

(3) This follows at once from (2) because $x^{rs} = (x^r)^s$. □

It is important to distinguish between the order of an element and the order of a group. There are infinite groups in which every element has finite order. For example, the group of all subsets of $\mathbb{Z}$ with group operation

$$A \oplus B = \{x \mid x \in A \cup B \text{ but } x \notin A \cap B\}$$

is infinite but every element in it has order two (except the identity which has order one).

6.2.1. *An infinite group all of whose elements have finite order.* Let G be the subgroup of $(\mathbb{Q}, +)$ consisting of those fractions a/b such that b is a power of 2; that is

$$G := \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b = 2^d \text{ for some } d \geq 0 \right\}.$$

Notice that $\mathbb{Z}$ is a subgroup of G. The quotient $G/\mathbb{Z}$ is an infinite group, and every element in it has finite order because if $g \in G$, then $g = a/2^d$

for some d and, if we set $n = 2^d$, $ng = g + \cdots + g \in \mathbb{Z}$. Hence, in $G/\mathbb{Z}$, $n(g + \mathbb{Z}) = ng + \mathbb{Z} = \mathbb{Z}$, the identity element to $G/\mathbb{Z}$.

6.3. A convenient notation: addition of subsets of $\mathbb{Z}$. If A and B are subsets of $\mathbb{Z}$ we introduce the notation

$$A + B := \{a + b \mid a \in A,\, b \in B\}.$$

This operation of $+$ is simply a notational convenience. It does *not* make the set of all subsets of $\mathbb{Z}$ a group (why not?). However, this addition is commutative, i.e., $A+B = B+A$, associative, i.e., $A+(B+C) = (A+B)+C$, and has an identity, namely the empty set $\{0\}$.

If A consists of a single element, say $A = \{a\}$, we write $a + B$ rather than $A + B$.

6.3.1. *Warning.* This notation for addition of subsets should not be confused with the notation for set difference. If A and B are subsets of a set X we write $A - B$ for $\{a \in A \mid a \notin B\}$. Sorry about this, but we have to deal with the language as it is spoken. Before getting too annoyed talk to a non-native English speaker about the words *thought, through, rough, and thorough*. Ugh!

6.4. The groups $\mathbb{Z}/d$. Let's start with the case $d = 5$. The group $\mathbb{Z}/5$ has five elements. (We pronounce $\mathbb{Z}/5$ as *zee mod five*.) Its elements are the following subsets of $\mathbb{Z}$:

$$\begin{aligned} 5\mathbb{Z} &:= \{5n \mid n \in \mathbb{Z}\}, \\ 1 + 5\mathbb{Z} &:= \{1 + 5n \mid n \in \mathbb{Z}\}, \\ 2 + 5\mathbb{Z} &:= \{2 + 5n \mid n \in \mathbb{Z}\}, \\ 3 + 5\mathbb{Z} &:= \{3 + 5n \mid n \in \mathbb{Z}\}, \\ 4 + 5\mathbb{Z} &:= \{4 + 5n \mid n \in \mathbb{Z}\}. \end{aligned}$$

For brevity we will write $r' = r + 5\mathbb{Z}$ for all integers r. Thus

$$\mathbb{Z}/5 := \{0', 1', 2', 3', 4'\}.$$

Note that $2' = 2+5\mathbb{Z} = 7+5\mathbb{Z} = 7'$, and so on. Notice too, that if $0 \le r \le 4$, then r' consists of the integers that leave a remainder of r when divided by 5. To make $\mathbb{Z}/5$ a group we define an addition on it by declaring that

$$r' + s' = (r + s)'.$$

The addition is well-defined (why?) and the addition table is

$+$	$0'$	$1'$	$2'$	$3'$	$4'$
$0'$	$0'$	$1'$	$2'$	$3'$	$4'$
$1'$	$1'$	$2'$	$3'$	$4'$	$0'$
$2'$	$2'$	$3'$	$4'$	$0'$	$1'$
$3'$	$3'$	$4'$	$0'$	$1'$	$2'$
$4'$	$4'$	$0'$	$1'$	$2'$	$3'$

To verify that this addition rule makes $\mathbb{Z}/5$ a group you must check associativity, the existence of an identity, and existence of inverses. Associativity follows from the associative law $A + (B + C) = (A + B) + C$ that was mentioned above for subsets of $\mathbb{Z}$. It is clear that $0'$ is an identity in $\mathbb{Z}/5$. Inverses exist because $r' + (5 - r)' = 0'$. Hence $(\mathbb{Z}/5, +)$ is a group. It is an abelian group because $r' + s' = s' + r'$, i.e., because the addition of subsets of $\mathbb{Z}$ is commutative.

In the preceding example no special properties of the number 5 were used. For every positive integer d we define

$$\mathbb{Z}/d := \{r + d\mathbb{Z} \mid 0 \leq r \leq d - 1\}$$

and make $\mathbb{Z}/d$ a group by defining

$$(r + d\mathbb{Z}) + (s + d\mathbb{Z}) := (r + s) + d\mathbb{Z}$$

PROPOSITION 2.28. *For every integer $d \geq 1$ there is an abelian group with d elements, namely $\mathbb{Z}/d$.*

Proof. Use the same arguments as those we used above to show that the addition on $\mathbb{Z}/5$ made it a group. □

PROPOSITION 2.29. *Let G be a cyclic group and d a positive integer. Then*

(1) *G is isomorphic to $\mathbb{Z}$ if it is infinite;*
(2) *G is isomorphic to $\mathbb{Z}/d$ if it has d elements.*

Proof. By hypothesis, $G = \langle x \rangle = \{x^n \mid n \in \mathbb{Z}\}$. Define $f : \mathbb{Z} \to G$ by

$$f(n) := x^n.$$

Notice that $f(m + n) = x^{m+n} = x^m x^n = f(m)f(n)$. If f is bijective it is an isomorphism so in that case $G \cong \mathbb{Z}$.

Now assume f is not bijective. In that case $x^n = x^m$ for some $n \neq m$. If follows that $x^{n-m} = x^{m-n} = 1$. Let d be the smallest positive integer such that $x^d = 1$. Define

$$g : \mathbb{Z}/d \to G \qquad \text{by} \quad f(r + d\mathbb{Z}) := x^r.$$

Then g is surjective, and therefor bijective because $|G| = d = |\mathbb{Z}/d|$. Also

$$\begin{aligned} f\big((r + d\mathbb{Z}) + (s + d\mathbb{Z})\big) &= f\big((r + s) + d\mathbb{Z}\big) \\ &= x^{r+s} \\ &= x^r x^s \\ &= f(r + d\mathbb{Z})f(s + d\mathbb{Z}) \end{aligned}$$

for all r and s. Hence f is an isomorphism; i.e., $G \cong \mathbb{Z}/d$. □

The previous result says the only cyclic groups are $\mathbb{Z}$ and $\mathbb{Z}/d$, $d > 0$. Up to isomorphism!

COROLLARY 2.30. *Let n be a positive integer. Then there is an isomorphism*

$$\mu_n \cong \mathbb{Z}/n$$

between the group of complex n^{th} roots of unity and $\mathbb{Z}/n$.

Proof. Since μ_n is cyclic, generated by $e^{2\pi i/n}$ for example, the corollary follows from Proposition 2.29. □

The next result says that if p is a prime number there is only one group having p elements, namely $\mathbb{Z}/p$.[4]

THEOREM 2.31. *If p is a positive prime number, there is, up to isomorphism, a unique group with p elements, namely $\mathbb{Z}/p$. In other words, if G is a group having p elements then $G \cong \mathbb{Z}/p$.*

Proof. Let p be a positive prime and G a group with p elements.

We fix an an element x in G that is not the identity. Then the order of x is ≥ 2 and divides $|G|$ by Proposition 2.27, so must be p. By Proposition 2.26, the order of x is the number of elements in $\langle x \rangle$. Thus

$$G = \{1, x, \ldots, x^{p-1}\}.$$

In particular, G is a cyclic group with p elements so G is isomorphic to $\mathbb{Z}/p$ by Proposition 2.29. □

7. The product of two groups

7.1. The definition. Let G and H be groups. We make their cartesian product

$$G \times H = \{(g, h) \mid g \in G,\, h \in H\}$$

into a group by declaring the product to be

$$(a, x).(b, y) := (ab, xy) \tag{2-3}$$

for $a, b \in G$ and $x, y \in H$. We call this the product of the groups G and H.

7.1.1. *Exercises.*

(1) Show (2-3) makes $G \times H$ a group.
(2) Show that $G \times H \cong H \times G$.
(3) If H_1 and H_2 are isomorphic groups show that $G \times H_1$ is isomorphic to $G \times H_2$.
(4) Let G_1, G_2, H_1, and H_2 be groups. Suppose that $G_1 \cong G_2$ and $H_1 \cong H_2$. Show that $G_1 \times H_1 \cong G_2 \times H_2$.

7.1.2. *Remark.* If G and H are abelian we often call their direct product the direct sum and denote it by $G \oplus H$.

[4] When I say *"only"* I should add the qualifier *"up to isomorphism"*, but I'm not doing that because it would make the sentence a little clunky and I want to encourage you to add that qualifier internally.

7.1.3. *An important isomorphism.* When you encountered complex numbers did you learn that every non-zero complex number can be written in a unique way as

$$re^{i\theta}$$

where r is a positive real number and $\theta \in [0, 2\pi)$? I hope so. We will review that in a moment but if you already know that fact are you aware that it is really saying that there is an isomorphism

$$f : (\mathbb{R}_{>0}, \cdot) \times U(1) \to (\mathbb{C} - \{0\}, \cdot)$$

given by the formula $f(r, e^{i\theta}) := re^{i\theta}$.

7.2. An example. Let $\mathbb{F} = (\{0, 1\}, +)$ be the group with addition $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$. Then $\mathbb{F} \times \mathbb{F}$ has four elements, $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. It is simpler to write these as 00, 01, 10, and 11. The group operation on $G \times G$ given by (**??**) is This is the group in

+	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

example 2.3.11. Now you know why we labeled it $\mathbb{F}^2$.

Of course, $\mathbb{F} \cong \mathbb{Z}/2$ so $\mathbb{F}^2 \cong (\mathbb{Z}/2) \times (\mathbb{Z}/2)$.

Suppose that X is a set and $X = Y \sqcup Z$, i.e., $X = Y \cup Z$ and $X \cap Z = \phi$. Then $X! \times Z!$ is a subgroup of $X!$.

More to say!

7.2.1. *Products of several groups.* Let G, H, and K, be groups. It is easy to show that

$$(G \times H) \times K \cong G \times (H \times K)$$

so we just write $G \times H \times K$ for this group. This idea extends to arbitrary finite collections of groups. If $G_1, \ldots, G_n$ are groups the elements of $G_1 \times \cdots \times G_n$ are ordered n-tuples $(x_1, \ldots, x_n)$ with $x_i \in G_i$ and this set is made into a group by declaring that

$$(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n).$$

7.3. The Chinese Remainder Theorem. We will now prove the Chinese Remainder Theorem, a result that is an important step towards the classification of all finite abelian groups (up to isomorphism, of course!).

THEOREM 2.32 (The Chinese Remainder Theorem). *Suppose d and e are positive integers. If $\gcd(d, e) = 1$, then there is an isomorphism*

$$f : \mathbb{Z}/de \xrightarrow{\sim} (\mathbb{Z}/d) \times (\mathbb{Z}/e)$$

given by $f(r + de\mathbb{Z}) := (r + d\mathbb{Z}, r + e\mathbb{Z})$.

Proof. First we will show f is well defined. If $r + de\mathbb{Z} = s + de\mathbb{Z}$, then de divides $r - s$ so $r - s \in d\mathbb{Z}$ and $r - s \in e\mathbb{Z}$. It follows that $r + d\mathbb{Z} = s + d\mathbb{Z}$ and $r + e\mathbb{Z} = s + e\mathbb{Z}$. Hence f is well-defined.

We will now show f is injective. It will follow from this that f is bijective because the sets $\mathbb{Z}/de$ and $(\mathbb{Z}/d) \times (\mathbb{Z}/e)$ have the same number of elements. By Lemma 2.9(2), to show f is injective we need only show that if $f(r + de\mathbb{Z}) = (d\mathbb{Z}, e\mathbb{Z})$, then $r \in de\mathbb{Z}$.

Suppose $f(r + de\mathbb{Z}) = (d\mathbb{Z}, e\mathbb{Z})$. Then $r \in d\mathbb{Z}$ and $r \in e\mathbb{Z}$; say $r = du = ev$. Because $\gcd(d, e) = 1$ there are integers a and b such that $ad + be = 1$. It follows that

$$u = adu + beu = aev + beu = e(av + bu).$$

Hence $r = du = de(av + bu) \in ed\mathbb{Z}$. Thus $r + ed\mathbb{Z} = ed\mathbb{Z}$. Thus, $f^{-1}\big((d\mathbb{Z}, e\mathbb{Z})\big) = \{de\mathbb{Z}\}$ and it follows from Lemma 2.9(2) that f is injective and therefore bijective.

Finally, the calculation

$$\begin{aligned} f\big((r + de\mathbb{Z}) + (s + de\mathbb{Z})\big) &= f\big((r + s) + de\mathbb{Z}\big) \\ &= (r + s + d\mathbb{Z}, r + s + e\mathbb{Z}) \\ &= (r + d\mathbb{Z}, r + e\mathbb{Z}) + (s + d\mathbb{Z}, s + e\mathbb{Z}) \\ &= f(r + de\mathbb{Z}) + f(s + de\mathbb{Z}) \end{aligned}$$

shows f is an isomorphism. □

Thus, for example,

$$\mathbb{Z}/60 \cong \mathbb{Z}/5 \times \mathbb{Z}/12 \cong \mathbb{Z}/5 \times \mathbb{Z}/4 \times \mathbb{Z}/3.$$

In contrast, we have already seen that

$$\mathbb{Z}/4 \ncong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

7.3.1. *History.* The Chinese Remainder Theorem appeared in ancient times in the following form: if $m_1, \ldots, m_n$ are pairwise relatively prime integers, and $a_1, \ldots, a_n$ are any integers, then there is an integer d such that $d \equiv a_i (\mathrm{mod}\, m_i)$ for all i. This statement appears in the manuscript *Mathematical Treatise in Nine Sections* written by Chin Chiu Shao in 1247 (search on the web if you want to know more).

7.4. How to recognize a product. If we have a group G and can then show it is a product of two smaller groups we have made progress: generally speaking smaller groups are easier to understand than large ones so we can understand G by understanding the two factors in its expression as a product. The next result gives a criterion for recognizing when a group can be written as a product of two of its subgroups.

THEOREM 2.33. *Let H and K be subgroups of a group G. Suppose that $hk = kh$ for all $h \in H$ and $k \in K$. If $HK = G$ and $H \cap K = \{e\}$, then*

$$G \cong H \times K.$$

The map $f : H \times K \to G$ given by $f(h,k) := hk$ is an isomorphism

Proof. Since $HK = G$, f is surjective. Furthermore, if $x = (h,k)$ and $y = (h',k')$, then

$$\begin{aligned} f(xy) &= f\big((h,k)\cdot(h',k')\big) \\ &= f(hh',kk') \\ &= hh'kk' \\ &= hkh'k' \\ &= f(h,k)f(h',k') \\ &= f(x)f(y). \end{aligned}$$

We will now use Lemma 2.9 to show that f is injective. If $f(h,k) = e$, then $hk = e$ so $h = k^{-1}$ and therefore $h \in H \cap K$. But $H \cap K = \{e\}$ so $h = e$ and $k = e$. Hence $f^{-1}(e) = (e,e)$ and Lemma 2.9 implies f is injective. □

In the situation of the Theorem 2.33 we usually write

$$G = H \times K.$$

This equality means, among other things, that every element in G can be written in a unique way as a product hk, i.e., given $g \in G$, there is a unique h in H and a unique k in K such that $g = hk$.

7.4.1. *The abelian case.* If H and K are subgroups of an abelian group $(G,+)$ such that $H + K = G$ and $H \cap K = \{0\}$, then Theorem 2.33 tells us that $G = H \times K$. We will make use of this in the proof of Theorem 2.36 in the next section.

8. Finite abelian groups

In the section we classify all finite abelian groups. Our treatment follows J.S. Milne's notes at `http://www.jmilne.org/math/CourseNotes/GT.pdf`.

First, because all the groups in this section are abelian we will use the symbol + for the group operation in each.

In the next section we introduce the idea of a basis for a finite abelian group. The analogy with the idea of a basis for a vector space should help you. First observe that $\mathbb{R}^n$ is an abelian group under the usual addition of vectors. Although the only element of finite order in $\mathbb{R}^n$ is the zero vector, i.e., the identity don't let that worry you. Recall that a basis for $\mathbb{R}^n$ is a subset $\{\underline{v}_i \mid i \in I\}$ that spans it and is linearly independent. The spanning condition means that every element in $\mathbb{R}^n$ can be written as a sum $\sum_i \lambda_i \underline{v}_i$ for some λ_is in $\mathbb{R}$, and the linear independence means that if $\sum_i \lambda_i \underline{v}_i = 0$, then all λ_i are zero. Of course a basis for $\mathbb{R}^n$ must consist of exactly n elements and leads to an isomorphism

$$\mathbb{R}^n = \mathbb{R}\underline{v}_1 \times \cdots \times \mathbb{R}\underline{v}_n.$$

8.1. The notion of a basis for a finite abelian group. A basis for an abelian group G is a subset $\{x_1, \ldots, x_n\}$ such that

$$G = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle.$$

LEMMA 2.34. *A set $\{x_1, \ldots, x_n\}$ is a basis for G if and only if*

(1) *every element in G can be written as $a_1x_1 + \cdots + a_nx_n$ for some $a_1, \ldots, a_n \in \mathbb{Z}$ and*
(2) *$a_1x_1 + \cdots + a_nx_n = 0$ implies $a_1x_1 = a_2x_2 = \cdots = a_nx_n = 0$.*

Proof. Exercise. □

LEMMA 2.35 (Milne). *Suppose G is generated by $\{x_1, \ldots, x_n\}$. Suppose $a_1, \ldots, a_n$ are integers whose greatest common divisor is 1. Then G is generated by a set $\{y_1, \ldots, y_n\}$ where $y_1 = a_1x_1 + \cdots + a_nx_n$.*

Proof. If some $a_i < 0$ we replace a_i by $-a_i$ and x_i by $-x_i$. This allows us to assume that all a_is are positive. We argue by induction on $s = a_1 + \cdots + a_n$. If $s = 1$, the hypotheses imply G is generated by x_1 and $y_1 = x_1$ so the result is obviously true.

Now suppose $s > 1$. Then at least two a_i are positive, say $a_1 \geq a_2 > 0$. It is clear that

$$G = \langle x_1, \ldots, x_n \rangle = \langle x_1, x_2 + x_1, x_3, \ldots, x_n \rangle$$

and $\gcd\{a_1 - a_2, a_2, a_3, \ldots, a_n\} = 1$. But the sum of these numbers is less than s so the induction hypothesis implies that G is generated by a set $\{y_1, \ldots, y_n\}$ where

$$\begin{aligned} y_1 &= (a_1 - a_2)x_1 + a_2(x_2 + x_1) + a_3x_3 + \cdots + a_nx_n \\ &= a_1x_1 + \cdots + a_nx_n. \end{aligned}$$

This completes the proof. □

THEOREM 2.36. *Every finite abelian group has a basis. In other words, if G is a finite abelian group, there are positive integers $n_1, \ldots, n_k$ such that*

$$G \cong (\mathbb{Z}/n_1) \times \cdots \times (\mathbb{Z}/n_k).$$

Proof. [Milne] Let G be a finite abelian group. We will argue by induction on the number of generators of G. If G has a single generator it is cyclic and therefore has a basis. We therefore assume that G needs $n > 1$ generators. We pick a generating set $\{x_1, \ldots, x_n\}$ with x_1 having minimal order. We will show that

$$\langle x_1 \rangle \cap \langle x_2, \ldots, x_n \rangle = 0$$

which, by the remarks in section 7.4.1, implies that $G = \langle x_1 \rangle \times \langle x_2, \ldots, x_n \rangle$, so proving the theorem.

Suppose the intersection is not zero. Then there are integers $a_1, \ldots, a_n$ such that

$$a_1x_1 + \cdots + a_nx_n = 0$$

and $a_1 x_1 \neq 0$. Of course, we can assume that a_1 is strictly smaller than the order of x_1. Let $d = \gcd\{a_1, \ldots, a_n\}$ and define $b_i := a_i/d_i$. Then $\gcd\{b_1, \ldots, b_n\} = 1$ so the lemma implies that $G = \langle y_1, y_2, \ldots, y_n \rangle$ with $y_1 = b_1 x_1 + \cdots + b_n x_n$. But

$$dy_1 = a_1 x_1 + \cdots + a_n x_n = 0$$

and d divides a_1 so is $\leq a_1 < \mathsf{order}(x_1)$. This contradicts the choice of x_1. □

The integers $n_1, \ldots, n_k$ are not uniquely determined by G. For example, the Chinese Remainder Theorem tells us that

$$\mathbb{Z}/mn \cong (\mathbb{Z}/m) \times (\mathbb{Z}/n)$$

if $\gcd(m, n) = 1$. However, using the Chinese Remainder Theorem and factoring the n_is and then using the Chinese remainder theorem to multiply the factors together in appropriate ways we can show there are prime numbers $p_1, \ldots, p_m$, possibly with repetitions, and integers $r_1, \ldots, r_m$ such that

$$G \cong \frac{\mathbb{Z}}{p_1^{r_1}} \times \cdots \times \frac{\mathbb{Z}}{p_m^{r_m}}$$

and the primes and the r_is are uniquely determined by G. We can also use the Chinese Chinese Remainder Theorem to deduce that

$$G \cong \frac{\mathbb{Z}}{a_1} \times \cdots \times \frac{\mathbb{Z}}{a_\ell}$$

where a_1 divides a_2, a_2 divides a_3, and so on and so on. The a_is are uniquely determined by G.

For example, if

$$G = \frac{\mathbb{Z}}{18} \times \frac{\mathbb{Z}}{40} \times \frac{\mathbb{Z}}{8} \times \frac{\mathbb{Z}}{48},$$

then

$$G \cong \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{8} \times \frac{\mathbb{Z}}{8} \times \frac{\mathbb{Z}}{16} \times \frac{\mathbb{Z}}{3} \times \frac{\mathbb{Z}}{9} \times \frac{\mathbb{Z}}{5}$$

and

$$G \cong \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{8} \times \frac{\mathbb{Z}}{24} \times \frac{\mathbb{Z}}{720}.$$

There are lots of other ways of writing this group. For example, using the Chinese Remainder Theorem we could remove the factor of 5 from 720 and then replace 24 by $24 \times 5 = 120$ to write

$$G \cong \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{8} \times \frac{\mathbb{Z}}{120} \times \frac{\mathbb{Z}}{144}.$$

Endless fun.