MATH 402 FINAL FALL 2005

I will write Z to denote the center of a group, or $Z(H)$ if I have to specify that it is the center of H.

I will write $\mathbb{R}^\times$ and $\mathbb{C}^\times$ to denote the groups of non-zero real and complex numbers under multiplication.

1. True/False

You get 2 points for each correct answer and -2 for each incorrect one. Just write T or F as your answer. My advice is to answer the questions you are sure of first, then do the rest of the exam and finally return to look at those you are not so sure about.

(1) Let G be the group of upper triangular 2×2 matrices and N the subgroup of matrices having determinant one. Then $G/N \cong \mathbb{R}^\times$.
(2) The subgroup of S_8 generated by (31) and (53) and (65) is isomorphic to S_4.
(3) The subgroup of S_8 generated by (25) and (325) and (56) isomorphic to S_4.
(4) The subgroup of S_8 generated by (14) and (15) and (26) is isomorphic to $S_3 \times \mathbb{Z}_2$.
(5) S_6 has cyclic subgroups of orders 8 and 9.
(6) S_6 has abelian subgroups of orders 8 and 9.
(7) The subgroup of S_8 generated by (123456) and (14)(25)(36) is isomorphic to $\mathbb{Z}_6$.
(8) G is abelian if $x^2 = 1$ for all $x \in G$.
(9) There is a surjective homomorphism $S_3 \to \mathbb{Z}_3$.
(10) There is an injective homomorphism $D_7 \to S_7$.
(11) There is a surjective homomorphism $S_7 \to D_7$.
(12) If x is conjugate to y, then x^2 is conjugate to y^2.
(13) If x is conjugate to y, then x^{-1} is conjugate to y^{-1}.
(14) The odd integers together with 0 form a subgroup of $\mathbb{Z}$.
(15) The subgroup of $(\mathbb{Q}, +)$ generated by $\frac{1}{3}$ and $\frac{1}{5}$ contains $\frac{1}{15}$.
(16) The subgroup of $(\mathbb{Q}, +)$ generated by $\frac{1}{3}$ and $\frac{1}{5}$ is generated by $\frac{1}{15}$.
(17) The subgroup of $(\mathbb{Q}, +)$ generated by $\frac{1}{2}$ and $\frac{1}{4}$ is isomorphic to $\mathbb{Z}$.
(18) Every subgroup of $\mathbb{Z}$ except the trivial subgroup is isomorphic to $\mathbb{Z}$.
(19) If $\phi \in \mathrm{Aut}(\mathbb{Z}_{10})$ is defined by $\phi(\bar{1}) = (\bar{7})$, then $\phi^2(\bar{x}) = -\bar{x}$ for all $\bar{x} \in \mathbb{Z}_{10}$.
(20) The order of $\phi \in \mathrm{Aut}(\mathbb{Z}_{10})$ defined by $\phi(\bar{1}) = (\bar{7})$ is 4.
(21) There is an element $\phi \in \mathrm{Aut}(\mathbb{Z}_{10})$ such that $\phi(\bar{1}) = \bar{2}$.
(22) Every group with n elements is isomorphic to a subgroup of S_n.
(23) The automorphism group of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ contains an element of order 2.
(24) The automorphism group of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ contains an element of order 3.
(25) There is an injective homomorphism $\phi : S_3 \to \mathrm{Aut}(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4)$.
(26) There is a non-abelian group having 47 elements.

(27) There is a non-abelian group having 49 elements.
(28) There is an injective homomorphism $\mathbb{Z} \to \mathbb{C}^\times$.
(29) There is an injective homomorphism $\mathbb{Z}_8 \to \mathbb{C}^\times$.
(30) There is an injective homomorphism $\mathbb{Z}_3 \times \mathbb{Z} \to \mathbb{C}^\times$.
(31) The automorphism group of $\mathbb{Z}_{17}$ has 16 elements.
(32) For every integer $n \geq 2$ there is an automorphism of $\mathbb{Z}_n$ having order 2.
(33) Let G act on the set X. If $a, b \in G$ move different numbers of elements of X they are not conjugate.
(34) Let G act on the set X. If $a, b \in G$ move the same numbers of elements of X they are conjugate.
(35) An abelian group has no inner automorphisms.
(36) There is an automorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $\phi(3) = -3$.
(37) The elements i and j are conjugate elements in the group of quaternions.
(38) The elements (316) and (241) are conjugate in S_8.
(39) The elements (12)(34) and (13)(24) are conjugate in S_5.
(40) For every group G, the function $\phi : G \to G$ defined by $\phi(x) = x^{-1}$ is a homomorphism.
(41) Let Z denote the center of a finite group G. Then $\alpha(Z) = Z$ for all $\alpha \in \mathrm{Aut}(G)$.
(42) There is a homomorphism $f : S_3 \times \mathbb{Z}_2 \to S_3 \times \mathbb{Z}_2$ such that $f(Z) \not\subset Z$.
(43) If G has a unique subgroup isomorphic to S_5, say H, then $\alpha(H) = H$ for all $\alpha \in \mathrm{Aut}(G)$.
(44) The subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of all matrices whose only non-zero entries lie on the diagonal is normal.
(45) Let $A \subset (\mathbb{Q}, +)$ be the set of elements that can be written as $n/5$ where n is an integer. Let B be the subgroup of $\mathbb{Z}$ consisting of the even integers. The groups A and B are not isomorphic.
(46) In S_6, (1234)(1352)(45) = (23514).
(47) In S_6, (163)(1354)(64) = (543).
(48) There is a group of order 63 that does not have an element of order 9.
(49) There is a group of order 63 that has an element of order 9.
(50) Let $\alpha : H \to \mathrm{Aut}\, N$ be a group homomorphism and $G := N \rtimes_\alpha H$ the semi-direct product. There is always a surjective homomorphism $\psi : G \to N$.

2. Complete the sentence

Complete the following sentences. DO NOT waste time by rewriting the part of the sentence that I have already written. Just write the rest of it.

Each question is worth 3 points.

(1) A homomorphism $f : G \to H$ is a function such that ....
(2) The subgroup of G generated by x, y, z is defined to be ...
(3) To show that an automorphism ϕ of a group G is inner one must show that...
(4) To show that elements $x, y \in G$ are NOT conjugate one must show that ...
(5) G is isomorphic to a product of two groups if it has subgroups A and B such that ...
(6) There is a surjective homomorphism $\Phi : G \to \mathrm{Inn}(G)$, the group of inner automorphisms, defined by ...
(7) To show that a subgroup H of G is not normal one must show there are elements .... such that ....
(8) Let $P(X)$ denote the group of all permutations of a set X. If $\phi : G \to P(X)$ is a group homomorphism we can define an action of G on X by $g.x =$ ...
(9) The action of a group G on a set X is transitive if ....
(10) To show that the action of a group G on a set X is not transitive one must show there are elements ... such that ....
(11) The quaternion group Q is not isomorphic to D_4 because ....
(12) The index of a subgroup H in G is defined to be ....
(13) To show that an element $g \in G$ is not in the center of G one must show there is an element .... such that ...
(14) If G acts on X and X is a single G-orbit, then there is a subgroup H in G such that the action of G on X is isomorphic to the action of G on ....; in fact, $H =$...
(15) If $xy \in Z(G)$, then $xy = yx$ because ...
(16) Let G be a non-abelian group with 22 elements and $x \in G$ an element such that $x^2 \neq 1$. The subgroup $\langle x \rangle$ is normal because ...
(17) In Question 16, If $a \notin \langle x \rangle$, then $a^2 \in \langle x \rangle$ because ...
(18) In Question 16, $a^2 x = x a^2$ because ....
(19) In Question 16, $axa^{-1} = x^i$ for some integer i because ...
(20) In Question 16, $axa^{-1} = x^{-1}$ because ...
(21) In Question 16, $G \cong D_{11}$, the dihedral group of order 22 because ....
(22) Let A and B be subgroups of a group G. Suppose that B is normal and that $AB = G$. Then there is an isomorphism between the groups $A/A \cap B$ and G/B given by the function ....
(23) The Class Equation for a finite group G says that ...
(24) If $\sigma = (2613)(4256)(25) \in S_6$ then $\sigma(2) =$ ... and $\sigma^{-1} =$ ...
(25) The subgroup of S_7 generated by (153) and (726) is isomorphic to ...
(26) Let $U \subset \mathbb{C}^\times$ be the subgroup consisting of the complex numbers of absolute value one. There is an isomorphism $\phi : U \times \mathbb{R}^\times \to \mathbb{C}^\times$ given by $\phi(u, r) =$...
(27) There is an injective homomorphism $\phi : H \to G \times H$ given by $\phi(x) =$ ...

(28) The smallest non-abelian group is ...
(29) The groups of order 8 up to isomorphism are ...
(30) The set $\mathbb{Z}$ with the "product" $a * b = a + b - 1$ is a group. Its identity element is ...
(31) The set $\mathbb{Z}$ with the "product" $a * b = a + b - 1$ is a group. The inverse of 5 is ...
(32) An element in the group $(\mathbb{Q}/\mathbb{Z}, +)$ having order 48 is ...
(33) If H is a subgroup of G and every left coset of H is a right coset, then ...
(34) If $\phi : G \to H$ is a homomorphism with $N = \ker \phi$ and $xN = yN$, then ...
(35) If $\phi : G \to H$ is a homomorphism with $N = \ker \phi$ and $\phi(x) = \phi(y)$, then ...
(36) There is an automorphism of the cyclic group $\mathbb{Z}_8$ that is not the identity map, namely ...
(37) If $x \in G$ has order $6n$, then G has a subgroup with $2n$ elements, namely ...
(38) There is a group G such that $G/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, namely $G = ...$
(39) There is an injective homomorphism ϕ from the dihedral group

$$D_n := \langle a, b \mid a^2 = b^n = 1,\ aba^{-1} = b^{-1} \rangle$$

to the dihedral group

$$D_{mn} := \langle x, y \mid x^2 = y^{mn} = 1,\ xyx^{-1} = y^{-1} \rangle$$

given by ...
(40) There is a transitive action of the group of quaternions Q on a set with four elements, namely the action of Q on ...