Non-commutative Algebraic Geometry

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0 Introduction

This is a reasonably faithful account of the five lectures I delivered at the summer course "Geometria Algebraica no Commutativa y Espacios Cuanticos" for graduate students, in Spain, July 25–29, 1994. The material covered was, for the most part, an abridged version of Artin and Zhang's paper [2].

Fix a field k. Given a $\mathbb{Z}$-graded k-algebra, A say, which for simplicity is assumed to be left noetherian and locally finite dimensional, its non-commutative projective scheme is defined to be the pair

$$\operatorname{proj}(A) := (\operatorname{tails}(A), \mathcal{A}),$$

where $\operatorname{tails}(A)$ is the quotient category of $\operatorname{grmod}(A)$, the category of finitely generated graded left A-modules, modulo its full subcategory of finite dimensional modules, and $\mathcal{A}$ is the image of the distinguished module ${}_AA$ in $\operatorname{tails}(A)$. If A is a quotient of a commutative polynomial ring generated in degree 1, Serre [4] proved that $\operatorname{proj}(A)$ is isomorphic (in an obvious sense) to the pair $(\operatorname{Coh}(\mathcal{O}_X), \mathcal{O}_X)$, where X denotes the projective scheme determined by A, $\mathcal{O}_X$ is the sheaf of regular functions on X, and $\operatorname{Coh}(\mathcal{O}_X)$ is the category of coherent $\mathcal{O}_X$-modules. Thus $\operatorname{tails}(A)$ is the non-commutative analogue of $\operatorname{Coh}(\mathcal{O}_X)$, and the objects in $\operatorname{tails}(A)$ are the non-commutative geometric objects analogous to sheaves of $\mathcal{O}_X$-modules.

For each $\mathcal{F} \in \operatorname{tails}(A)$ there are cohomology groups $H^q(\mathcal{F})$, $q \geq 0$, which generalize the Čech cohomology groups—if A is commutative as above, then $H^q(\mathcal{F})$ coincides with $H^q(X, \mathcal{F})$ for $\mathcal{F} \in \operatorname{Coh}(\mathcal{O}_X)$. The functors $H^q(-)$ have the properties one would want/expect for a satisfactory generalization of $H^q(X,-)$. In particular, there is a version of Serre's Finiteness Theorem (4.16) provided a certain technical condition χ holds (see Definition 4.11). Every commutative algebra satisfies χ, but there exist rather nice non-commutative algebras which do not (Example 4.15). We compute the cohomology groups $H^q(\mathcal{A}[d])$, $d \in \mathbb{Z}$, when A is an Artin-Schelter regular algebra. This family of algebras includes the commutative polynomial ring, and in that case $H^q(\mathcal{A}[d]) \cong H^q(\mathbb{P}^n, \mathcal{O}(d))$. The Artin-Schelter regular algebras are non-commutative algebras which enjoy many of the properties of polynomial rings; amongst the non-commutative Artin-Schelter regular algebras are most graded iterated Ore extensions, homogenizations of enveloping algebras, and Sklyanin algebras. Artin-Schelter regular algebras always satisfy the condition χ.

The functorial behavior of tails(A), and maps between proj(A) and proj(B) are discussed in Section 5.

The polarized projective scheme associated to A is the triple (tails$(A), \mathcal{A}$, [1]), where [1] is the degree shift functor on grmod(M), namely $M[1]_i = M_{i+1}$. Including [1] with proj(A) is analogous to specifying a line bundle on a scheme X; it is natural to ask whether that line bundle is very ample, i.e., whether it determines an embedding of X in some projective space (or, equivalently, whether it arises from an embedding of X in some $\mathbb{P}^n$). This leads to the notion of ampleness for [1] on proj(A) (see Definition 5.18). Whether or not [1] is ample in proj(A) is closely related to the condition χ.

Polarized projective schemes are objects in a category of triples $(\mathcal{C}, \mathcal{O}, s)$ where $\mathcal{C}$ is a k-linear category, $\mathcal{O}$ a distinguished object in $\mathcal{C}$, and s is an auto-equivalence of $\mathcal{C}$. The notion of ampleness is defined in this larger context and plays a key role in whether a given triple $(\mathcal{C}, \mathcal{O}, s)$ arises from a graded algebra A. Indeed, if s is ample, and $(\mathcal{C}, \mathcal{O}, s)$ satisfies some finiteness conditions, then $(\mathcal{C}, \mathcal{O}, s) \cong (\text{tails}(A), \mathcal{A}, [1])$ for some left noetherian, locally finite, $\mathbb{N}$-graded algebra A which satisfies χ_1. This result gives some idea of the scope of non-commutative algebraic geometry because it says (roughly) which $\mathcal{C}$ can be non-commutative schemes. The result may be used to show that some non-commutative algebras behave as if they are commutative from the point of view of tails( ); for example, if A is a twisted homogeneous coordinate ring (see Example 5.16), usually written $A = B(X, \sigma, \mathcal{L})$, where X is a projective scheme, $\sigma \in \text{Aut}(X)$ and $\mathcal{L}$ is a σ-ample line bundle on X, then $(\text{tails}(A), \mathcal{A}, [1]) \cong (\text{Coh}(\mathcal{O}_X), \mathcal{O}_X, s)$ for a suitable s (the hypothesis that $\mathcal{L}$ is σ-ample guarantees that s is ample). In particular, tails(A) is equivalent to

$\mathrm{Coh}(\mathcal{O}_X)$, which allows A to be studied via the methods of algebraic geometry. The utility of this result arises because twisted homogeneous coordinate rings turn up rather often in the theory of non-commutative graded algebras.

1 Graded Algebras and Modules

In all that follows,

- k is a field, and
- A is a $\mathbb{Z}$-graded k-algebra; that is $A = \bigoplus_{n \in \mathbb{Z}} A_n$ and $A_m A_n \subset A_{m+n}$.

An A-module, M say, is *graded* if it has a vector space decomposition $M = \bigoplus_{n \in \mathbb{Z}} M_n$ such that

$$A_i M_j \subset M_{i+j}$$

for all i and j. Elements in M_j are *homogeneous* of *degree* j, and M_j is the degree j *homogeneous component* of M. The graded A-modules are the objects in the category $\mathrm{GrMod}(A)$. The full subcategory of $\mathrm{GrMod}(A)$ consisting of finitely generated modules is denoted by $\mathrm{grmod}(A)$. The morphisms in these categories, denoted $\mathrm{Hom}_{\mathrm{Gr}}(N, M)$, are the A-module maps $f : N \to M$ such that $f(N_i) \subset M_i$ for all i. More generally, if $f : N \to M$ satisfies $f(N_i) \subset M_{i+d}$ for all $i \in \mathbb{Z}$, we say that the *degree* of f is d.

We need to consider several other Hom spaces:

- $\mathrm{Hom}_A(N, M) := \{\text{all } A\text{-module homomorphisms } f : N \to M\}$;
- $\mathrm{Hom}_A(N, M)_d := \{f \in \mathrm{Hom}_A(N, M) \mid \deg(f) = d\}$
- $\underline{\mathrm{Hom}}_A(N, M) := \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_A(N, M)_d$.

Lemma 1.1 *If N is finitely generated, then*

$$\underline{\mathrm{Hom}}_A(N, M) = \mathrm{Hom}_A(N, M).$$

Example 1.2 *Let V be a graded vector space such that $\dim_k V_n \geq 1$ for all $n \in \mathbb{Z}$. If $f : V \to k$ is such that $f(V_n) \neq 0$ for infinitely many n, then $f \notin \underline{\mathrm{Hom}}_k(V, k)$. Thus $\underline{\mathrm{Hom}}_k(V, k) \neq \mathrm{Hom}_k(V, k)$.*

The field k itself is a graded algebra concentrated in degree zero. If V is a graded vector space, the graded dual of V is

$$V^* := \underline{\mathrm{Hom}}_k(V, k).$$

Thus $V^*_{-d} = \mathrm{Hom}_k(V_d, k)$. A graded vector space, V say, is *locally finite* if $\dim_k V_n < \infty$ for all n. A graded k-algebra generated by a finite number of elements of positive degree is locally finite. Finitely generated modules over a locally finite algebra are locally finite.

We use the notation

$$M_{\geq n} = \bigoplus_{d \geq n} M_d \qquad \text{and} \qquad M_{\leq n} = \bigoplus_{d \leq n} M_d.$$

We say that M is *left* (respectively, *right*) *bounded* if $M_{\leq n} = 0$ (respectively, $M_{\geq n} = 0$) for some n. From Section 2 onwards our attention is restricted to $\mathbb{N}$-graded algebras. Such an algebra, A say, is left bounded, and so are its finitely generated modules. Further, if $M \in \mathrm{GrMod}(A)$, so is $M_{\geq n}$.

$\mathrm{GrMod}(A)$ is an abelian category and one's intuition from the category of ungraded modules carries over. A small difference is that A is rarely a generator in $\mathrm{GrMod}(A)$; for example $\mathrm{Hom}_{\mathrm{Gr}}(A, M_{\geq 1}) = 0$ for all M. This minor irritation is alleviated by introducing the *shift functor* $[1] : \mathrm{GrMod}(A) \to \mathrm{GrMod}(A)$ defined as follows: as an A-module $M[1]$ equals M, but the grading is now $M[1]_n = M_{n+1}$. The pair $(A, [1])$ now acts somewhat like a generator; more precisely $P = \bigoplus_{n \in \mathbb{Z}} A[n]$ is a generator. It is an easy but worthwhile exercise to check that $\underline{\mathrm{Hom}}_A(N[i], M[j]) \cong \underline{\mathrm{Hom}}_A(N, M)[j-i]$ as graded vector spaces.

The algebra A is *connected* if it is $\mathbb{N}$-graded and $A_0 = k$. In this case there is a distinguished A-module, namely $A/A_{\geq 0}$; it is the only irreducible object in $\mathrm{GrMod}(A)$, and is called the *trivial* module. For connected algebras there is a useful analogue of Nakayama's Lemma.

Lemma 1.3 *Let A be connected. If $M \in \mathrm{GrMod}(A)$ is left bounded, then $M = 0$ if and only if $k \otimes_A M = 0$.*

Proof. Suppose that $M \neq 0$. Since M is bounded below, we can choose $0 \neq m \in M$, homogeneous of minimal degree. Such m cannot belong to

$A_{\geq 1}M$. This is absurd, since $k \otimes_A M = 0$ implies that $A_{\geq 1}M = M$, so we conclude that $M = 0$. ■

Since $\mathrm{Hom}_A(A, -)$ is exact, so is $\mathrm{Hom}_{\mathrm{Gr}}(A[n], -)$ for all $n \in \mathbb{Z}$. Thus $A[n]$ is projective in $\mathrm{GrMod}(A)$, whence $\mathrm{GrMod}(A)$ has enough projectives. A module M is *free* if it is a direct sum of shifts of A.

Lemma 1.4 *Let A be connected, and $M \in \mathrm{GrMod}(A)$. If M is bounded below, then*

1. *M is free if and only if $\mathrm{Tor}_1^A(k, M) = 0$*
2. *M is projective if and only if M is free.*

Proof. (1) ($\Leftarrow$) Choose a graded vector space V such that $V \oplus A_{\geq 1}M = M$. Then $k \otimes_A (M/AV) = 0$ so, by Nakayama's Lemma $M = AV$. Let $\psi : A \otimes_k V \to M$ be the multiplication map. Since $\mathrm{Tor}_1^A(k, M) = 0$, there is an exact sequence

$$0 \to k \otimes_A \ker(\psi) \to k \otimes_A A \otimes_k V \xrightarrow{1 \otimes \psi} k \otimes_A M \to 0.$$

Since $1 \otimes \psi$ is an isomorphism, $k \otimes_A \ker(\psi) = 0$. But $\ker(\psi)$ is bounded below so, by Nakayama's Lemma, ψ is an isomorphism.

(2) This follows immediately from (1). ■

The existence of injectives in $\mathrm{GrMod}(A)$ is more complicated than the existence of projectives, but we have the following positive result.

Proposition 1.5 $\mathrm{GrMod}(A)$ *has enough injectives.*

As in the category of ungraded modules, $E \in \mathrm{GrMod}(A)$ is injective if and only if it has no essential extensions. There is an obvious notion of the injective envelope of a module, and it may be characterized as the largest essential extension. Hence we have injective resolutions. If $0 \to M \to E^0 \xrightarrow{d} E^1 \to \cdots$ is an injective resolution, we say it is *minimal* if E^j is the injective envelope of dE^{j-1} for all $j \geq 0$.

For each $q \geq 0$ we may define $\mathrm{Ext}^q_{\mathrm{Gr}}(N, -)$ as the right derived functors of $\mathrm{Hom}_{\mathrm{Gr}}(N, -)$, and compute these by taking injective resolutions in the usual way; $\mathrm{Ext}^q_{\mathrm{Gr}}(N, M)$ can also be computed by taking projective resolutions of N. We will use the following notation:

$$\begin{aligned} \mathrm{Ext}^q_A(N,M) &= \text{the usual Ext groups in } \mathrm{Mod}(A), \\ \mathrm{Ext}^q_A(N,M)_d &= \text{the derived functors of } \mathrm{Hom}_A(N,-)_d, \\ \underline{\mathrm{Ext}}^q_A(N,M) &= \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^q_A(N,M)_n. \end{aligned}$$

Example 1.6 *If A is connected, the injective envelope of ${}_Ak = A/A_{\geq 1}$ is $A^* = \underline{\mathrm{Hom}}_k(A,k)$ with left action of $x \in A$ given by $(x.\lambda)(a) = \lambda(ax)$ for $\lambda \in A^*$. The copy of k inside A^* is $k\epsilon$, where $\epsilon : A \to k$ is the projection with kernel $A_{\geq 1}$. It is easy to see that A^* is an essential extension of $k\epsilon$.*

The injectivity of A^* follows from the projectivity of A_A as follows. If $f : N \to M$ is injective and $\alpha : N \to A^*$ are maps in GrMod(A)*, then $\beta : M \to A^*$ is defined by*

$$\beta(m)(a) = \theta(a)(m),$$

where $\theta : A \to M^*$ is a right A-module map such that $f^* \circ \theta = \alpha^*$, and $\alpha^* : A \to N^*$, $f^* : M^* \to N^*$ are the maps dual to α and f. It is easy to check that β is a left A-module map satisfying $\beta \circ f = \alpha$, showing that A^* is injective.

If A, B, C are graded algebras, and ${}_AM_B$ and ${}_AN_C$ are graded bimodules, then $\underline{\mathrm{Ext}}^q_A(N,M)$ is a graded C-B-bimodule.

The Ext-groups inherit good properties from their second argument.

Proposition 1.7 *Let A be left noetherian, and $\mathbb{N}$-graded. If $N \in$* grmod(A) *and $M \in$* GrMod(A)*, then*

1. *if M is left (or right) bounded, so is $\underline{\mathrm{Ext}}^q_A(N,M)$;*
2. *if M is locally finite, so is $\underline{\mathrm{Ext}}^q_A(N,M)$;*
3. *if M is a graded A-B bimodule, where B is a right noetherian graded algebra, and $M \in$* grmod(B)*, then $\underline{\mathrm{Ext}}^q_A(N,M) \in$* grmod$(B)$ *too.*

Proof. Take a projective resolution for N, each term of which is a finite direct sum of shifts of A. Apply $\underline{\mathrm{Hom}}_A(-,M)$ to get a complex in which each term is a finite direct sum of shifts of M. Each $\underline{\mathrm{Ext}}^q_A(N,M)$ is a subquotient of these terms, so inherits the relevant property from M. ■

2 Torsion

From now on A is a locally finite, left noetherian, $\mathbb{N}$-graded algebra over a field k. Hence each $M \in \mathrm{grmod}(A)$ is left bounded and locally finite.

Definition 2.1 The torsion submodule *of* $M \in \mathrm{GrMod}(A)$ *is*

$$\tau M := \textit{the sum of all finite dimensional submodules of } M.$$

We say that M *is* torsion *(respectively,* torsion-free*) if* $\tau M = M$ *(respectively,* $\tau M = 0$*). We define* $\mathrm{Tors}(A)$ *(respectively,* $\mathrm{tors}(A)$*) to be the full subcategory of* $\mathrm{GrMod}(A)$ *(respectively,* $\mathrm{grmod}(A)$*) consisting of the torsion modules.*

It follows from the definition that $M/\tau M$ is torsion-free.

A module $M \in \mathrm{grmod}(A)$ is torsion if and only if $\dim_k M < \infty$ (since M is noetherian, an ascending sum of finite-dimensional submodules stabilizes after finitely many terms). Thus

$$\mathrm{tors}(A) = \{\text{finite dimensional modules}\}.$$

A useful reformulation of this is that $\tau M = \varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, M)$.

Proposition 2.2 $\mathrm{Tors}(A)$ *and* $\mathrm{tors}(A)$ *are dense subcategories of* $\mathrm{GrMod}(A)$*; that is, if* $0 \to L \to M \to N \to 0$ *is exact in* $\mathrm{GrMod}(A)$*, then* M *is torsion if and only if* L *and* N *are.*

Proof. ($\Rightarrow$) Suppose M is a sum of finite dimensional modules. Then N is the sum of their images, so is torsion. Also, each $m \in M$ belongs to a finite sum of finite dimensional modules, so $\dim_k(Am) < \infty$, whence every submodule of M is a sum of finite dimensional modules, so is torsion.

($\Leftarrow$) Suppose L and N are torsion. For $m \in M$, we have an exact sequence

$$0 \to Am \cap L \to Am \to Am/Am \cap L \to 0.$$

By the first part of the proof, $Am \cap L$ is torsion since L is, and so is $Am/Am \cap L$ since it is isomorphic to $Am + L/L$, which is a submodule of N. But $Am \cap L$ and $Am/Am \cap L$ are noetherian, since A is, whence they are finite dimensional. Thus $\dim_k(Am) < \infty$ also. Hence M is a sum of finite dimensional modules, as required. ■

The relation between injective envelopes and torsion is described by the next result.

Lemma 2.3 *An essential extension of a torsion (respectively, torsion-free) module is torsion (respectively, torsion-free).*

Proof. Let $M \subset E$ be an essential extension. If $\tau E \neq 0$, then $\tau E \cap M = 0$, so $\tau M \neq 0$. Thus M torsion-free implies E is too. Conversely, suppose that $M = \tau M$. Let $e \in E$. Then $Ae \cap M$ is torsion, hence finite dimensional since A is noetherian. Thus $A_{\geq n} e \cap M = 0$ for $n \gg 0$, whence $\dim_k(Ae) < \infty$, since A is locally finite. Thus E is a sum of finite dimensional modules, hence torsion. ■

3 Tails

Since $\text{Tors}(A)$ and $\text{tors}(A)$ are dense, there are quotient categories

$$\begin{aligned} \text{Tails}(A) &:= \text{GrMod}(A)/\,\text{Tors}(A) \\ \text{tails}(A) &:= \text{grmod}(A)/\,\text{tors}(A). \end{aligned}$$

We write

$$\pi : \text{GrMod}(A) \to \text{Tails}(A)$$

for the quotient functor, and

$$\mathcal{A} = \pi A.$$

The objects in a quotient category are the same as those in the original category so, to avoid confusion we will write πM for the image of $M \in \text{GrMod}(A)$ in $\text{Proj}(A)$.

The basic properties of quotient categories may be found in [3].

Theorem 3.1 *(Serre) If A is a quotient of the polynomial ring $k[X_0, \ldots, X_n]$ with $\deg(X_i) = 1$ for all i, there is an equivalence of categories*

$$\text{tails}(A) \simeq \text{Coh}(\mathcal{O}_X),$$

the category of coherent $\mathcal{O}_X$-modules, where $X \subseteq \mathbb{P}^n$ is the closed subscheme cut out by the ideal defining A, and $\mathcal{O}_X$ is the sheaf of regular functions on X.

Thus the objects in tails(A) are the non-commutative analogues of sheaves of $\mathcal{O}_X$-modules—they are the objects of non-commutative geometry, and the category tails(A) is the main object of study in non-commutative geometry. To reinforce the analogy with sheaves of $\mathcal{O}_X$-modules, we will use script letters to denote objects in Tails(A).

Since [1] sends Tors(A) to Tors(A), the functor [1] passes to Tails(A). Under Serre's equivalence of categories we have the correspondence

$$\begin{array}{rcl} \mathcal{A} & \leftrightarrow & \mathcal{O}_X \\ \mathcal{A}[d] & \leftrightarrow & \mathcal{O}_X(d), \end{array}$$

where $\mathcal{O}_X(d)$ is the line bundle on X induced from the degree d line bundle on $\mathbb{P}^n$ (by definition $\mathcal{O}_X(d)(X_f)$ is the degree d component of $k[X_0, \ldots, X_n][f^{-1}]$, where $X_f = \{p \in \mathbb{P}^n \mid f(p) \neq 0\}$).

A scheme is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X, and a sheaf of rings $\mathcal{O}_X$ on X, this data being subject to certain axioms. The space X can be recovered from the pair $(\mathrm{Coh}(\mathcal{O}_X), \mathcal{O}_X)$, so in a sense the objects of algebraic geometry are pairs $(\mathcal{C}, \mathcal{O})$ consisting of a category together with a distinguished object. Hence we make the following definition.

Definition 3.2 The (noetherian) projective scheme *associated to a graded algebra A is the pair*

$$\mathrm{proj}(A) := (\mathrm{tails}(A), \mathcal{A}).$$

The general projective scheme *associated to A is*

$$\mathrm{Proj}(A) = (\mathrm{Tails}(A), \mathcal{A}).$$

The morphisms in Tails(A) can be a little tricky to understand; by definition

$$\mathrm{Hom}_{\mathrm{Tails}}(\pi N, \pi M) = \varinjlim \mathrm{Hom}_{\mathrm{Gr}}(N', M/M')$$

where the direct limit is taken over all pairs (N', M') of submodules of N and M such that N/N' and $M' \in \mathrm{Tors}(A)$, and $(N', M') \leq (N'', M'')$ if $N'' \subseteq N'$ and $M' \subseteq M''$. The hypotheses on A allow us to simplify this description.

Proposition 3.3 *If $N \in \mathrm{grmod}(A)$ and $M \in \mathrm{GrMod}(A)$, then*

$$\mathrm{Hom}_{\mathrm{Tails}}(\pi N, \pi M) = \varinjlim \mathrm{Hom}_{\mathrm{Gr}}(N_{\geq n}, M).$$

This direct limit is similar to a union, with the proviso that it is not really a union since the restriction of $f : N_{\geq n} \to M$ to $N_{\geq n+1}$ may be zero, even if $f \neq 0$.

The main properties of Tails, and the functor π, which we need are contained in the next few results.

Proposition 3.4 *If $f : N \to M$ is a morphism in* $\mathrm{GrMod}(A)$*, then*

1. $\ker(\pi f) = \pi(\ker f)$ *and* $\mathrm{coker}(\pi f) = \pi(\mathrm{coker}\, f)$*;*
2. $\pi f = 0$ *if and only if* $\mathrm{Im}(f)$ *is torsion;*
3. πf *is a monomorphism if and only if* $\ker(f)$ *is torsion;*
4. πf *is an epimorphism if and only if* $\mathrm{coker}(f)$ *is torsion;*
5. πf *is an isomorphism if and only if* $\ker(f)$ *and* $\mathrm{coker}(f)$ *are torsion.*

Proposition 3.5 1. $\mathrm{Tails}(A)$ *is an abelian category and* π *is exact.*

2. *If $\mathcal{D}$ is another abelian category, and $F : \mathrm{GrMod}(A) \to \mathcal{D}$ is an exact functor such that $FN = 0$ for all $N \in \mathrm{Tors}(A)$, then there is a unique functor $G : \mathrm{Tails}(A) \to \mathcal{D}$ such that $F = G \circ \pi$.*
3. *A functor $G : \mathrm{Tails}(A) \to \mathcal{D}$ is exact if and only if $G \circ \pi$ is.*

We mention two applications of Proposition 3.5. First, $\pi(M/\tau M) \cong \pi M$, so given $\mathcal{F} \in \mathrm{Tails}(A)$, $\mathcal{F} \cong \pi N$ for some torsion free N. Second, since A is $\mathbb{N}$-graded each $M_{\geq n}$ is a submodule of M, and $\pi M \cong \pi M_{\geq n}$; we call $M_{\geq n}$ a *tail* of M, and this explains the name of the quotient category—its objects are determined by the tails of A-modules. More precisely, we have the following result.

Proposition 3.6 *If $M, N \in \mathrm{grmod}(A)$, then $\pi M \cong \pi N$ if and only if $M_{\geq n} \cong N_{\geq n}$ for some n.*

Proof. Suppose that $\pi M \cong \pi N$. By Proposition 3.3, the isomorphism is given by πf for some $f : N_{\geq n} \to M$. Thus $\ker(f)$ and $\mathrm{coker}(f)$ are torsion, and hence finite dimensional by the noetherian hypotheses. It follows that for $r \gg 0$, $f : N_{\geq r} \to M_{\geq r}$ is an isomorphism, as required. The converse is trivial. ∎

Theorem 3.7 *The functor π has a right adjoint $\omega : \mathrm{Tails}(A) \to \mathrm{GrMod}(A)$.*

We will make frequent use of the adjoint isomorphism

$$\mathrm{Hom}_{\mathrm{Tails}}(\pi N, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Gr}}(N, \omega \mathcal{F}).$$

This implies that $\omega\mathcal{F}$ is torsion-free since, if N is torsion then $\pi N = 0$, which ensures that both the above homomorphism groups are zero.

Proposition 3.8 $\omega\pi M \cong \varinjlim \underline{\mathrm{Hom}}_A(A_{\geq n}, M)$

Proof. The proof is a "finger exercise":

$$\begin{aligned}
\omega\pi M &= \underline{\mathrm{Hom}}_A(A, \omega\pi M) && \text{because } {}_AA \text{ is finitely generated,}\\
&= \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}}(A, \omega\pi M[d]) \\
&= \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Tails}}(\pi A, \pi M[d]) && \text{by the adjoint isomorphism,}\\
&= \bigoplus_{d\in\mathbb{Z}} \varinjlim \mathrm{Hom}_{\mathrm{Gr}}(A_{\geq n}, M[d]) && \text{by Proposition 3.3,}\\
&= \varinjlim \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}}(A_{\geq n}, M[d]) \\
&= \varinjlim \underline{\mathrm{Hom}}_A(A_{\geq n}, M).
\end{aligned}$$

■

Notation. It is convenient to write

$$\underline{\mathrm{Hom}}_{\mathrm{Tails}}(\mathcal{F}, \mathcal{G}) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Tails}}(\mathcal{F}, \mathcal{G}[d]).$$

With this notation, the proof of Proposition 3.8 says that $\omega\mathcal{F} \cong \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{F})$; in fact, there is a natural equivalence of functors

$$\omega \simeq \underline{\mathrm{Hom}}(\mathcal{A}, -).$$

We also note that there is a natural map

$$\rho : A \to \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{A}) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}(\mathcal{A}, \mathcal{A}[d])$$

sending $a \in A_d$ to $\pi\rho_a$, where $\rho_a : A \to A$ is right multiplication by a. It is easy to check that ρ is an anti-homomorphism of graded algebras, so each $\underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{F})$ has a natural left A-module structure. Of course,

$$\underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{F}) \cong \varinjlim \underline{\mathrm{Hom}}_A(A_{\geq n}, \omega\mathcal{F})$$

already has a natural left A-module structure coming from the right action of A on $A_{\geq n}$. These two actions of A on $\underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{F})$ coincide.

Although Proposition 3.8 gives an explicit description of ω, its existence and basic properties are usually established by defining ω as follows. Given $M \in \mathrm{GrMod}(A)$, let E denote the injective envelope of $\overline{M} = M/\tau M$. Then $\omega\pi M$ is defined to be the largest graded submodule, H say, of E such that $\overline{M} \subset H$ and $H/\overline{M}$ is torsion. Thus $H/\overline{M} = \tau(E/\overline{M})$, and there is an exact sequence

$$0 \to \tau M \to M \to \omega\pi M \to \text{torsion} \to 0;$$

the last term in this sequence will be described in Proposition 4.6.

Example 3.9 *Let $A = k[x]$. One can check directly that $E = k[x, x^{-1}]$ is an injective A-module, and hence is the injective envelope of A in* $\mathrm{GrMod}(A)$. *(Notice this shows that, in contrast to projectives, injectives in* $\mathrm{GrMod}(A)$ *need not be injective in* $\mathrm{Mod}(A)$.*) Since E/A is torsion it follows that $\omega\pi A \cong E$. (We will see later that for the polynomial ring in ≥ 2 variables, $\omega\pi A \cong A$.) In particular, $\omega\pi A$ is not a finitely generated A-module.*

The following result is crucial.

Proposition 3.10 $\pi \circ \omega \simeq \mathrm{Id}$.

Proof. We must show that the natural map $\pi\omega\mathcal{F} \to \mathcal{F}$ is an isomorphism for all $\mathcal{F} \in \mathrm{Tails}(A)$. By Yoneda's lemma, it is enough to show that the map

$$\mathrm{Hom}_{\mathrm{Tails}}(\pi N, \pi\omega\mathcal{F}) \to \mathrm{Hom}_{\mathrm{Tails}}(\pi N, \mathcal{F})$$

is an isomorphism for all $N \in \mathrm{GrMod}(A)$; in fact, it suffices to do this for finitely generated N, by writing an arbitrary module as a direct limit of finitely generated ones. The map in question is the horizontal map in the following diagram

$$
\begin{array}{ccc}
\varinjlim \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, \omega\mathcal{F}) \cong \operatorname{Hom}_{\mathrm{Tails}}(\pi N, \pi\omega\mathcal{F}) & \longrightarrow & \operatorname{Hom}_{\mathrm{Tails}}(\pi N, \mathcal{F}) \\
\uparrow \pi & \nearrow_{\text{adjoint isomorphism}} & \\
\operatorname{Hom}_{\mathrm{Gr}}(N, \omega\mathcal{F}) & &
\end{array}
$$

where the isomorphism on the left is a consequence of Proposition 3.3 and the torsion-freeness of $\omega\mathcal{F}$. It suffices to show that the vertical map, which is π on morphisms, is an isomorphism. The functoriality of the adjoint isomorphism yields a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Hom}_{\mathrm{Gr}}(N, \omega\mathcal{F}) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathrm{Tails}}(\pi N, \mathcal{F}) \\
\downarrow & & \downarrow \\
\operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, \omega\mathcal{F}) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathrm{Tails}}(\pi N_{\geq n}, \mathcal{F})
\end{array}
$$

where the vertical maps are restriction, and the horizontal maps are the adjoint isomorphisms. Since the inclusion map $N_{\geq n} \to N$ induces an isomorphism $\pi N_{\geq n} \to \pi N$ the right hand vertical map is an isomorphism, hence so is the left hand one. Hence all maps in $\varinjlim \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, \omega\mathcal{F})$ are isomorphisms, so

$$\varinjlim \operatorname{Hom}_{\mathrm{Gr}}(N_{\geq n}, \omega\mathcal{F}) \cong \operatorname{Hom}_{\mathrm{Gr}}(N, \omega\mathcal{F}),$$

which is precisely what we required. ■

4 Cohomology

We will define the cohomology groups $H^q(\mathcal{F})$ for $\mathcal{F} \in \mathrm{Tails}(A)$, and prove a version of Serre's Finiteness Theorem. An essential preliminary step is to understand injectives in $\mathrm{Tails}(A)$.

Proposition 4.1 *1.* $\mathrm{Tails}(A)$ *has enough injectives.*

2. If $\mathcal{Q} \in \mathrm{Tails}(A)$ *is injective, then* $\omega\mathcal{Q}$ *is a torsion-free injective.*

3. If $Q \in \mathrm{GrMod}(A)$ *is torsion-free injective, then* πQ *is injective and* $Q \cong \omega\pi Q$.

Proof. (2) The adjoint isomorphism gives the natural equivalence

$$\mathrm{Hom}_{\mathrm{Tails}}(-,\mathcal{Q})\circ\pi \simeq \mathrm{Hom}_{\mathrm{Gr}}(-,\omega\mathcal{Q}).$$

The left hand side is a composition of exact functors, so we conclude that $\omega\mathcal{Q}$ is injective. It is torsion-free since $\omega\mathcal{F}$ is always torsion-free.

(3) By definition of ω, $Q \cong \omega\pi Q$, so

$$\mathrm{Hom}_{\mathrm{Tails}}(\pi N,\pi Q) \cong \mathrm{Hom}_{\mathrm{Gr}}(N,\omega\pi Q) \cong \mathrm{Hom}_{\mathrm{Gr}}(N,Q)$$

for all $N \in \mathrm{GrMod}(A)$. That is

$$\mathrm{Hom}_{\mathrm{Tails}}(-,\pi Q)\circ\pi \simeq \mathrm{Hom}_{\mathrm{Gr}}(-,Q).$$

By hypothesis the right hand side is exact, hence so is $\mathrm{Hom}_{\mathrm{Tails}}(-,\pi Q)$ by Proposition 3.5. Thus πQ is injective.

(1) Let $\mathcal{F} \in \mathrm{Tails}(A)$, and let $f:\omega\mathcal{F}\to Q$ be the inclusion of $\omega\mathcal{F}$ in its injective envelope. Since $\omega\mathcal{F}$ is torsion-free, so is Q. Thus πQ is injective by (3). Also $\ker(\pi f)=\pi(\ker f)$, so $\pi f:\pi\omega\mathcal{F}\simeq\mathcal{F}\to\pi Q$ is injective, which shows $\mathcal{F}$ embeds in an injective, as required. ■

Lemma 4.2 *1. Each injective in* $\mathrm{GrMod}(A)$ *decomposes as a direct sum of a torsion injective and a torsion-free injective.*

2. If A is connected, then every torsion injective is a direct sum of shifts of $A^* = \underline{\mathrm{Hom}}_k(A,k)$.

Proof. (1) Let E an injective. Being injective it contains a copy of the injective envelope of τE, say I. Since I is injective, $E = I \oplus Q$ for some other submodule Q; being a summand of an injective, Q is also injective, and torsion-free since $\tau E \subset I$. Finally, by Lemma 2.3, I is torsion.

(2) Let I be a torsion injective in $\mathrm{GrMod}(A)$. If $0 \neq M \in \mathrm{Tors}(A)$, then $\underline{\mathrm{Hom}}_A(k,M) \neq 0$. We may consider $S = \underline{\mathrm{Hom}}_A(k,I)$ as a submodule of I; it is a (possibly infinite) direct sum of shifts of ${}_Ak$. If M is a non-zero submodule of I then, since M is torsion, $\mathrm{Hom}_A(k,M) \neq 0$, whence $M \cap S \neq 0$, so S is essential in I; thus $I = E(S)$. Since A is left noetherian, a direct sum of injective modules is injective, whence $E(S)$ is a (possibly infinite) direct sum of shifts of $E({}_Ak) \cong A^*$. ■

If $\mathcal{F} \in \text{Tails}(A)$, then $\text{Hom}_{\text{Tails}}(\mathcal{F}, -)$ is left exact, so we may define its right derived functors, and compute them via injective resolutions. That is, if $\mathcal{G} \to \mathcal{E}^\bullet$ is an injective resolution in Tails(A), then

$$\text{Ext}^q(\mathcal{F}, \mathcal{G}) := h^q(\text{Hom}_{\text{Tails}}(\mathcal{F}, \mathcal{E}^\bullet)),$$

the q^{th} homology group of the complex. We also define

$$\underline{\text{Ext}}^q(\mathcal{F}, \mathcal{G}) := \bigoplus_{d \in \mathbb{Z}} \text{Ext}^q(\mathcal{F}, \mathcal{G}[d]).$$

These Ext groups are k-vector spaces. We will show that they can be computed in terms of Ext groups in GrMod(A) by using ω.

Proposition 4.3 *Let $N \in \text{grmod}(A)$ and $M \in \text{GrMod}(A)$. Let $E^\bullet M$ be a minimal injective resolution of M, and write $E^\bullet M = I^\bullet M \oplus Q^\bullet M$, where $I^\bullet M$ is the torsion part of $E^\bullet M$ (it is a subcomplex) and $Q^\bullet M$ is a torsion-free complement. Write $\mathcal{N} = \pi N$ and $\mathcal{M} = \pi M$. Then*

1. $\text{Ext}^q(\mathcal{N}, \mathcal{M}) = h^q(\text{Hom}_{\text{Gr}}(N, Q^\bullet M))$
2. $\underline{\text{Ext}}^q(\mathcal{N}, \mathcal{M}) \cong \varinjlim \underline{\text{Ext}}^q_A(N_{\geq n}, M)$

Proof. (1) Although $Q^\bullet M$ is not usually a subcomplex of $E^\bullet M$, we may identify it with the complex $E^\bullet M / I^\bullet M$. The exactness of π implies that $\mathcal{M} \to \pi E^\bullet \simeq \pi Q^\bullet$ is an injective resolution of $\mathcal{M}$ in Tails(A). But

$$\text{Hom}(\mathcal{N}, \pi Q^\bullet) \cong \text{Hom}_{\text{Gr}}(N, \omega\pi Q^\bullet \cong Q^\bullet),$$

so the result follows.

(2) First observe that $\varinjlim \underline{\text{Hom}}_A(N_{\geq n}, I^\bullet) = 0$: if $f : N_{\geq n} \to I^\bullet$, then $N_{\geq n}/\ker f$ is finite dimensional because $I^\bullet$ is torsion and N is noetherian, whence $N_{\geq r} \subseteq \ker f$ for $r \gg 0$, which implies that in the direct limit f becomes zero. Therefore

$$\begin{aligned}
\varinjlim \underline{\text{Ext}}^q_A(N_{\geq n}, M) &= \varinjlim h^q(\underline{\text{Hom}}_A(N_{\geq n}, I^\bullet \oplus Q^\bullet)) \\
&= h^q(\varinjlim \underline{\text{Hom}}_A(N_{\geq n}, Q^\bullet)) \\
&\cong h^q(\underline{\text{Hom}}(\mathcal{N}, \pi Q^\bullet)) \\
&= \underline{\text{Ext}}^q(\mathcal{N}, \mathcal{M}),
\end{aligned}$$

as required. ∎

Definition 4.4 *For* $\mathcal{F} \in \mathrm{Tails}(A)$ *we define the* cohomology groups

$$H^q(\mathcal{F}) := \mathrm{Ext}^q(\mathcal{A}, \mathcal{F})$$

and the cohomology modules

$$\underline{H}^q(\mathcal{F}) := \underline{\mathrm{Ext}}^q(\mathcal{A}, \mathcal{F}),$$

which are graded by

$$\underline{H}^q(\mathcal{F})_d := \underline{\mathrm{Ext}}^q(\mathcal{A}, \mathcal{F}[d]).$$

We have already observed that $\underline{\mathrm{Hom}}(\mathcal{A}, -) \simeq \omega$, so the $\underline{H}^q(-)$ are the right derived functors of ω. In particular, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, there is a long exact cohomology sequence

$$0 \to H^0(\mathcal{F}') \to H^0(\mathcal{F}) \to H^0(\mathcal{F}'') \to H'(\mathcal{F}') \to H'(\mathcal{F}) \to \cdots.$$

The Čech cohomology groups $H^q(X, -)$, defined for $\mathcal{O}_X$-modules, are the derived functors of the global section functor $\Gamma(X, -)$. But $\Gamma(X, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$, so $H^q(X, -)$ are the derived functors of $\mathrm{Hom}(\mathcal{O}_X, -)$. Hence by Serre's equivalence of categories (Theorem 3.1), this definition of cohomology reduces to the classical one for projective schemes.

The following result is mostly a specialization of earlier results.

Proposition 4.5 *Let* $M \in \mathrm{GrMod}(A)$*, and write* $\mathcal{M} = \pi M$*. Then*

1. $\underline{H}^0(\mathcal{M}) \cong \omega\pi M$;
2. $\underline{H}^q(\mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^q_A(A_{\geq n}, M)$;
3. $\underline{H}^q(\mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^{q+1}_A(A/A_{\geq n}, M)$ *for* $q \geq 1$;
4. $\underline{H}^q(\mathcal{M}) \cong h^{q+1}(I^\bullet M)$ *for* $q \geq 1$, *where* $I^\bullet M$ *is the torsion part of the injective resolution of* M.

Proof. (1) and (2) follow from the previous result and Proposition 3.8.

(3) For $q \geq 1$, the long exact sequence for $\underline{\mathrm{Ext}}_A(-, M)$ gives

$$\underline{\mathrm{Ext}}^q_A(A_{\geq n}, M) \cong \underline{\mathrm{Ext}}^{q+1}_A(A/A_{\geq n}, M)$$

since A is projective, so (3) follows from (2).

(4) Consider the exact sequence of complexes

$$0 \to I^\bullet M \to E^\bullet M \to Q^\bullet M \to 0.$$

Since $Q^\bullet M$ is torsion free, and $A/A_{\geq n}$ is torsion, there is an isomorphism of complexes

$$\underline{\mathrm{Hom}}_A(A/A_{\geq n}, I^\bullet M) \cong \underline{\mathrm{Hom}}_A(A/A_{\geq n}, E^\bullet M).$$

Taking direct limits and homology yields

$$h^{q+1}(\varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, I^\bullet M)) \cong \underline{H}^q(\mathcal{M}).$$

But $I^\bullet M$ is torsion, so the sum of its finite dimensional submodules, whence

$$\varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, I^\bullet M) \cong I^\bullet M. \qquad \blacksquare$$

Each $\underline{H}^q(\mathcal{M})$ has a natural left A-module structure arising from the right action of A on $A_{\geq n}$ (in Proposition 4.5(2) say). In fact, $\underline{H}^q(\mathcal{M})$ becomes a graded left A-module with degree d component being $H^q(\mathcal{M}[d])$.

Although $\varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)$ does not appear in the statement of the previous Proposition, it is an important object as the next, and later, results show.

Proposition 4.6 *For each $M \in \mathrm{GrMod}(A)$, there is an exact sequence*

$$0 \to \tau M \to M \to \omega\pi M \to \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M) \to 0.$$

Proof. Over directed sets $\varinjlim$ is an exact functor, so taking direct limits of the exact sequences

$$0 \to \underline{\mathrm{Hom}}_A(A/A_{\geq n}, M) \to \underline{\mathrm{Hom}}_A(A, M) \to \underline{\mathrm{Hom}}_A(A_{\geq n}, M) \to \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M) \to 0$$

yields the result, because $\varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, M) = \tau M$. $\blacksquare$

After defining sheaf cohomology, one of the first exercises is to compute the cohomology groups of line bundles on $\mathbb{P}^n$, i.e. $H^q(\mathbb{P}^n, \mathcal{O}(d))$. We will now carry out a slight generalization of this. The non-commutative algebras in the next definition are good analogues of polynomial rings.

Definition 4.7 A locally finite connected k-algebra, A say, is Artin-Schelter *regular of dimension $n+1$ if*

- $\operatorname{gl.dim}(A) = n+1 < \infty$,
- $\operatorname{GK}\dim(A) < \infty$, *and*
- $\operatorname{Ext}^i_A({}_Ak, A) = \begin{cases} 0 & \text{if } i \neq n+1 \\ k & \text{if } i = n+1 \end{cases}$, *i.e. A is Gorenstein.*

Polynomial rings, and more generally iterated Ore extensions

$$k[X_0][X_1;\sigma_1,\delta_1]\cdots[X_n;\sigma_n,\delta_n]$$

where each σ_i is an automorphism and $\deg(X_i) = 1$ for all i, are Artin-Schelter regular; so too are the Sklyanin algebras.

Example 4.8 *Let A be Artin-Schelter regular of dimension $n+1$. We compute $H^q(\mathcal{A}[d])$ for $d \in \mathbb{Z}$. For simplicity suppose that $n+1 \geq 2$.*

First we show that $\underline{H}^0(\mathcal{A}) = \omega\pi A$. The Gorenstein property ensures that $\underline{\operatorname{Hom}}_A(A/A_{\geq 1}, M) = 0$, whence $\tau A = 0$. Also, $\underline{\operatorname{Ext}}^1_A(A/A_{\geq 1}, M) = 0$ by the Gorenstein property, whence $\underline{\operatorname{Ext}}^1_A(A/A_{\geq n}, M) = 0$ for all n (by induction). Hence by Proposition 4.6, $A \cong \omega\pi A$. That is,

$$\underline{H}^0(\mathcal{A}) = A \qquad \text{and} \qquad H^0(\mathcal{A}[d]) = A_d.$$

Now suppose that $q \geq 1$. Since $\underline{\operatorname{Ext}}^{n+1}_A(k, A) \cong k[l]$, the trivial right A-module shifted by some integer l, it follows that for any finite dimensional A-module T, $\underline{\operatorname{Ext}}^{n+1}_A(T, A) \cong T^*[l]$; one argues by induction on the length of T, the case of a shift of k being obviously true. Hence

$$\begin{aligned} \underline{H}^q(\mathcal{A}) &= \varinjlim \underline{\operatorname{Ext}}^{q+1}_A(A/A_{\geq n}, A) \\ &= \varinjlim \begin{cases} 0 & q \neq n, \\ (A/A_{\geq n})^*[l] & q = n. \end{cases} \\ &= \begin{cases} 0 & q \neq n \\ A^*[l] & q = n. \end{cases} \end{aligned}$$

Thus $H^n(\mathcal{A}[d]) = (A^*)_{l+d} = (A_{-l-d})^*$.

When A is a polynomial ring on $n+1$ generators the Koszul complex gives a linear resolution of the trivial module ${}_Ak$, so $l = n+1$, whence we recover the usual result for $H^q(\mathbb{P}^n, \mathcal{O}(d))$.

Before proving Serre's Finiteness Theorem, we need some technical results.

Lemma 4.9 *Write* $[l,r] = \{T \in \mathrm{GrMod}(A) \mid T_{<l} = T_{>r} = 0\}$.
If $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M) \in [l', r']$ *for all* $j \leq i$, *and* $T \in [l,r]$, *then*

$$\underline{\mathrm{Ext}}^j_A(T, M) \in [l' - r, r' - l]$$

for all $j < i$.

Proof. By induction on $r - l$, we reduce to $r - l = 1$, in which case T is a direct sum of shifts of $A/A_{\geq 1}$; the lemma is easy for such T. ■

Proposition 4.10 *Let* $M \in \mathrm{grmod}(A)$ *and fix* $i \geq 0$. *The following are equivalent:*

1. *for all* $j \leq i$, $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, M)$ *is finite dimensional;*
2. *for all* $j \leq i$, $\underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)$ *is finite dimensional for all* n*;*
3. *for all* $j \leq i$ *and all* $N \in \mathrm{grmod}(A)$, $\underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M)$ *has a right bound independent of* n*;*
4. *for all* $j \leq i$ *and all* $N \in \mathrm{grmod}(A)$, $\varinjlim \underline{\mathrm{Ext}}^j_A(N/N_{\geq n}, M)$ *is right bounded.*

Proof. First, by Proposition 1.7, if $T \in \mathrm{grmod}(A)$, $\underline{\mathrm{Ext}}^q_A(T, M)$ is a subquotient of a finite direct sum of shifts of M, so is left bounded and locally finite.

We will prove the result by induction on i. For $i = 0$, (1)–(4) all hold because $\dim_k T < \infty$ implies that $\underline{\mathrm{Hom}}_A(T, M) \subseteq \underline{\mathrm{Hom}}_A(T, \tau M)$ which is finite dimensional since $\dim_k(\tau M) < \infty$; notice that (4) holds because $\varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geq n}, M) = \tau M$. So suppose the Proposition is true for $i - 1$; i.e., the four conditions are equivalent.

(1) $\Leftrightarrow$ (2) If (1) holds, the previous lemma implies that $\underline{\mathrm{Ext}}^j_A(A/A_{\geq n}, M)$ is bounded, and hence finite dimensional by the first paragraph; thus (2) holds. The converse is a tautology.

(1) $\Rightarrow$ (3) The exact sequence $0 \to T \to N/N_{\geq n+1} \to N/N_{\geq n} \to 0$ yields an exact sequence.

$$\underline{\mathrm{Ext}}_A^{j-1}(T, M) \to \underline{\mathrm{Ext}}_A^j(N/N_{\geq n}, M) \to \underline{\mathrm{Ext}}_A^j(N/N_{\geq n+1}, M) \to \underline{\mathrm{Ext}}_A^j(T, M).$$

But $T \in [n, n]$, so by Lemma 4.9, the first and last terms are bounded, and their right bounds approaches $-\infty$ as $n \to \infty$. Hence, given $d \in \mathbb{Z}$, there is a natural isomorphism

$$\underline{\mathrm{Ext}}_A^j(N/N_{\geq n}, M)_{\geq d} \xrightarrow{\sim} \underline{\mathrm{Ext}}_A^j(N/N_{\geq n+1}, M)_{\geq d} \tag{1}$$

for $n \gg 0$. By Lemma 4.9, these are right bounded, so have a right bound which is independent of n.

(3) $\Rightarrow$ (4) This is immediate.

(4) $\Rightarrow$ (1) Consider the exact sequence

$$\underline{\mathrm{Ext}}_A^{i-1}(A_{\geq 1}/A_{\geq n}, M) \to \underline{\mathrm{Ext}}_A^i(A/A_{\geq 1}, M) \to \underline{\mathrm{Ext}}_A^i(A/A_{\geq n}, M).$$

By hypothesis the direct limit of the last term is right bounded. Since (4) holds for i, and hence for $i-1$, the direct limit of the first term is right bounded. Hence so is the direct limit of the middle term. But that is simply $\underline{\mathrm{Ext}}_A^i(A/A_{\geq 1}, M)$, which we already know is left bounded and locally finite, whence it is finite dimensional. Thus (1) is true. ∎

Definition 4.11 Let $M \in \mathrm{grmod}(A)$. *We say that*

- $\chi_i(M)$ holds *if the equivalent conditions of Proposition 4.10 hold;*
- $\chi(M)$ holds *if* $\chi_i(M)$ *holds for all* i;
- A satisfies χ *if* $\chi(M)$ *holds for all* $M \in \mathrm{grmod}(A)$.

Proposition 4.12 *If* $M \in \mathrm{grmod}(A)$, *the following are equivalent:*

1. $\chi_1(M)$ *holds;*
2. $\mathrm{coker}(M \to \omega\pi M)$ *is right bounded;*
3. $(\omega\pi M)_{\geq d}$ *is finitely generated for all* $d \in \mathbb{Z}$.

Proof. We will use the exact sequence

$$0 \to \tau M \to M \to \omega\pi M \to \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M) \to 0.$$

The equivalence of (1) and (2) is a restatement of the equivalence of (1) and (4) in Proposition 4.10, noting that the proof of (4) implies (1) only used the truth of (4) for $N = A$.

(1) $\Rightarrow$ (3). Fix $d \in \mathbb{Z}$, and consider

$$M_{\geq d} \to (\omega\pi M)_{\geq d} \to \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geq n}, M)_{\geq d} \to 0. \tag{2}$$

By hypothesis the first term is finitely generated. Since $\chi_1(M)$ holds, part (3) of Proposition 4.10 ensures that the last term of (2) is right bounded and hence finite dimensional. It follows that $(\omega\pi M)_{\geq d}$ is finitely generated too.

(3) $\Rightarrow$ (1). The hypothesis ensures that the last term of (2) is finitely generated, but it is also torsion, hence finite dimensional. Therefore part (4) of Proposition 4.10 holds for $i = 1$ (with $N = A$) and, as noted, this ensures that part (1) of Proposition 4.10 holds too; i.e., $\chi_1(M)$ holds. ■

Rephrasing part (2) of Proposition 4.12, if A satisfies χ_1, then $\omega\pi M$ is finitely generated up to torsion whenever $M \in \mathrm{grmod}(A)$ (Example 3.9 showed that $\omega\pi M$ is not generally finitely generated). Part (3) of Proposition 4.12 says that if $\omega\pi M$ is considered as a rather nice module with respect to torsion, then M is not too far from being nice—at least $M_{\geq d} \cong (\omega\pi M)_{\geq d}$ for $d \gg 0$.

The condition χ is a non-commutative phenomenon. The next two results show that quotients of polynomial rings satisfy it, and the example which follows these positive results shows a rather nice non-commutative algebra which does not satisfy χ_1.

Theorem 4.13 *Noetherian Artin-Schelter regular algebras satisfy* χ.

Proof. Let A be such an algebra, and $M \in \mathrm{grmod}(A)$. We proceed by induction on $\mathrm{pdim}(M)$. If $\mathrm{pdim}(M) = 0$, then A is a finite direct sum of shifts of A; but $\underline{\mathrm{Ext}}^j_A(A/A_{\geq 1}, A)$ is finite dimensional by the Gorenstein hypothesis, so $\chi_1(M)$ holds. If $\mathrm{pdim}(M) > 0$, write $0 \to K \to P \to M \to 0$ with P projective, and $\mathrm{pdim}(K) = \mathrm{pdim}(M) - 1$. By the induction hypothesis, the last term of the exact sequence

$$\underline{\mathrm{Ext}}^j_A(k, P) \to \underline{\mathrm{Ext}}^j_A(k, M) \to \underline{\mathrm{Ext}}^{j+1}_A(k, K)$$

is finite dimensional, as is the first term, whence so is the middle term. ■

Proposition 4.14 *If A is noetherian and satisfies χ_i, so does A/I for all ideals I.*

Proof. Write $B = A/I$ and let $M \in \text{grmod}(B)$. We will proceed by induction on i; since B satisfies χ_0, we will assume the result is true for $i-1$. Thus B satisfies χ_{i-1}, and we must show B satisfies χ_i.

Consider the spectral sequence

$$E_2^{pq} = \underline{\text{Ext}}_B^p(\text{Tor}_q^A(B, A/A_{\geq n}), M) \Rightarrow \underline{\text{Ext}}_A^{p+q}(A/A_{\geq n}, M).$$

Since A is noetherian, each term in the minimal projective resolution of B_A is a finite direct sum of shifts of A, whence each $\text{Tor}_q^A(B, A/A_{\geq n})$ is finite dimensional. In particular, it is right bounded. Since A is projective,

$$\text{Tor}_q^A(B, A/A_{\geq n}) \cong \text{Tor}_{q-1}^A(B, A_{\geq n})$$

for $q \geq 2$, and

$$\text{Tor}_1^A(B, A/A_{\geq n}) \subseteq B \otimes_A A_{\geq n}.$$

By taking a minimal resolution of $A_{\geq n}$, it is easy to see that

$$\text{Tor}_{q-1}^A(B, A_{\geq n}) \in [n, \infty)$$

for all $q \geq 1$, whence

$$\text{Tor}_q^A(B, A/A_{\geq n}) \in [n, \infty)$$

for all $q \geq 1$. Since B satisfies χ_{i-1}, Lemma 4.9 with $T = \text{Tor}_q^A(B, A/A_{\geq n})$ implies that, for all $p \leq i-1$, the right bound of E_2^{pq} tends to $-\infty$ as $n \to \infty$. Thus, given $d \in \mathbb{Z}$, $p \leq i-1$, and $q \geq 1$,

$$(E_2^{pq})_{\geq d} = 0$$

for $n \gg 0$. Hence, for all $p \leq i$,

$$(E_2^{p0})_{\geq d} \cong \underline{\text{Ext}}_A^p(A/A_{\geq n}, M)_{\geq d}$$

for all $n \gg 0$. That is, for all $p \leq i$, and all $n \gg 0$,

$$\underline{\text{Ext}}_B^p(B/B_{\geq n}, M)_{\geq d} \cong \underline{\text{Ext}}_A^p(A/A_{\geq n}, M)_{\geq d}.$$

But A satisfies χ_i, so the condition in part (3) of Proposition 4.10 implies that B satisfies χ_i too. ■

Example 4.15 *Fix $0 \neq q \in k$, and suppose that q is not a root of unity. Let $B = k[x,y]$, with defining relation $xy - qyx = y^2$. (It is easy to show that $B \cong k[u,v]$ with relation $vu = quv$.) Define $A = k + xB$.*

It is standard that B is (right and left) noetherian, and not too difficult to deduce from that that A is also noetherian. As a right A-module, B is finitely generated, namely $B = A + yA$. In contrast, as a left A-module, B is not finitely generated: indeed, as a left A-module,

$$B/A \cong k[-1] \oplus k[-2] \oplus \cdots$$

is an infinite direct sum of shifts of the trivial A-module ${}_Ak = A/A_{\geq 1}$. To see this, simply observe that B/A has a basis given by the images of $\{y^i | i \geq 1\}$, and that $A_{\geq 1}y = xBy \subseteq A$. Since A is a domain, $\tau A = 0$, whence $A \subseteq \omega\pi A$. Since $\operatorname{Fract}(A) = \operatorname{Fract}(B)$, *$B$ is an essential extension of A; since ${}_A(B/A)$ is torsion, it follows from the definition of ω that $A \subset B \subset \omega\pi A$. Thus* $\operatorname{coker}(A \to \omega\pi A)$ *is not right bounded, so $\chi_1(A)$ does not hold. Alternatively, one can see from the description of B/A that $\underline{\operatorname{Ext}}^1_A(A/A_{\geq 1}, A)$ is not finite dimensional.*

Theorem 4.16 *(Serre's Finiteness Theorem). Let $\mathcal{F} \in$* tails(A)*. If A satisfies χ, then*

1. $\dim_k H^q(\mathcal{F}) < \infty$ *for all q, and*
2. *if $q \geq 1$, then $H^q(\mathcal{F}[n]) = 0$ for $n \gg 0$.*

Conversely, if A satisfies χ_1, and (2) holds for all $\mathcal{F} \in$ tails(A)*, then A satisfies χ.*

Proof. Write $\mathcal{F} = \pi M$ where $M \in \operatorname{grmod}(A)$.

Suppose that $q = 0$. Since $\chi_1(M)$ holds, $(\omega\pi M)_{\geq 0}$ is finitely generated, hence locally finite. In particular, $(\omega\pi M)_0 = H^0(\mathcal{F})$ is finite dimensional.

Suppose that $q \geq 1$. Since A satisfies χ_{q+1},

$$\varinjlim \underline{\operatorname{Ext}}^{q+1}_A(A/A_{\geq n}, M)$$

is right bounded; but this equals $\underline{H}^q(\mathcal{F})$, so (2) follows because $\underline{H}^q(\mathcal{F})_n = H^q(\mathcal{F}[n])$. The proof of Proposition 4.10 showed that, given $d \in \mathbb{Z}$,

$$\varinjlim \underline{\operatorname{Ext}}^{q+1}_A(A/A_{\geq n}, M)_{\geq d} \cong \underline{\operatorname{Ext}}^{q+1}_A(A/A_{\geq r}, M)_{\geq d}$$

for $r \gg 0$; in particular, this is locally finite, which proves (1) for $q \geq 1$.

Conversely, (2) implies that $\underline{H}^{i-1}(\mathcal{F})$ is right bounded for $i \geq 2$, but this is isomorphic to $\varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, M)$; thus, since $\chi_1(M)$ holds, condition (4) in Proposition 4.10 is satisfied for all i. Thus A satisfies χ. ∎

5 Non-commutative Schemes

Let A be a left noetherian, locally finite, $\mathbb{N}$-graded k-algebra. We already defined the projective scheme associated to A as the pair $\mathrm{proj}(A) = (\mathrm{tails}(A), \mathcal{A})$. Such pairs are the objects in the category Pairs: the objects are pairs $(\mathcal{C}, \mathcal{O})$ consisting of a k-linear abelian category $\mathcal{C}$, together with a distinguished object $\mathcal{O}$; the morphisms are pairs

$$(f, \theta) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$$

consisting of a covariant k-linear functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ and a morphism $\theta : f\mathcal{O}_1 \to \mathcal{O}_2$.

Definition 5.1 A map $F : \mathrm{proj}(B) \to \mathrm{proj}(A)$ *of schemes is a natural equivalence class of morphisms*

$$(f, \theta) : (\mathrm{tails}(A), \mathcal{A}) \to (\mathrm{tails}(B), B)$$

such that f is right exact, and $\theta : f\mathcal{A} \to \mathcal{B}$ is an isomorphism.

There is a similar notion of map between general projective schemes $\mathrm{Proj}(A) = (\mathrm{Tails}(A), \mathcal{A})$.

Warning: The map F goes in the opposite direction to the functor f. We have deliberately not defined a category of schemes—possibly the notion of map is too restrictive, and other morphisms should be permitted; in any case, whatever the appropriate definition should be, the maps above should be allowed.

Given a homomorphism $f : A \to B$ of graded algebras, we have the induction and restriction functors

$$f^* : \mathrm{GrMod}(A) \to \mathrm{GrMod}(B),$$
$$f_* : \mathrm{GrMod}(B) \to \mathrm{GrMod}(A),$$

defined by

$$\begin{aligned} f^*M &= B \otimes_A M, \qquad \text{and} \\ f_*N &= {}_AN. \end{aligned}$$

These are an adjoint pair:

$$\mathrm{Hom}(f^*M, N) \cong \mathrm{Hom}(M, f_*N).$$

Proposition 5.2 *If $f : A \to B$ is a homomorphism of graded algebras, there are induced functors*

1. $f^* : \mathrm{tails}(A) \to \mathrm{tails}(B)$ *which is exact;*
2. $f_* : \mathrm{tails}(B) \to \mathrm{tails}(A)$ *if ${}_AB$ is finitely generated up to torsion (i.e., $\pi B \in \mathrm{tails}(A)$);*
3. $f^* : \mathrm{Tails}(A) \to \mathrm{Tails}(B)$ *and* $f^* : \mathrm{tails}(A) \to \mathrm{tails}(B)$ *if either B_A is finitely generated, or* $\mathrm{coker}(f)$ *is right bounded; in this case, we obtain a map* $\mathrm{proj}(B) \to \mathrm{proj}(A)$.

Proof. The existence of f^* or f_* at the level of Tails or tails is proved by checking that induction or restriction of a torsion module is again torsion. The details are straightforward. ■

If $f : A \to A/I$ is the natural map to a quotient of A, then $f_* : \mathrm{Tails}(A/I) \to \mathrm{Tails}(A)$ is fully faithful, and we think of the induced map $\mathrm{proj}(A/I) \to \mathrm{proj}(A)$ as being a closed embedding. We usually identify $\mathrm{proj}(A/I)$ with its image in $\mathrm{proj}(A)$.

If u is a homogeneous regular normalizing element of A and $g : A \to A[u^{-1}]$ is the natural map, then $g_* : \mathrm{Tails}(A[u^{-1}]) \to \mathrm{Tails}(A)$ is fully faithful, and we should think of $\mathrm{proj}(A[u^{-1}]_0)$ as being the (open) complement to $\mathrm{proj}(A/(u))$ in $\mathrm{proj}(A)$. Suppose that u is not in A_0. Then $A[u^{-1}]$ cannot have any torsion modules (because there is a unit of positive degree), so $\mathrm{tails}(A[u^{-1}]) \simeq \mathrm{grmod}(A[u^{-1}])$. If u is of degree one, or if A is generated by A_0 and A_1, then $A[u^{-1}]$ is a strongly graded algebra, meaning that the product of the degree i and j components equals the degree $i + j$ component, and therefore has the property that $\mathrm{grmod}(A[u^{-1}]) \simeq \mathrm{mod}(A[u^{-1}]_0)$, the equivalence being given by $M \mapsto M_0$. Thus the open complement to $\mathrm{proj}(A/(u))$ is the 'affine scheme' $\mathrm{mod}(A[u^{-1}]_0)$.

Although we have restricted our attention to $\mathbb{N}$-graded algebras in these talks, the main ideas extend to $\mathbb{Z}$-graded algebras. In particular, there is a $\mathbb{Z}$-graded version of Proposition 5.2. In particular, we have the next Proposition which establishes an equivalence of categories $\text{Tails}(A_{\geq 0}) \simeq \text{Tails}(A)$. Thus, as far as projective schemes are concerned, we can replace A by the $\mathbb{N}$-graded algebra $A_{\geq 0}$; it is for this reason that our restriction to $\mathbb{N}$-graded algebras is reasonable.

Proposition 5.3 *If $f : A \to B$ is a homomorphism of graded algebras such that* $\ker(f)$ *is torsion and* $\text{coker}(f)$ *is right bounded, then f^* and f_* induce equivalences* $\text{Tails}(A) \simeq \text{Tails}(B)$ *and* $\text{tails}(A) \simeq \text{tails}(B)$. *In particular,* $\text{proj}(A) \simeq \text{proj}(B)$.

We omit the proof of the next two results, which may be found in [2] and [5] respectively.

Proposition 5.4 *If A is left noetherian, and generated over A_0 by A_1, then* $\text{proj}(A) \cong \text{proj}(A^{(d)})$, *where $A^{(d)} = \oplus_{n\in\mathbb{Z}} A_{nd}$ is the d^{th} Veronese subalgebra of A, with grading defined by $A^{(d)}_n = A_{nd}$.*

Proposition 5.5 *Let A and B be $\mathbb{N}$-graded k-algebras, generated over A_0 by A_1. Define their Segre product*

$$A \circ B = \bigoplus_{n\in\mathbb{Z}} A_n \otimes_k B_n$$

with grading $(A \circ B)_n = A_n \otimes B_n$, and multiplication inherited from that on $A \otimes_k B$. Then there are maps

$$\text{proj}(A \circ B) \to \text{proj}(A) \qquad \textit{and} \qquad \text{proj}(A \circ B) \to \text{proj}(B).$$

The maps in the previous proposition are *not* induced by algebra homomorphisms since the natural map $A \to A \otimes_k B$ does not have image in $A \circ B$.

Twisting. We now describe an important construction which gives rise to a map of schemes which does not arise from an algebra homomorphism. In particular, it shows that there may be a wide range of algebras having the same scheme associated to them—for example, a non-commutative algebra may determine the same scheme as a commutative algebra.

If σ is a graded algebra automorphism, $\sigma \in \mathrm{Aut}_k(A)$, then the *twisted algebra* ${}^\sigma A$ is ${}^\sigma A = A$ as a graded vector space, but with multiplication

$$a \odot b = a^{\sigma^n} b$$

if $a \in A_m$, $b \in A_n$.

Proposition 5.6 *There is an isomorphism*

$$\mathrm{proj}(A) \cong \mathrm{proj}({}^\sigma A).$$

Proof. In fact there is an equivalence of categories

$$\Phi : \mathrm{GrMod}(A) \to \mathrm{GrMod}({}^\sigma A)$$

sending A to ${}^\sigma A$ which is defined as follows. if $M \in \mathrm{GrMod}(A)$, then ΦM is the ${}^\sigma A$-module defined by $\Phi M = M$ as a graded vector space, and

$$a \odot m = a^{\sigma^n} m$$

if $a \in {}^\sigma A_i$, $m \in M_n$. The details are easy to check (see [7]). ■

Notation *We usually write ${}^\sigma M$ for the ${}^\sigma A$-module ΦM defined in the proof of Proposition 5.6.*

Example 5.7 *Let $A = k[x, y]$ be the commutative polynomial ring , and σ the automorphism defined by $x^\sigma = x$ and $y^\sigma = qy$ where q is some fixed non-zero scalar. Then ${}^\sigma A = k[u, v]$ with defining relation $vu = quv$. Here* $\mathrm{proj}({}^\sigma A) \cong \mathrm{Coh}(\mathbb{P}^1)$.

Continuing with this idea, if $A = k[x, y, z]$ with defining relations $zy = \alpha yz$, $xz = \beta zx$, $yx = \gamma xy$ then $\mathrm{proj}(A)$ *contains three copies of $\mathbb{P}^1$, namely the three coordinate axes* $\mathrm{proj}(A/(x))$, $\mathrm{proj}(A/(y))$ *and* $\mathrm{proj}(A/(z))$.

In the proof of Proposition 5.6, we usually have

$$({}^\sigma M)[1] \not\cong {}^\sigma(M[1]);$$

the [1] on the left is the shift functor for ${}^\sigma A$, whereas the [1] on the right is the shift functor for A. To see this let $a \in {}^\sigma A_i$ and $m \in M_{j+1}$. If we consider $m \in ({}^\sigma M)[1]_j$, then $a \odot m = a^{\sigma^{j+1}} m$; on the other hand, if we consider $m \in {}^\sigma(M[1])_j$, then $a \odot m = a^{\sigma^j} m$. The relation between the shift functors for ${}^\sigma A$ and A is described by the next lemma.

Lemma 5.8 *Let $\Phi : \mathrm{GrMod}(A) \to \mathrm{GrMod}({}^\sigma A)$ be the functor in Proposition 5.6. Let $\sigma_* : \mathrm{GrMod}(A) \to \mathrm{GrMod}(A)$ be the restriction functor arising from the algebra homomorphism $\sigma : A \to A$. There is a natural equivalence*

$$\Phi \circ \sigma_* \circ [1] \simeq [1] \circ \Phi.$$

Proof. (Probably this should be done behind closed doors.) Let $M \in \mathrm{GrMod}(A)$; then $\sigma_* M = M$ as a graded vector space, and the A-action on $\sigma_* M$ is given by

$$a \cdot m = a^\sigma m.$$

for $a \in A$ and $m \in \sigma_* M$. As vector spaces, both $(\Phi \circ \sigma_* \circ [1])M$ and $([1] \circ \Phi)M$ equal M; let

$$\rho : (\Phi \circ \sigma_* \circ [1])M \to ([1] \circ \Phi)M$$

be the identity map. We will show that ρ is an isomorphism of ${}^\sigma A$-modules. First the degree j component of $(\Phi \circ \sigma_* \circ [1])M$ equals $\sigma_*(M[1])_j = M_{j+1}$, as does the degree j component of $([1] \circ \Phi)M$. Thus ρ is a graded vector space map. If $m \in {}^\sigma(\sigma_* M[1])_j$, then

$$a \odot m = a^{\sigma^j} \cdot m = a^{\sigma^{j+1}} m;$$

on the other hand, $\rho(m) \in ({}^\sigma M)[1]_j = ({}^\sigma M)_{j+1}$, so

$$a \odot \rho(m) = a^{\sigma^{j+1}} m.$$

That is, $\rho(a \odot m) = a \odot \rho(m)$, as required. ■

Remark 5.9 *The twist $({}^\sigma A, \odot)$ defined prior to Proposition 5.6 should really be called the* left twist *of A. We may also define the* right twist *$(A^\sigma, *)$; as a graded vector space $A^\sigma = A$, and it is endowed with multiplication*

$$a * b = ab^{\sigma^m}$$

for $a \in A_m$, $b \in A_n$. It is an easy exercise to show that the map $\theta : {}^\sigma A \to A^{\sigma^{-1}}$ defined by $\theta(a) = \sigma^{-i}(a)$ for $a \in A_i$ is a graded algebra isomorphism.

Definition 5.10 Given a graded algebra A, we call

$$(\mathrm{tails}(A), \mathcal{A}, [1]) \text{ or } (\mathrm{Tails}(A), \mathcal{A}, [1])$$

the polarized projective scheme *associated to A.*

These are objects of the following category.

Definition 5.11 The category of triples, Trip, *has as its objects triples* $(\mathcal{C}, \mathcal{O}, s)$ *where*

- $\mathcal{C}$ *is a* k*-linear abelian category,*
- $\mathcal{O}$ *is a distinguished object of* $\mathcal{C}$*, and*
- $s : \mathcal{C} \to \mathcal{C}$ *is an auto-equivalence,*

and morphisms

$$(f, \theta, \mu) : (\mathcal{C}_1, \mathcal{O}_1, s_1) \to (\mathcal{C}_2, \mathcal{O}_2, s_2),$$

where

- $f : \mathcal{C}_1 \to \mathcal{C}_2$ *is a* k*-linear covariant functor,*
- $\theta : f\mathcal{O}_1 \to \mathcal{O}_2$ *is a morphism, and*
- $\mu : f \circ s_1 \to s_2 \circ f$ *is a natural transformation.*

Definition 5.12 A map $F : (\text{Proj}(B), s_2) \to (\text{Proj}(A), s_1)$ *of polarized schemes is a natural equivalence class of morphisms*

$$(f, \theta, \mu) : (\text{Tails}(A), \mathcal{A}, s_1) \to (\text{Tails}(B), B, s_2)$$

such that f *is right exact,* θ *is an isomorphism, and* μ *is a natural equivalence.*

For example, there are isomorphisms of polarized schemes

$$(\text{Tails}({}^{\sigma}A), {}^{\sigma}\mathcal{A}, [1]) \xrightarrow{\sim} (\text{Tails}(A), \mathcal{A}, \sigma_* \circ [1])$$

and

$$(\text{Proj}(A), [d]) \xrightarrow{\sim} (\text{Proj}(A^{(d)}), [1])$$

by Lemma 5.8 and Proposition 5.4 respectively.

A graded algebra associated to $(\mathcal{C}, \mathcal{O}, s)$**.** We may associate to a triple $(\mathcal{C}, \mathcal{O}, s)$ a $\mathbb{Z}$-graded algebra

$$B(\mathcal{C}, \mathcal{O}, s) = \bigoplus_{n \in \mathbb{Z}} B_n$$

where

$$B_n = \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^n(\mathcal{O})),$$

and the product rule $B_m \times B_n \to B_{m+n}$ is given by composition:

$$(f, g) \mapsto (s^n f) \circ g.$$

It is easy to see this is associative.

Proposition 5.13 *The rule above gives a covariant functor*

$$B : \mathrm{Trip} \to \mathrm{GrAlg}$$

to the category of graded k-algebras.

Proof. This is straightforward, although the notation can get a little unwieldy. ■

Example 5.14 *If R is a k-algebra and $\sigma \in \mathrm{Aut}_k(R)$, then*

$$B(\mathrm{Mod}(R^{\mathrm{op}}), R_R, \sigma^*) \cong R[x, x^{-1}; \sigma],$$

the skew Laurent polynomial ring, in which $\deg(R) = 0$ and $\deg(x) = 1$.

Example 5.15 *Let X be a projective scheme, and $\mathcal{L}$ a coherent $\mathcal{O}_X$-module. Let $s = \mathcal{L} \otimes_{\mathcal{O}_X} -$. Since $\mathrm{Hom}(\mathcal{O}_X, \mathcal{F}) \cong H^0(X, \mathcal{F})$ for any $\mathcal{O}_X$-module $\mathcal{F}$, we have*

$$B(\mathrm{Coh}(\mathcal{O}_X), \mathcal{O}_X, s) \cong \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{L}^{\otimes m})$$

with its natural commutative multiplication.

Example 5.16 *We now combine the ideas in the previous two examples. Let X be a projective scheme, $\mathcal{L}$ a coherent $\mathcal{O}_X$-module, and $\sigma \in \mathrm{Aut}_k X$. For an $\mathcal{O}_X$-module $\mathcal{F}$, we write $\mathcal{F}^\sigma = \sigma^* \mathcal{F}$; we have $\sigma^* \mathcal{F} \cong \sigma_*^{-1} \mathcal{F}$, and also $\mathcal{O}_X^\sigma \cong \mathcal{O}_X$. Now define $s = (\mathcal{L} \otimes_{\mathcal{O}} -) \circ \sigma^*$, and consider $B(\mathrm{Coh}(\mathcal{O}_X), \mathcal{O}_X, s)$. Then*

$$s^n \mathcal{O} = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}},$$

so

$$B_n = H^0(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}),$$

and the product rule $B_m \otimes B_n \to B_{m+n}$ is given by

$$(f,g) \mapsto s^n(f) \otimes g.$$

Thus B is a twisted homogeneous coordinate ring in the sense of Artin-van den Bergh [1]; to be consistent with their notation, we have

$$B(\mathrm{Coh}(\mathcal{O}_X), \mathcal{O}_X, s) \cong B(X, \sigma^{-1}, \mathcal{L})$$

(this isomorphism is analogous to the isomorphism ${}^{\sigma}A \cong A^{\sigma^{-1}}$ remarked on earlier). Equivalently,

$$B(\mathrm{Coh}(\mathcal{O}_X), \mathcal{O}_X, (\mathcal{L} \otimes_{\mathcal{O}} -) \circ \sigma_*) \cong B(X, \sigma, \mathcal{L}).$$

The next result is an important special case of the functoriality of B in Proposition 5.13.

Proposition 5.17 *Let A be left noetherian, locally finite and $\mathbb{N}$-graded.*

1. $B(\mathrm{GrMod}(A^{\mathrm{op}}), A_A, [1]) \cong A$.
2. $B(\mathrm{Tails}(A^{\mathrm{op}}), \mathcal{A}, [1]) \cong \omega\pi A$ *as a graded vector space; thus $\omega\pi A$ has a graded algebra structure.*
3. *Let $f = B(\pi)$, where $\pi : \mathrm{GrMod}(A^{\mathrm{op}}) \to \mathrm{Tails}(A^{\mathrm{op}})$ is the quotient functor. Then $f : A \to \omega\pi A$ is a graded algebra homomorphism.*
4. *If A satisfies χ_1, then f^* induces isomorphisms of polarized schemes*

$$(\mathrm{Proj}(A), [1]) \cong (\mathrm{Proj}(\omega\pi A), [1])$$

and

$$(\mathrm{proj}(A), [1]) \cong (\mathrm{proj}(\omega\pi A), [1]).$$

Proof. (1) Just use the definition of B.

(2) We have

$$\begin{aligned} B(\mathrm{Tails}(A^{\mathrm{op}}), \mathcal{A}, [1]) &= \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Tails}}(\mathcal{A}, \mathcal{A}[n]) \\ &= \underline{\mathrm{Hom}}(\pi A, \pi A) \\ &\cong \underline{\mathrm{Hom}}_{\mathrm{Gr}}(A, \omega\pi A) \\ &\cong \omega\pi A. \end{aligned}$$

(3) The map, f say, arising from the functoriality of B is

$$\begin{aligned} A_n = \mathrm{Hom}_{\mathrm{Gr}}(A, A[n]) &\rightarrow \mathrm{Hom}_{\mathrm{Tails}}(\pi A, \pi A[n]) \\ &= \mathrm{Hom}_{\mathrm{Gr}}(A, \omega\pi A[n]) \\ &= (\omega\pi A)_n \end{aligned}$$

which coincides with the map $A \rightarrow \omega\pi A$ in the definition of ω.

(4) If A satisfies χ_1, then $\mathrm{coker}(f)$ is right bounded by Proposition 4.12. On the other hand $\ker(f) = \tau A$ is torsion, so the result follows from Proposition 5.3. ■

Part (4) of this proposition says that as far as the projective scheme $\mathrm{proj}(A)$ is concerned, one may replace A by $\omega\pi A$ (at least if A satisfies χ_1), which is generally a better algebra than A (as in (as in (as in (as in (as in (as in (as in (as in (as in (as in (as in (as in Example 4.15).

Notation *Given a triple* $(\mathcal{C}, \mathcal{O}, s)$*, we write*

- $\mathcal{F}(n) = s^n\mathcal{F}$ *for objects* $\mathcal{F}$ *in* $\mathcal{C}$*, and*
- $H^0(\mathcal{F}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F})$.

Definition 5.18 Let $(\mathcal{C}, \mathcal{O}, s)$ *be a triple. We say that* s *is* ample *if*

1. *for each* $\mathcal{F}$ *in* $\mathcal{C}$*, there exist positive integers* $n_1, \ldots, n_p$ *and an epimorphism*

$$\bigoplus_{i=1}^{p} \mathcal{O}(-n_i) \rightarrow \mathcal{F},$$

and

2. *if* $f : \mathcal{F} \rightarrow \mathcal{G}$ *is an epimorphism in* $\mathcal{C}$*, then the induced map*

$$H^0(\mathcal{F}(n)) \rightarrow H^0(\mathcal{G}(n))$$

is surjective for $n \gg 0$.

Condition (1) says that $\mathcal{O}$ (with the help of the shift s) is something like a generator in $\mathcal{C}$, and that the objects are in a sort of finitely generated, and condition (2) says that $\mathcal{O}$ (with the help of s) is something like a projective object.

Proposition 5.19 *If A satisfies χ_1, then [1] is ample for* $(\text{tails}(A), \mathcal{A}, [1])$.

Proof. Let $M \in \text{grmod}(A)$, and define $\mathcal{F} = \pi M$. Then $\mathcal{F} \cong \pi(M_{\geq 1})$. Since A is left noetherian, $M_{\geq 1} \in \text{grmod}(A)$ too, whence there is a surjection

$$\bigoplus_{i=1}^{p} A[-n_i] \to M_{\geq 1}$$

for some positive integers $n_1, \dots, n_p$. Applying π, this gives an epimorphism

$$\bigoplus_{i=1}^{p} \mathcal{A}[-n_i] \to \mathcal{F}$$

so condition (1) of Definition 5.18 is satisfied.

Let $f : \mathcal{F} \to \mathcal{G}$ be an epimorphism in tails(A), and write $\mathcal{G} = \pi N$ where $N \in \text{grmod}(A)$. Now

$$\begin{aligned} \bigoplus_{n\in\mathbb{Z}} H^0(\mathcal{F}[n]) &= \bigoplus_{n\in\mathbb{Z}} \text{Hom}_{\text{Tails}}(\mathcal{A}, \mathcal{F}[n]) \\ &= \bigoplus_{n\in\mathbb{Z}} \text{Hom}_{\text{Tails}}(\pi A, \pi M[n]) \\ &= \bigoplus_{n\in\mathbb{Z}} \text{Hom}_{\text{Gr}}(A, \omega\pi M[n]) \\ &= \omega\pi M. \end{aligned}$$

Hence we must show that the map $\omega\pi M \to \omega\pi N$ induced by f is surjective in high degree. By Proposition 3.3, f is of the form πg for some A-module map $g : M_{\geq n} \to N_{\geq n}$, for $n \gg 0$. Hence we must show that g induces a surjection $\omega\pi(M_{\geq n}) \to \omega\pi(N_{\geq n})$. Since $f = \pi g$ is an epimorphism, $\ker(g)$ and $\text{coker}(g)$ are torsion, hence finite dimensional as $M_{\geq n}, N_{\geq n} \in \text{grmod}(A)$. Thus, for $n \gg 0$, $g : M_{\geq n} \to N_{\geq n}$ is surjective. But A satisfies χ_1, so for $n \gg 0$ $(\omega\pi M)_{\geq n} = M_{\geq n}$ and $(\omega\pi N)_{\geq n} = N_{\geq n}$, whence the result. ■

Example 5.20 *Let A and B be the algebras in Example 4.15, and define $R = A[t]$, the polynomial extension with* $\deg(t) = 1$. *Then [1] is* not *ample for* $(\text{tails}(R), \mathcal{R}, [1])$. *Let $N = R/(z)$. The point of Example 4.15 is that N has an essential extension by a torsion module which is not right bounded; thus* $\text{coker}(N \to \omega\pi N)$ *is not right bounded. Let $M = R$, let $g : M \to N$ be the natural map, and let $f : \mathcal{F} = \pi M \to \mathcal{G} = \pi N$ denote πg. Certainly f is an epimorphism because g is surjective, so if $\omega\pi M \to \omega\pi N$ is not surjective in high degree, then [1] is not ample. However, $\underline{\text{Ext}}^1_R(k, R) \cong \underline{\text{Hom}}_A(k, A) = 0$, so $\chi_1(M)$ holds. Thus $\omega\pi M/M$ is right bounded; since $\omega\pi N/N$ is not right bounded, the map $\omega\pi M \to \omega\pi N$ cannot be surjective in high degree.*

The next Theorem is one of the main results in [2]; it gives some idea of which categories can arise as non-commutative schemes.

Theorem 5.21 *Let $(\mathcal{C}, \mathcal{O}, s)$ be a triple such that*

1. *s is ample,*
2. *$\mathcal{O}$ is a noetherian object in $\mathcal{C}$, and*
3. *$\mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{F})$ is a finite dimensional for all $\mathcal{F}$ in $\mathcal{C}$.*

Then $A := B(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$ is

- *right noetherian,*
- *locally finite,*
- *satisfies χ_1, and*
- *$(\mathcal{C}, \mathcal{O}, s) \cong (\mathrm{tails}(A^{\mathrm{op}}), \mathcal{A}, [1])$.*

The only comment we will make concerning the proof of Theorem 5.21 is to describe the functor implementing the equivalence of categories between $\mathcal{C}$ and $\mathrm{tails}(A^{\mathrm{op}})$. Let $B = B(\mathcal{C}, \mathcal{O}, s)$; a graded B-module is of course an A-module. The equivalence is given by

$$\mathcal{F} \mapsto \pi \Gamma \mathcal{F}$$

where $\pi : \mathrm{GrMod}(A^{\mathrm{op}}) \to \mathrm{Tails}(A^{\mathrm{op}})$ is the quotient functor and

$$\Gamma : \mathcal{C} \to \mathrm{GrMod}(B^{\mathrm{op}})$$

is the functor defined by

$$\Gamma \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} (\Gamma \mathcal{F})_n = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^n \mathcal{F})$$

with right B-module structure

$$f.b = s^n(f) \circ b$$

for $f \in (\Gamma \mathcal{F})_m$, $b \in B_n$.

Perhaps the simplest illustration of Theorem 5.21 is the following: if $(\mathcal{C}, \mathcal{O}, s) = (\text{Mod}(R^{\text{op}}), R_R, \text{Id})$, then $B = R[x, x^{-1}]$ the Laurent polynomial extension, and $\Gamma M = M[x, x^{-1}]$; since B is strongly graded $\text{Mod}(R^{\text{op}})$ is equivalent to $\text{GrMod}(B)$ which is equivalent to $\text{Tails}(B)$. Since $R[x] \to B$ has right bounded cokernel $\text{Tails}(R[x])$ is equivalent to $\text{Tails}(B)$. Thus $\text{Mod}(R^{\text{op}})$ is equivalent to $\text{Tails}(R[x])$, and $R[x]$ is the ring A in the statement of the Theorem. Thus the theorem confirms what we already know.

Although Proposition 5.17, which says that $\text{tails}(A)$ is equivalent to $\text{tails}(\omega\pi A)_{\geq 0}$ when χ_1 holds, has a simple proof, it can also be deduced by applying Theorem 5.21 to $(\text{tails}(A), \mathcal{A}, [1])$.

Another important consequence of Theorem 5.21 is Theorem 5.24 below on twisted homogeneous coordinate rings.

Definition 5.22 Let X be a noetherian scheme and $\sigma \in \text{Aut}\, X$. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is σ-ample *if, for every $\mathcal{F} \in \text{Coh}(\mathcal{O}_X)$*

$$H^q(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^n} \otimes \mathcal{F}) = 0$$

for all $q > 0$ and all $n \gg 0$.

The point is that σ-ampleness ensures that a certain shift functor is ample—the key to proving this is the next Lemma.

Lemma 5.23 *[1, Lemma 3.2] Let $\mathcal{L}$ be a σ-ample line bundle on a scheme X. If $\mathcal{F} \in \text{Coh}(\mathcal{O}_X)$, then $s^n\mathcal{F}$ is generated by its global sections for $n \gg 0$.*

Theorem 5.24 *Let $\mathcal{L}$ be a σ-ample line bundle on a scheme X. Then $B = B(X, \sigma, \mathcal{L})_{\geq 0}$ is noetherian and*

$$\text{tails}(B) \simeq \text{Coh}(\mathcal{O}_X).$$

Proof. Apply the Theorem to the triple

$$(\text{Coh}(\mathcal{O}_X), \mathcal{O}_X, s)$$

where $s = (\mathcal{L} \otimes_{\mathcal{O}_X} -) \circ \sigma^*$, as in Example 5.16. By standard commutative theory, $\mathcal{O}_X$ is a noetherian object in $\text{Coh}(\mathcal{O}_X)$ and $H^0(\mathcal{F}) = \text{Hom}(\mathcal{O}_X, \mathcal{F}) = H^0(X, \mathcal{F})$ is finite dimensional for all $\mathcal{F} \in \text{Coh}(\mathcal{O}_X)$.

Next we show that s is ample. By applying s^{-n} to the result in Lemma 5.23, it follows that there is an epimorphism $(s^{-n}\mathcal{O})^p \to \mathcal{F}$ for some large p. Hence condition (1) in Definition 5.18 holds. Now let $f : \mathcal{F} \to \mathcal{G}$ be an epimorphism in $\mathrm{Coh}(\mathcal{O}_X)$. Write $\mathcal{K} = \ker(f)$; since s is an exact functor, there is a long exact sequence in cohomology

$$0 \to H^0(s^n\mathcal{K}) \to H^0(s^n\mathcal{F}) \to H^0(s^n\mathcal{G}) \to H^1(s^n\mathcal{K}).$$

Since $\mathcal{L}$ is σ-ample, $H^1(s^n\mathcal{K}) = 0$ for $n \gg 0$, whence

$$H^0(s^n\mathcal{F}) \to H^0(s^n\mathcal{G})$$

is surjective; thus condition (2) in Definition 5.18 holds.

Hence the hypotheses of Theorem 5.21 hold. By its conclusion B is right noetherian and $\mathrm{tails}(B^{\mathrm{op}}) \cong \mathrm{Coh}(\mathcal{O}_X)$. ■

References

[1] M. Artin and M. van den Bergh, Twisted Homogeneous Coordinate Rings, *J. Algebra* **133** (1990) 249–271.

[2] M. Artin and J. Zhang, Non-commutative Projective Schemes, *Adv. in Math.*, to appear.

[3] P. Gabriel, Des Catégories Abéliennes, *Bull. Soc. Math. France*, **90** (1962) 323–448.

[4] J. P. Serre, Faisceaux Algébriques Cohérents, *Ann. of Math.*, **61** (1955) 197–278.

[5] K. van Rompay, Segre product of Artin-Schelter regular algebras of dimension 2 and embeddings in quantum spaces of dimension 4, preprint (1994).

[6] J. T. Stafford and J. Zhang, Examples in Non-commutative Projective Geometry, *Proc. Camb. Phil. Soc.*, to appear.

[7] J. Zhang, Twisted Graded Algebras and Equivalences of Graded Categories, submitted.