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SELF-INJECTIVE CONNECTED ALGEBRAS

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Throughout this paper A is an algebra over a field k. We say that A is a connected k-algebra if there is a vector space decomposition $A = \oplus_{n \geq 0} A_n$, such that $A_i A_j \subset A_{i+j}$, and $A_0 = k$. The **augmentation ideal** is the unique maximal graded ideal $\mathfrak{m} := \oplus_{i>0} A_i$. We call $A/\mathfrak{m}$ the **trivial module**, and usually denote it by k. A finite dimensional k-algebra A is **Frobenius** if $\operatorname{Hom}_k(A, k)$ is isomorphic to A as a left and as a right A-module.

A Frobenius algebra is **quasi-Frobenius**, meaning that it is left and right artinian and left and right self-injective. For trivial reasons, a quasi-Frobenius algebra need not be Frobenius: take a division algebra infinite dimensional over its center k. Also, a self-injective ring need not be quasi-Frobenius: there is a local, commutative and self-injective ring which is neither artinian nor noetherian [2, page 214].

Frobenius and quasi-Frobenius algebras have been studied in great detail, but rarely under the additional hypothesis that they are connected graded. Our interest in connected graded Frobenius algebras arises from our interest in various non-commutative analogues of polynomial rings because these analogues are often Koszul algebras whose dual algebra is a connected graded Frobenius algebra of finite dimension (see [5] for details).

We will prove the following result.

Key words and phrases. Frobenius algebra, connected algebra.

THEOREM. *A connected self-injective ring is finite dimensional and Frobenius.*

J. Lawrence [4] has shown that a self-injective algebra of countable dimension is quasi-Frobenius, so one can get part-way to our result if A has a countable number of generators.

The proof of the theorem is given in two steps. The next lemma proves the result under an additional hypothesis. Then we show the hypothesis is already a consequence of the other hypotheses.

LEMMA 1. *Let A be a connected algebra with $\mathfrak{m} \neq 0$. Suppose there is a non-zero homogeneous element $x \in \mathfrak{m}$ such that $x\mathfrak{m} = 0$. If* $\operatorname{injdim} A_A = 0$, *then A is finite dimensional and Frobenius.*

PROOF. We will apply the hypothesis that A_A is injective in the following situation. If I is a right ideal of A, and $i : I \to A^{\oplus t}$ is an injective homomorphism as pictured

$$\begin{array}{ccc} I & \xrightarrow{\ i\ } & A^{\oplus t} \\ {\scriptstyle f}\big\downarrow & & \\ A & & \end{array} \qquad (1)$$

then there is an A-module map g from $A^{\oplus t}$ to A such that $gi = f$. Every homomorphism $A^{\oplus t} \to A$ is of the form $(a_1, \cdots, a_t) \longmapsto \sum_j c_j a_j$ for some elements $c_j \in A$.

Suppose that $0 \neq x \in \mathfrak{m}_e$ is such that $x\mathfrak{m} = 0$. Let $a \in A$ be homogeneous, and write $I = aA$. Since $x\mathfrak{m} = 0$, there is a well-defined A-module homomorphism $f : I \longrightarrow A$ defined by $f(a) = x$. Let $i : I \to A$ be the inclusion map. By (1), there is a $c \in A$ such that $x = ca$. Since both x and a are homogeneous, there is a homogeneous such c. In particular, $\deg a \leq \deg x = e$. Thus $A_n = 0$ if $n > e$.

If $\deg a = \deg x$, then $x = ca$ implies that $c \in k$ and $a \in kx$. Hence $A_e = kx$. Fix an integer n between 0 and e. Let $\{a_j\}_{j\in\Omega}$ be a k-linear basis of A_n and let I be the right ideal $A_nA = \sum_j a_jA$. Let $\{l_j \mid j \in \Omega\} \subset k$ be any sequence of scalars. We define f from $I = A_nA$ to A by $f(a_j) = l_jx$ for all $j \in \Omega$. By (1), there is a homogeneous element $c \in A_{e-n}$ such that $ca_j = l_jx$ for all $j \in \Omega$. In other words, $A_{e-n}/K_{e-n} \cong A_n^*$ induced by the multiplication where $K_{e-n} := \{c \in A_{e-n} \mid cA_n = 0\}$. If $\dim A_n$ is infinite, then $\dim A_n < \dim A_n^*$ (see [3, Ex. 12, p 207] for example), so

$$\dim A_n < \dim A_n^* \leq \dim A_{e-n} < \dim A_{e-n}^* \leq \dim A_n,$$

where the last inequality follows from the isomorphism $A_n/K_n \cong A_{e-n}^*$. But this is a contradiction, so we conclude that $\dim A_n < \infty$ for all n. Hence

$$\dim A_n = \dim A_{e-n} - \dim K_{e-n} = \dim A_n - \dim K_n - \dim K_{e-n}.$$

Thus $K_n = 0$ for all n and $A_{e-n} \cong A_n^*$. As a consequence, the multiplication $A_n \times A_{e-n} \longrightarrow A_e = kx$ induces a perfect k-bilinear pairing. By [5,3.2], A is Frobenius. ■

We can now prove our Theorem.

PROOF. If $A = k$ the result is true, so suppose that $\mathfrak{m} \neq 0$. By Lemma 2, it suffices to show that there is a nonzero homogeneous element x such that either $x\mathfrak{m} = 0$ or $\mathfrak{m}x = 0$. Suppose this is false. Then A is not bounded, meaning that $A_n \neq 0$ for infinitely many n, and

$$\bigcap_{a\in\mathfrak{m}} l(a) = \bigcap_{a\in\mathfrak{m}} r(a) = \{0\}$$

where $l(a) = \{b \in A \mid ba = 0\}$ and $r(a)$ is defined similarly. Let $\{a_j\}_{j=1}^t$ be any finite set of homogeneous elements in $\mathfrak{m}$. We claim that $\cap_{j=1}^t r(a_j)$ is not bounded. If it is bounded by degree m, then we define an injection $i : A_{\geq m} \longrightarrow A^{\oplus t}$ by $i(a) = (a_1 a, \cdots, a_t a)$. Let $f : A_{\geq m} \to A$ be the inclusion. By (1), there are elements $\{c_j\}$ such that for every homogeneous element $a \in A_{\geq m}$, $a = \sum c_j a_j a$. This is a contradiction because the degrees of the a_j are positive. Therefore $\cap_{j=1}^t r(a_j)$ is not bounded. Similarly $\cap_{j=1}^t l(a_j)$ is not bounded.

We choose two sequences of nonzero homogeneous elements in $\mathfrak{m}$, $\{b_j\}_{j=1}^\infty$ and $\{a_j\}_{j=1}^\infty$ inductively as follows. Pick $b_1 \in \mathfrak{m} - l(a_1)$ for some $a_1 \in \mathfrak{m}$ initially. Suppose we have picked $\{b_1, \cdots, b_n\}$ and $\{a_1, \cdots, a_n\}$ such that

(i) $b_i \in \cap_{j=1}^{i-1} l(a_i) - \cap_{j=1}^{i} l(a_j)$, and

(ii) $\{\deg b_i \mid i = 1, 2 \cdots\}$ is strictly increasing.

In the previous paragraph we have proved that $\cap_{j=1}^n l(a_j)$ is not bounded and assumed that $\cap_{a\in\mathfrak{m}} l(a) = \{0\}$. Hence there is a_{n+1} such that $\cap_{j=1}^n l(a_j) \neq \cap_{j=1}^{n+1} l(a_j)$ in some degree bigger than the degree of b_n. Pick $b_{n+1} \in \cap_{j=1}^n l(a_j) - \cap_{j=1}^{n+1} l(a_j)$ and $\deg b_{n+1} > \deg b_n$. Therefore $\{b_j\}_{j=1}^{n+1}$ satisfies (i) and (ii). By definition we see that $b_j a_j \neq 0$ and $b_i a_j = 0$ for all $i > j$. Using $\{b_i\}_{i=1}^\infty$ and $\{a_j\}_{j=1}^\infty$, we define an A-homomorphism f from $I = \sum_j a_j A$ to A by $f(\sum_j a_j d_j) = (\sum_j b_j)(\sum_j a_j d_j)$. To see this is well-defined we note that $\sum_j a_j d_j$ a finite sum and product $(\sum_j b_j)(\sum_j a_j d_j)$ is also a finite sum because $b_i a_j = 0$ for all $i > j$. Let $i : I \to A$ be the inclusion. By (1), there is $c \in A$ such that $f(a_i) = ca_i$ for all i, i.e., $(\sum_j b_j)a_i = ca_i$. Since a_i and b_j are homogeneous, if we let m be higher than the degree of all nonzero homogeneous components of c, we have $(\sum_{j\geq m} b_j)a_i = 0$ for all i. But this contradicts to the fact $b_m a_m \neq 0$ and $b_j a_m = 0$ for all $j > m$. Therefore we have proved our claim: there is a nonzero homogeneous element x such that $x\mathfrak{m} = 0$ or $\mathfrak{m}x = 0$. ■

The rest of the paper contains some further observations on homological properties of connected algebras.

PROPOSITION 2. *Let A be a connected algebra with $\mathfrak{m} \neq 0$. If there is a non-zero homogeneous element $x \in \mathfrak{m}$ such that $x\mathfrak{m} = 0$, then* gldim $A = \infty$.

PROOF. Consider the minimal projective resolution of the trivial module ${}_A k$, say

$$\cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_0 \longrightarrow {}_A k.$$

Since $x\mathfrak{m} = 0$, A is not a domain, and hence not a free algebra, whence A has at least one relation. This implies that $P_2 \neq 0$, whence the map $P_1 \to P_0$ is not injective. By minimality, the kernel of $\delta_i : P_i \to P_{i-1}$ is a submodule of $\mathfrak{m}P_i$. Since $x\mathfrak{m} = 0$, $x \ker \delta_i = 0$, whence $\ker \delta_i$ cannot contain a non-zero free module. Therefore, if δ_i is not injective, then δ_{i+1} is not injective. The statement follows by induction on i. ■

PROPOSITION 3. *Let A be connected graded, or a local ring with augmentation ideal $\mathfrak{m}$ and residue field k. Suppose that $\mathfrak{m}^l = 0$ for some $l > 0$. If M is a non-zero right A-module, then*

1. injdim $M = \max\{i \mid \operatorname{Ext}^i(k, M) \neq 0\}$, *and*

2. injdim $M < \infty$ *if and only if* injdim $M = 0$.

PROOF. 1. Let N be a right A-module. Then there is a finite chain of submodules

$$N \supset N\mathfrak{m} \supset \cdots \supset N\mathfrak{m}^{l-1} \supset N\mathfrak{m}^l = 0,$$

and the subquotients $N\mathfrak{m}^{i-1}/N\mathfrak{m}^i$ are isomorphic to direct sums of copies of k. By an induction argument using the long exact sequence for Ext, $\max\{i \mid \operatorname{Ext}^i(N, M) \neq 0\} \leq \max\{i \mid \operatorname{Ext}^i(k, M) \neq 0\}$, whence injdim $M = \max\{i \mid \operatorname{Ext}^i(k, M)\}$.

2. Suppose $0 < i = \operatorname{injdim} M < \infty$. Then $A \neq k$. Since $\mathfrak{m}^l = 0$, $\mathfrak{m}$ contains a copy of k and we have an exact sequence

$$0 \longrightarrow k \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{m}/k \longrightarrow 0.$$

Applying $\operatorname{Ext}^*(-, M)$, we obtain an exact sequence

$$\operatorname{Ext}^i(\mathfrak{m}, M) \longrightarrow \operatorname{Ext}^i(k, M) \longrightarrow \operatorname{Ext}^{i+1}(\mathfrak{m}/k, M) = 0.$$

Applying $\operatorname{Ext}^*(-, M)$ to the short exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow k \longrightarrow 0$$

we obtain that $\operatorname{Ext}^i(\mathfrak{m}, M) = \operatorname{Ext}^{i+1}(k, M) = 0$. Hence $\operatorname{Ext}^i(k, M) = 0$, which contradicts the first part of the proposition. Therefore injdim M is either 0 or ∞. ∎

EXAMPLE. Let A be the commutative graded k-algebra generated by the elements $\{x_i \mid i \geq 0\}$ of degree one, and defined by the relations

$$x_i^2 - x_j^2 = x_i x_j = x_i^3 = 0 \qquad (i \neq j).$$

Notice that $\dim A_2 = 1$, $A_3 = 0$, and A is infinite dimensional. By our theorem, injdim $A \neq 0$, so injdim $A = \infty$ by Proposition 3. Clearly A is the union of its subalgebras $k[x_0, \dots, x_n]$. The multiplication in A induces a perfect pairing $A_i \times A_{2-i} \to A_2$ which restricts to perfect pairings on each of these finite dimensional subalgebras. Therefore each of these subalgebras is Frobenius by [5, 3.2], so A is the direct limit of self-injective algebras. But injdim $A = \infty$, so Berstein's formula [1, Cor. 1] for the global dimension of a direct limit does not have an anologue for injective dimension.

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