

OBERWOLFACH MAY 2006

NOTES BY S. PAUL SMITH

ABSTRACT. These are the notes I typed during the talks. I haven't spent much time after the lectures cleaning them up so there are probably lots of typos and even some more serious errors. The hardest thing was getting the quiver diagrams done in real time. I often had to give up! Anyway, there may be some value to this so I'm placing it in the public domain. feedback is welcome.

1. KING—MODULI OF SHEAVES FROM MODULI OF KRONECKER MODULES

with Alvarez-Consul

1.1. **Basics.** Let X be a projective scheme over $k = \bar{k}$.

Important features of $\text{coh}X$: monoidal category with identity \mathcal{O} , simple objects \mathcal{O}_x , $x \in X$, and automorphisms $(n) : E \mapsto E(n)$ given by tensoring n times with an ample invertible sheaf $\mathcal{O}(1)$.

Notation:

- $H = H(n_0, n_1) := H^0(\mathcal{O}(n_1 - n_0)) = \text{Hom}(\mathcal{O}(-n_1), \mathcal{O}(-n_0))$; typically $n_1 \geq n_0$.
- $\mathfrak{M}_X(P)$:= moduli space of semistable sheaves with Hilbert polynomial P . It is a *projective* scheme.

Motivating question: why is $\mathfrak{M}_X(P)$ projective? Classical answer: because it is constructed by GIT. Not very helpful answer!

1.2. **Modern answer.** First, why is X projective? Because when $n_1 - n_0 \gg 0$ (i.e., $\exists N \forall n_0, n_1$ with $n_1 - n_0 \geq N$...) the map

$$x \mapsto \phi_x : H^0(\mathcal{O}_x(n_0)) \otimes H \rightarrow H^0(\mathcal{O}_x(n_1))$$

is an embedding $X \rightarrow \mathbb{P}(H^*) = \mathfrak{M}_H(1, 1)$ = the moduli space of strictly non-zero H -Kronecker modules¹!

Theorem 1.1. $\mathfrak{M}_X(P)$ is projective because for $n_1 \gg n_0 \gg 0$ (i.e., $\exists N_0 \forall n_0 \geq N_0 \exists N_1 \forall n_1 \geq N_1$...) the functor

$$E \mapsto \phi_E : H^0(E(n_0)) \otimes H \rightarrow H^0(E(n_1))$$

gives an embedding $\mathfrak{M}_X(P) \rightarrow \mathfrak{M}_H(P(n_0), P(n_1))$ = the moduli space of semistable H -Kronecker modules which is obviously (!) projective.

Remarks:

- (1) The proof gives a “new” construction of the $\mathfrak{M}_X(P)$ (cf. Simpson).
- (2) the embedding is scheme-theoretic, except possibly at strictly semistable points when $\text{char } k = p$;

¹Representations of dimension $(n_0, n_1) = (1, 1)$ of the Kronecker quiver with $\dim H$ arrows.

- (3) functoriality implies there should be an embedding of “non-commutative moduli”, whatever that means.
- (4) how do we put the “right” non-commutative structure on $\mathfrak{M}_H(P(n_0), P(n_1))$ and thus on $\mathfrak{M}_X(P)$ (non-commutative necessary when dimensions are not co-prime)??
- (5) if $\text{End } \mathcal{O} = k$ and $T = \mathcal{O}(n_0) \oplus \mathcal{O}(n_1)$, then

$$A := \text{End } T = \begin{pmatrix} k & 0 \\ H & k \end{pmatrix}$$

and obtain

$$\Phi := \text{Hom}(T^\vee, -) : \text{coh } X \rightarrow \text{mod } A$$

with adjoint $\Phi^\vee : V \mapsto T^\vee \otimes_A V$.

1.3. Regularity. E has Hilbert polynomial

$$P(n) := \sum (-1)^i \dim H^i(E(n)) = rn^s + \text{lower degree terms.}$$

where $s = \dim \text{Supp } E$ and r is like $\text{rank } E$. Note $P(n) = \dim H^0(E(n))$ for $n \gg 0$ because $\mathcal{O}(1)$ is ample. (in fact, get = when E is n -regular². Regularity plays a key role.

Theorem 1.2. *If $\mathcal{O}(n_1 n_0)$ is 0-regular, then Φ is fully faithful on all n_0 -regular E , i.e., the evaluation*

$$\varepsilon_E : T^\vee \otimes_A \text{Hom}(T^\vee, E) \rightarrow E$$

is an isomorphism.

Does this mean there is some sort of embedding of the stack of n -regular sheaves?

1.4. Stability. An A -module V (i.e., a representation of the H -Kronecker quiver $V_0 \otimes H \rightarrow V_1$) is semistable if for all $W \subset V$,

$$\frac{\dim W_0}{\dim W_1} \leq \frac{\dim V_0}{\dim V_1}$$

Note this is essentially the only (sensible) stability condition because the rank of the Grothendieck group is 2.

A sheaf E is semistable if for all $E' \subsetneq E$ such that $s_E = s_{E'}$,

$$\frac{P_{E'}(n)}{r_{E'}} \leq \frac{P_E(n)}{r_E}$$

Note: when $\dim X = 1$ this reduces to

$$\frac{\deg_{E'}}{r_{E'}} \leq \frac{\deg_E}{r_E}.$$

Boundedness: given P , for all $n \gg 0$ if E is semistable for P , then E is n -regular.

Set of Kronecker modules with fixed dimension vector is “bounded”. What does he mean??

Can write stability condition in a better way:

² E is n -regular if $H^i(E(n-i)) = 0$ for all $i > 0$; if E is n -regular it is m -regular for all $m \geq n$.

Lemma 1.3 (Rudakov). *The stability condition is equivalent to the condition that*

$$\frac{P_{E'}(n_0)}{P_{E'}(n_1)} \leq \frac{P_E(n_0)}{P_E(n_1)}$$

for $n_1 \gg n_0 \gg 0$. In other words, $\Phi_{E'}$ does not destabilize $\Phi(E)$ for large n .

Theorem 1.4. *Given P , for $n_1 \gg n_0 \gg 0$, for any E with Hilbert polynomial P , E is semistable if and only if E is n_0 -regular and pure (i.e., support of all submodules of E is same as that for E) and $\Phi(E)$ is semistable.*

Corollary 1.5. *There is an embedding $\mathfrak{M}_X(P) \rightarrow M_H(P(n_0), P(n_1))$ for $n_1 \gg n_0 \gg 0$.*

1.5. **Theta functions.** Why is $M_H(P(n_0), P(n_1))$ projective?

Theorem 1.6 (Schofield, Van den Bergh, Derksen-Weyman). *An A -module V is semi-stable if and only if there is a $\gamma : P_1^{k_1} \rightarrow P_0^{k_0}$ such that $\text{Hom}(\gamma, V) : V_0^{k_0} \rightarrow V_1^{k_1}$ is an isomorphism, i.e., $\theta_\gamma(E) := \det \text{Hom}(\gamma, E)$ is non-zero. Further, the θ_γ embeds \mathfrak{M}_A .*

Adjunction $\text{Hom}(\gamma, \Phi(E)) = \text{Hom}(\Phi^\vee \gamma, E)$

i.e., $\Phi(E)$ is semistable if and only if there is a map $\delta : \mathcal{O}(-n_1)^{k_1} \rightarrow \mathcal{O}(-n_0)^{k_0}$ such that $\theta_\delta = \det \text{Hom}(\delta, E) \neq 0$.

2. LUNTS—DG-DEFORMATION

$X \mapsto D^b(\text{coh}X)$ is a 2-functor. How might one try to reconstruct X from $\text{coh}X$? Moduli space of point-like objects in $D^b(X)$ But needed a good theory of moduli spaces for families in derived categories.

2.1. Review of deformation functor. Fix a field k . Let \mathcal{C} :=the category of local commutative artin algebras (R, \mathfrak{m}) with $R/\mathfrak{m} = k$

X/k =algebra over k , or a compact complex manifold, or a principal G -bundle, a repon of a group Γ etc.

$$\text{Def}(X) : \mathcal{C} \rightarrow \text{Gpd}$$

Objects in $\text{Def}_R(X)$ are pairs (\mathcal{X}, σ) where \mathcal{X} is a flat family of objects such that

$$\begin{array}{ccc} \mathcal{X} & \longleftarrow & X \\ \sigma \downarrow & & \downarrow \\ \text{Spec } R & \xleftarrow{i} & \text{Spec } k \end{array}$$

is a Cartesian square. Morphisms are maps $f : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & \swarrow \sigma_x & \downarrow f & \nwarrow & \\ \text{Spec } R & & & & X \\ & \swarrow \sigma_y & \downarrow & \nwarrow & \\ & & \mathcal{Y} & & \end{array}$$

commutes.

$\text{Def}_R(X)$ is pro-representable by a complete local \hat{R} if

$$\text{Hom}_{\text{alg}}(\hat{R}, R) = \pi_0 \text{Def}_R(X).$$

functorially in R .

2.2. Universal approach: DG-Lie algebras. (Grothendieck, Drinfeld, Deligne)

$\text{Def}(X)$ is controlled by a DG-Lie algebra $\mathfrak{g} = \mathfrak{g}(X)$ =the Lie algebra of infinitesimal derived automorphisms of X , i.e., deformations of X =inner deformations of the differential in \mathfrak{g} .

The Deligne or Maurer-Cartan groupoid $\text{Def}(\mathfrak{g}) : \mathcal{C} \rightarrow \text{Gpd}$. Since R is commutative, $\mathfrak{g} \otimes \mathfrak{m}$ is also a DG-Lie algebra (nilpotent because \mathfrak{m} is a nilpotent ideal).

$$\text{Ab } \text{Def}_R(\mathfrak{g}) = \text{MC}(\mathfrak{g} \otimes \mathfrak{m}) = \{ \alpha \in \mathfrak{g} \otimes \mathfrak{m} \mid d\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \}$$

a super-bracket. The MC condition is equivalent to the condition that $(d + \alpha)^2 = 0$ where $d + \alpha := d + [\alpha, -] : \mathfrak{g} \otimes R \rightarrow \mathfrak{g} \otimes R$.

Only works in $\text{char } k = 0$

Morphisms in $\text{MC}(\mathfrak{g} \otimes \mathfrak{m})$ is a pair (α, p) $\exp(\mathfrak{g} \otimes \mathfrak{m})^\circ$ contains $f(d + \alpha)f^{-1} = d + \beta$.

Example: X =compact complex manifold, $\text{Def}_{\mathbb{C}[t]/(t^2)}(X) = H^1(X, T_X)$.

$\mathfrak{g}(X) = \Gamma(X, \mathcal{C}_U(T_X))$ and $\mathfrak{m} = \mathbb{C}.t$ and $\mathfrak{m} \otimes \mathfrak{g} = t.\mathfrak{g} = \mathfrak{g}$

Amnon says this won't work because \mathfrak{g} not abelian????

Properties:

- (1) Invariance Theorem: $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ $\mathfrak{g}^{<0} = \mathfrak{h}^{<0} = 0$ $\varphi^* : \text{Def}(\mathfrak{g}) \rightarrow \text{Def}(\mathfrak{h})$ is an isomorphism of functors
- (2) Representability Theorem. Assume $\mathfrak{g}^{<0} = 0$ and $\dim \mathfrak{g}^1 < \infty$. Then $\text{Def}(\mathfrak{g})$ is prorepresentable by $H_0(\mathfrak{g})^*$ which is a cocommutative coalgebra.
- (3)

2.3. Linear Deformation Theory. Want to deform modules. Given $E \in \mathcal{T}$ where \mathcal{T} comes from a DG-category.

dgart=DG artinian algebras (R/\mathfrak{m}) such that $R/\mathfrak{m} = k$ dgart_=DG artinian algebras living in non-positive degree.

\mathcal{A} a DG-category $E \in \mathcal{A}^0\text{-mod}$.

$\mathcal{A}_R := \mathcal{A} \otimes_k R$

$$\begin{array}{ccc} & \xleftarrow{i^*} & \\ \text{Mod } \mathcal{A}_R^0 & \xrightarrow{i_*} & \text{mod } \mathcal{A}_R \\ & \xleftarrow{i^!} & \end{array}$$

$i^*M = M \otimes_R k$ and $i^! = \text{Hom}_R(k, M)$

Ob $\text{Def}_R^k(E)$ has objects (S, σ) $S \in \mathcal{A}_R^0$ where $\sigma : i^*S \rightarrow E$ is an isom
 \exists an isom $\eta : (E \otimes R)^{\text{gr}} \rightarrow S^{\text{gr}}$.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & X \\ \downarrow f & & \downarrow \\ \mathcal{Y} & \longrightarrow & Y \end{array}$$

2.4. DG algebra.

2.5. Example.

2.6. Point-like objects. E such that $\text{RHom}^\bullet(E, E) \cong \Lambda \cdot \text{Ext}^1(E, E)$

3. BONDAL—DERIVED CATEGORIES OF TORIC VARIETIES AND MIRROR SYMMETRY

3.1. Weighted \mathbb{P}^n s and toric stacks. Let X be a projective toric stack over \mathbb{C} with fan Σ_X and dense torus $T \cong (\mathbb{C}^\times)^n$.

If X is smooth, then Σ_X is simplicial i.e., all polytopes are simplices. We should also allow X to be a stack—then Σ_X is still simplicial but can have more general simplices.

Write $\text{Pic}_T X$ for the T -equivariant Picard group. The forgetful functor $\text{coh}_T X \rightarrow \text{coh} X$ on invertible sheaves gives a homomorphism between the two Picard groups:

$$0 \rightarrow M \rightarrow \text{Pic}_T X \rightarrow \text{Pic} X \rightarrow 0$$

and the kernel M is a lattice naturally isomorphic to the character group T^\vee . Write $N = \text{Hom}(M, \mathbb{Z})$ for the dual lattice

Theorem 3.1 (Bondal-Ruan). *There is a fully faithful functor*

$$D_{\text{coh}}^b(X) \rightarrow D^b(\mathbb{T}, \Sigma)$$

where the RHS is the category of constructible sheaves on a real torus $\mathbb{T} = (S^1)^n$ with respect to a stratification Σ . This is an equivalence under some conditions (*) (which imply X is Fano); the conditions hold in $\dim X = 2$ when X is del Pezzo. In $\dim 3$ there are 18 Fanos, and the conditions hold for 16 of them.

Should think of $D^b(\mathbb{T}, \Sigma)$ as mirror

$$D_{\text{coh}}^b(X) \equiv D^b(\mathbb{T}, \Sigma) \equiv \text{Fuk}(Y, \omega).$$

Mirror symmetry is a relation between algebraic geometry and symplectic geometry

$$D_{\text{coh}}^b(X) \equiv \text{Fuk}(Y, \omega)$$

e.g., when (X, J) is Calabi-Yau variety there is a (Y, ω) on the RHS

Fano varieties are NOT Calabi-Yau. For Fano varieties we use a Landau-Ginzburg potential W and associate to (Y, ω, W) a category $\underline{\text{Fuk}}(Y, \omega, W)$ that is a replacement for the Fukaya category.

Theorem 3.2 (Kawamata). *If X is a toric stack, then $D^b(X)$ has a full exceptional collection of coherent sheaves.*

Conjecture: there is a full exceptional collection of *line bundles*.

Exceptional collection corresponds to critical points for W but sometimes Y is affine so some critical points may be missing so don't get line bundles????

3.2. Define the real torus $\mathbb{T} := M_{\mathbb{R}}/M$.

For each $\ell \in \mathbb{N}$, define G_ℓ to be the kernel in the exact sequence

$$0 \longrightarrow G_\ell \longrightarrow T \xrightarrow{F_\ell} T \longrightarrow 1$$

where $F_\ell(t_1, \dots, t_n) = (t_1^\ell, \dots, t_n^\ell)$. Since G_ℓ is the ℓ -torsion subgroup it is isomorphic to $(\mu_\ell)^n$. Applying $\text{Hom}(-, \mathbb{C}^\times)$ to this gives an exact sequence

$$0 \longrightarrow M \xrightarrow{\ell} M \longrightarrow G_\ell^\vee \cong \frac{M}{\ell M} \longrightarrow 0$$

in which G_ℓ^\vee is the character group of G_ℓ . The last isomorphism is natural so composing with the multiplication map by ℓ^{-1} , namely

$$\frac{1}{\ell} : \frac{M}{\ell M} \rightarrow \frac{M \otimes \mathbb{R}}{M} \cong \mathbb{T}$$

we get a natural inclusion $G_\ell^\vee \rightarrow \mathbb{T}$ with image the set of rational points in \mathbb{T} with denominator ℓ .

Now want to define a map $G_\ell^\vee \rightarrow \text{Pic } X$.

The map³ $F_\ell : T \rightarrow T$ extends to $F_\ell : X \rightarrow X$. There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_\ell} & X \\ \uparrow & & \uparrow \\ T & \xrightarrow{F_\ell} & T \end{array}$$

We can decompose

$$(F_\ell)_* \mathcal{O}_X = \bigoplus_{\chi \in G_\ell^\vee} \mathcal{L}_\chi$$

as a direct sum of line bundles. The rule $\chi \mapsto \mathcal{L}_\chi$ gives us $G_\ell^\vee \rightarrow \text{Pic } X$.

Taking the dual of $G_\ell \rightarrow T$ gives a map $G_\ell^\vee \rightarrow T^\vee = M \rightarrow M_\mathbb{R} \rightarrow \mathbb{T}$. There is unique Φ making

$$\begin{array}{ccc} G_\ell^\vee & \longrightarrow & \text{Pic } X \\ \downarrow & \nearrow \Phi & \\ \mathbb{T} & & \end{array}$$

commute.

Let D_1, \dots, D_N be T -invariant prime divisors giving a basis for $\text{Pic}_T X$. If $D = \sum a_i D_i \in M_\mathbb{R}$ then

$$\Phi(D) := \sum [a_i] D_i \in \text{Pic } X$$

where $[a_i]$ is the integer part. Clearly, $B = \text{Im } \Phi$ is a finite set. Define the strata of \mathbb{T} to be the level sets of Φ . Associate to each stratum the line bundle that is its image.

(My comment: all this seems to depend on choice of ℓ ...??) Bondal says independent of ℓ and under some conditions B provides a complete strong exceptional collection.

Now associate a quiver to the stratification: pick a point x_S in each stratum S , and draw one arrow $S \rightarrow S'$ for each homotopy class of oriented paths from x_S to $x_{S'}$ that are compatible with Σ in the sense that a path can pass through a point on the boundary of a stratum only on the way inside the stratum.

3.3. The case of \mathbb{P}^1 . For $X = \mathbb{P}^1$, $\mathbb{T} = S^1$. Then Σ is the stratification of S^1 with two strata, a point p and its complement $S^1 - \{p\}$. Say $p = 0$ and pick $\infty \in S^1 - \{p\}$. The quiver has vertices 0 and ∞ and two arrows from ∞ to 0 , one arrow for each of the two paths from ∞ to 0 .

Associate \mathcal{O} to the point 0 and $\mathcal{O}(-1)$ to the other stratum. Two T -invariant divisors on \mathbb{P}^1 , 0 and ∞ and $\Phi(a_0 D_0 + a_\infty D_\infty) = \mathcal{O}(a_0 + a_\infty)$.

Let $i : \{p\} \rightarrow S^1$ and $j : S^1 - \{p\} \rightarrow S^1$ be the inclusions. Let \mathcal{F} be a constructible sheaf of \mathbb{C} -vector spaces⁴ on S^1 with respect to Σ . Then $j^* \mathcal{F}$ is a sheaf on the contractible space $S^1 - \{p\}$ so is constant, say V_1 . Let $V_0 = i^* \mathcal{F}$.

³cf. the Frobenius map

⁴A sheaf on a stratified space is **constructible** if its restriction to each stratum is locally constant—e.g., the solution complex of a holonomic \mathcal{D} -module.

For an open set $U \subset S^1$,

$$\mathcal{F}(U) \cong \begin{cases} V_0 & \text{if } p \in U, \\ V_1 & \text{if } p \notin U. \end{cases}$$

Take an open set U containing p , and open sets $U_1, U_2 \subset U$ that do not contain p , one to the right of p , the other to the left. There are restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(U_i)$ that then give two maps

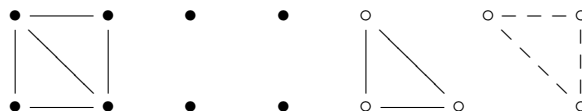
$$V_0 \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} V_1$$

In this way we get a functor from $D^b(\mathbb{T}, \Sigma)$ to the derived category of representations of the Kronecker quiver, but the latter is equivalent to $D^b(\mathbb{P}^1)$.

Which sheaves are sent to the indecomposable projectives? Maybe $i_*\mathbb{C}_p$ and $j_*\mathbb{C}_{S^1-p}$??

More generally, if $i : S \rightarrow \mathbb{T}$ is the inclusion of a stratum I think $i_*\mathbb{C}_S$ is injectives and $i_*\mathbb{C}_S$ is projective.

3.4. The case of \mathbb{P}^2 . Now $\mathbb{T} = S^1 \times S^1$, the 2-torus, and he draws it as $\mathbb{R}^2/\mathbb{Z}^2$. There are three strata.



The four bullets are identified to a single point in \mathbb{T} forming a stratum, S_0 . The lower left triangle, without its vertices but with its edges and interior forms a stratum S_1 . The interior of the top right triangle forms a stratum, S_2 .

The quiver is

$$S_2 \rightrightarrows S_1 \rightrightarrows S_0$$

How to get this? No oriented paths leaving S_0 are compatible with the stratification. Picking x_1 in the interior of S_1 , there are three homotopy classes from x_1 to the three vertices of that triangle; since the edges of S_1 belong to S_1 , *not* to S_2 , there can be no paths leaving S_1 . There was lots here that I did not type!

3.5. Vanishing Cycles. Let $f : M \rightarrow S$ be a proper holomorphic map from an n -dimensional complex manifold with boundary to a Riemann surface. Suppose f has no critical points on the boundary of M and only non-degenerate critical points on the interior with distinct critical values. Let $\gamma : [0, 1] \rightarrow S$ be a path such that $\gamma(0)$ is a critical value of f and $\gamma(t)$ is a regular value for all $t \in (0, 1]$ For each $U \subset [0, 1]$ define

$$M_U := \{(x, t) \in M \times U \mid f(x) = \gamma(t)\}.$$

Then $H_n(M_{[0,1]}, M_1) \cong \mathbb{Z}$. An n -chain Δ on $M_{[0,1]}$ is called a **Lefschetz thimble** if it generates this group, and its boundary $\partial\Delta$ is then called a **vanishing cycle**. It is uniquely determined by γ up to sign.

3.6. The LG-potential for \mathbb{P}^2 . The fan for \mathbb{P}^2 has rays spanned by $(0, 1)$, $(1, 0)$, $(-1, -1)$. Define a LG-potential⁵

$$W = x + y + \frac{1}{xy}.$$

⁵Notice W encodes the rays of the fan.

lots here i didn't type
Fuk depends only on $[\omega] \in H^2(Y)$.

4. BEREST—CHEREDNIK ALGEBRAS, CALOGERO-MOSER SPACES, ...

Work in progress with O. Chalykh

4.1. Calogero-Moser spaces. Fix $n \in \mathbb{N}$ and define

$$\widetilde{CM}_n := \left\{ (\bar{X}, \bar{Y}) \in M_n(\mathbb{C})^2 \mid \text{rank}([\bar{X}, \bar{Y}] + \text{id}) = \begin{cases} 0 & n=0, \\ 1 & \text{otherwise.} \end{cases} \right\}$$

Then $\text{GL}(n, \mathbb{C})$ acts on \widetilde{CM}_n by simultaneous conjugation—it is a free action so can form the quotient

$$CM_n := \widetilde{CM}_n // \text{GL}(n, \mathbb{C}).$$

Facts: [See Wilson 1998]

- (1) CM_n is a smooth connected affine variety of dimension $2n$
- (2) it has a natural symplectic structure
- (3) $\cong \text{Hilb}_n(\mathbb{C}^2)$ as a real C^∞ manifold
- (4) can interpret CM_n as completed phase space of the rational Calogero-Moser system of n particles on \mathbb{C} with Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^n p_i^2 - \sum_{i < j} \frac{1}{(x_i - x_j)^2}.$$

- (5) can interpret $\sqcup_{i=1}^\infty CM_n$ as the space of rational solutions (decaying at ∞ in x) of the KP hierarchy of non-linear PDEs, the first of which is

$$\frac{3}{4} u_{yy} = (u_t - \frac{1}{4}(u_{xxx} + 6uu_x))_x$$

4.2. $\Pi_\lambda(Q)$ and the Weyl algebra. For any finite quiver Q with vertex set I and any $\lambda : I \rightarrow \mathbb{C}$ define the deformed pre-projective algebra

$$\Pi_\lambda(Q) := \frac{\mathbb{C}\bar{Q}}{(\sum_a [a, a^*] - \sum_{i \in I} \lambda_i e_i)}.$$

Let Q be the quiver

$$e_0 \longrightarrow e_1 \curvearrowright$$

So its double \bar{Q} is

$$\begin{array}{ccc} & & x \\ & & \downarrow \\ e_0 & \xrightarrow{i} & e_1 \\ & \xleftarrow{j} & \uparrow \\ & & y \end{array}$$

For $\lambda = (\lambda_0, \lambda_1)$, $\Pi_\lambda(Q)$ is the path algebra of \bar{Q} modulo the relation

$$[x, y] + ij - \lambda_1 e_1 = ji + \lambda_0 e_0.$$

Proposition 4.1. *The Weyl algebra is $A = A_1(\mathbb{C}) = \mathbb{C}[X, Y]$ with relation $XY - YX = 1$. Let $\Pi = \Pi_\lambda(Q)$ be the deformed pre-projective algebra above. Then*

$$\frac{\Pi}{\Pi e_0 \Pi} \cong A_1(\mathbb{C}).$$

4.3. A Rational Cherednik algebra.

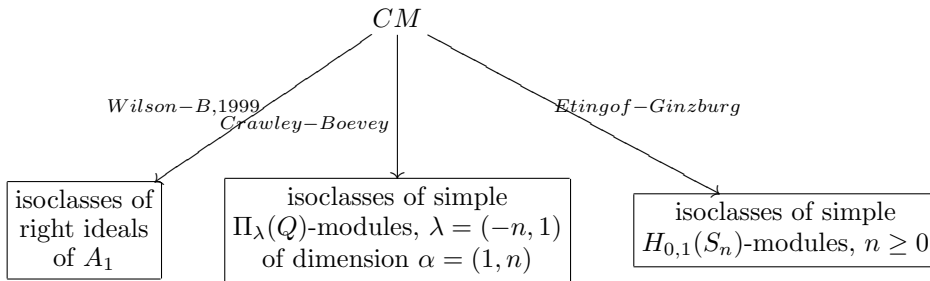
$$W = S_n, \quad \mathfrak{h} = \mathbb{C}^n, \quad t = 0, \quad t = 1.$$

is $H = H_{0,1}(S_n) = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n][s_{ij}]$ with relations

$$[y_i, x_j] = s_{ij} \quad (i \neq j), \quad [y_k, x_k] = \sum_{i \neq k} s_{ik}.$$

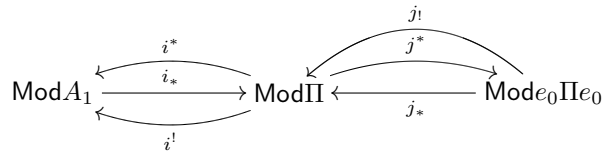
Define

$$CM := \bigsqcup_{i=0}^{\infty} CM_n$$



Question: explain the mysterious bijection between simple $H_{0,1}(S_n)$ -modules and left ideals of A_1 . More generally, explain the relation between simple modules over $H_{0,1}(S_n)$ and $\Pi_\lambda(Q)$ and left ideals of A_1 .

4.4. Functors. There are adjoint triples $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ of functors

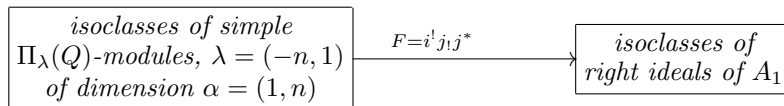


The first three functors are induced by the homomorphism $\Pi_\lambda(Q) \rightarrow A_1$ in the obvious way. The Π - $e_0\Pi e_0$ -bimodule Πe_0 gives an exact functor $j^* = - \otimes_{\Pi} \Pi e_0$ which has right adjoint $j_* \text{Hom}_{e_0\Pi e_0}(\Pi e_0, -)$. If X is a right Π -module, then

$$j^* X = X \otimes_{\Pi} \Pi e_0 = X e_0 \cong \text{Hom}_{\Pi}(e_0 \Pi, X).$$

So j^* also has a left adjoint $j_! = - \otimes_{e_0 \Pi e_0} e_0 \Pi$.

Theorem 4.2. *There is an injection*



4.5. First step of the proof. Let $X \in \text{Mod}\Pi_\lambda(Q)$ be simple. There is a natural transformation $j_!j^* \rightarrow \text{id}_{\text{Mod}e_0\Pi e_0}$ and the associated map

$$\phi : j_!j^* X \rightarrow X$$

is non-zero, so surjective because X is simple. Since $\dim X < \infty$ and A_1 has no non-zero finite dimensional modules, $i^!X = 0$. Hence if we apply $i^!$, which is left exact, to the exact sequence

$$0 \rightarrow M \rightarrow j_!j^* X \rightarrow X \rightarrow 0$$

we get $M \cong i^!j_!j^* X$.

More to do here!

4.6. Relation to $H_{0,1}(S_n)$.

Theorem 4.3. *There is an algebra isomorphism $\theta : e_0\Pi_\lambda(Q)e_0 \rightarrow eHe$ given by*

$$ja(X, Y)i \mapsto -nea(x_1, y_1)e.$$

The composition functor $\text{Mod}H \rightarrow \text{Mod}(eHe) \rightarrow \text{Mod}B \rightarrow \text{Mod}A$ maps simple H -modules injectively to isoclasses of right ideals in A_1 .

5. HILLE—STRONG EXCEPTIONAL SEQUENCES OF LINE BUNDLES ON TORIC VARIETIES

5.1. Motivation. Work over \mathbb{C} .

Theorem 5.1 (Beilinson). *Define*

$$\Phi = \text{Hom}_{\mathbb{P}^n}(\oplus_{i=0}^n \mathcal{O}(i), -) : \text{coh}\mathbb{P}^n \rightarrow \text{mod}A.$$

Then $R\Phi : D^b(\mathbb{P}^n) \rightarrow D^b(A)$ is an equivalence, and where $A = \text{End}(\oplus_{i=0}^n \mathcal{O}(i))$.

Want to extend this to smooth projective toric varieties.

5.2. Let $\mathcal{E} = (\mathcal{L}_1, \dots, \mathcal{L}_n)$ be a sequence of line bundles on X .

We say \mathcal{E} is

- (1) complete if $n = \text{rank } K_0(X)$;
- (2) full if $\langle \mathcal{L}_1, \dots, \mathcal{L}_n \rangle = D^b(X)$; full \Rightarrow complete
- (3) strongly exceptional if $\text{Ext}^k(\mathcal{L}_i, \mathcal{L}_j) = 0$ for all $k \neq 0$; in that case one can change the ordering so that $\text{Hom}(\mathcal{L}_j, \mathcal{L}_i) = 0 \forall j > i$ ⁶
- (4) exceptional if $\text{Ext}^k(\mathcal{L}_j, \mathcal{L}_i) = 0 \forall j > i$.

5.3. Conjectures. Let X be a smooth projective toric variety from now on!

1. (A. King) (unpubl paper on his home page) Every such X has a full, strongly exceptional sequence.

Kawamata proved X always has a full exceptional sequences of *sheaves*.

2. On any Fano X there exists a helix: i.e., if $\mathcal{E} = (\mathcal{L}_1, \dots, \mathcal{L}_n)$ is a strong exc. seq., so is each length n subsequence of

$$S(\mathcal{E}) = (\dots, \mathcal{L}_{n-1} \otimes \omega, \mathcal{L}_n \otimes \omega, \mathcal{L}_1, \dots, \mathcal{L}_n, \mathcal{L}_1 \otimes \omega^{-1}, \dots)$$

3. On X a full exceptional sequence of line bundles exists.

5.4. Toric surfaces. Write Σ_X for the fan of the toric variety X .

Write $n = \text{rank } K_0(X)$, $d = \dim X$ and $t = \text{rank}$ of the equivariant Picard group. Then $\text{rank Pic}(X) = t - d$ and there are T -invariant prime divisors D_1, \dots, D_t that provide a basis for the equivariant Picard group.

$$\omega^{-1} = \mathcal{O}(\sum_{i=1}^t D_i).$$

Draws a picture of a fan for a toric surface...the cones correspond to the orbits... 0 corresponds to the dense orbit, the rays correspond to the T -invariant divisors, and the big cones correspond to the fixed points. He labels the rays $\mathbb{R}_{\geq 0}v_i$, $i = 1, 2, \dots$, moving counterclockwise round the origin.

Write $P(\Sigma) = \cup_i \text{convexhull}\{v_i, v_{i+1}, 0\}$. Then X is Fano if and only if $P(\Sigma)$ is strictly convex. In that case, one has a *polytope* from which one can recover the fan.

Define a graph $\Gamma_X = (\Sigma \cap S^{d-1})$ with vertices the D_i s, and an edge between i and j if $\text{codim}_{D_i} D_i \cap D_j = 1$. When $\dim X = 2$, this graph is \tilde{A}_{t-1} (and $t = n$).

Example: The Hirzebruch surface \mathbb{F}_n : rays through $(0, 1)$, $(1, 0)$, $(0, -1)$, $(-1, n)$...etc

⁶Can do this because if \mathcal{L} and \mathcal{L}' are non-isomorphic bundles only one of $\text{Hom}(\mathcal{L}, \mathcal{L}')$ and $\text{Hom}(\mathcal{L}', \mathcal{L})$ can be non-zero—enough to prove this when $\mathcal{L}' = \mathcal{O}$; think about $\mathcal{O} \rightarrow \mathcal{L} \rightarrow \mathcal{O}$

5.5. Constructions. For simplicity will focus on line bundles of the form $\mathcal{L} = \mathcal{O}(\sum a_i D_i)$ $a_i \in \{0, \pm 1\}$.

Theorem 5.2. Let $\mathcal{E} = (\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \dots, \mathcal{O}(D_1 + \dots + D_{t-1}))$.

- (1) \mathcal{E} exceptional $\Rightarrow v_1, \dots, v_t, v_1$ is a Hamiltonian cycle in Γ_X
- (2) $\dim X \leq 3$ then iff in (1)
- (3) \mathcal{E} is complete $\Leftrightarrow \mathcal{E}$ is full $\Leftrightarrow t = n \Leftrightarrow X = \mathbb{P}^n$ or $\dim X = 2$
- (4) If X Fano and \mathcal{E} exceptional, then \mathcal{E} is strongly exceptional.

Theorem 5.3. Let $\dim X = 2$ and $\mathcal{E} = (\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \dots, \mathcal{O}(D_1 + \dots + D_{t-1}))$. $\omega^{-1} = \mathcal{O}(\sum_{i=1}^t D_i)$.

- (1) Then \mathcal{E} is full exceptional if we take the cyclic orientation on the divisors
- (2) \mathcal{E} is strongly exceptional iff $D_i^2 \geq -1$ for all $i = 1, \dots, t-1$
- (3) $S(\mathcal{E})$ is a helix iff $D_i^2 \geq -1$ for all i

Take X equal to \mathbb{F}_2 blown up at 3 points, three additional rays through $(1, -1)$, $(2, -1)$, $(3, -1)$ has no strongly exceptional sequence of length 7.

Theorem 5.4. Suppose X is Fano and $\dim X = 3$. Then strongly exceptional sequence exists (complete) and can be completed to a helix.

strongly exceptional sequence exists (complete) on the maximal Fanos. Craw

5.6.

6. B. KELLER—ACYCLIC CALABI-YAU CATEGORIES ARE CLUSTER CATEGORIES

(Joint with I. Reiten.)

6.1. Cluster Categories. Fix field k , Q a finite quiver without oriented cycles. kQ =path algebra, $\dim kQ < \infty$, $\text{mod } kQ$ =category of finite dimensional right kQ -modules. $D^b(kQ)$ =bounded derived category⁷ of $\text{mod } kQ$. S =suspension, $M \mapsto M[1]$. Σ =Serre functor=the unique auto-equivalence such that

$$D \text{Hom}(X, Y) \cong \text{Hom}(Y, \Sigma X)$$

where $D = \text{Hom}_k(-, k)$.

\mathcal{C}_Q =the cluster category = the orbit category of $D^b(kQ)/(\Sigma^{-1} \circ [2])$. Objects in \mathcal{C}_Q are same as those in $D^b(kQ)$ but

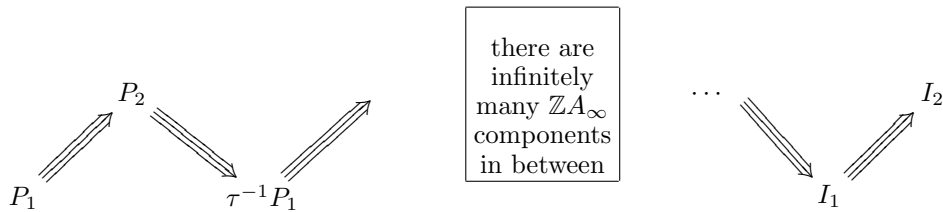
$$\text{Hom}_{\mathcal{C}_Q}(X, Y) := \bigoplus \text{Hom}_{D^b(kQ)}(X, (\Sigma^{-1} \circ [2])^n Y)$$

\mathcal{C}_Q was invented by Buan-Marsh-Reineke-Reiten-Todorov in the general case, and by Caldero-Chapoton-Schiffer for Q of type A_n . Their motivation was to categorify the cluster algebras of Fomin-Zelevinsky (2000)

Example. Let Q be the quiver

$$1 \rightrightarrows 2$$

The AR-quiver for $\text{mod } kQ$ is



The picture of $D^b(\text{mod } kQ)$ should go here



$$F = \Sigma^{-1} \circ [2] = \tau^{-1} \circ [1]$$

The picture of \mathcal{C}_Q should go here. \diamond

⁷Says that formally inverting all quisms gives the same category whether or not one first mods out homotopies.

6.2. Properties of \mathcal{C}_Q .

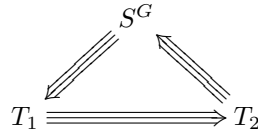
- (1) \mathcal{C}_Q is triangulated, even algebraic triangulated, i.e., $\mathcal{C}_Q \equiv \underline{E}$ = the stable category for some Frobenius category E .
- (2) Hom-finite
- (3) 2-Calabi-Yau, i.e., it has a Serre functor induced by Σ , still call it Σ , so on \mathcal{C}_Q we have an isomorphism of triangle functors $\Sigma \cong [2]$ as triangle functors.
- (4) T_Q = (image of kQ under $\pi : D^b(kQ) \rightarrow \mathcal{C}_Q$) is a cluster-tilting object, i.e.,
 - $\text{Hom}(T_Q, T_Q[1]) = 0$
 - $\text{Hom}(T_Q, X[1]) = 0$ implies $X \in \text{add } T_Q$
- (5) $\text{End}_{\mathcal{C}_Q}(T_Q) \cong kQ$. Hence the quiver of $\text{End}(T_Q)$ has no oriented cycles.

6.3. Main Theorem.

Theorem 6.1. *Suppose k is algebraically closed. If \mathcal{C} is an algebraic 2-CY-category having a cluster-tilting object T such that the quiver of $\text{End } T$ has no oriented cycles, then there is a triangle equivalence $\mathcal{C}_Q \rightarrow \mathcal{C}$ such that $T_Q \mapsto T$.*

Let $G = \mu_3$ act on $S = k[[x, y, z]]$ via multiplication on x, y, z . Then $R = S^G$ is an isolated singularity and Gorenstein, so $\underline{E} := \text{MCM}(R)$ is Frobenius and $\mathcal{C} = \underline{E}$ is 2-CY (Auslander) $T = S$ is cluster-tilting (Iyama), $\text{End}_R T = S \rtimes G$

(6-1)



$\text{End}_{\mathcal{C}}(T) = \underline{\text{End}}_R(T)$, $T_1 \xrightarrow{\text{triple}} T_2$ Main Theorem implies $\mathcal{C} \equiv \mathcal{C}_Q$ where Q is the quiver $1 \xrightarrow{\text{triple}} 2$

Terminology: An acyclic 2-CY-category is a category \mathcal{C} as in the theorem.

Conjecture: If \mathcal{C} is a 2-CY-category with a cluster tilting object T' having ≤ 3 non-isomorphic indecomposable summands (and assume the quiver of $\text{End } T'$ has no loops and no 2-cycles), then \mathcal{C} is an acyclic 2-CY-category.

6.4. The proof. Uses DG-algebra $B = kQ \oplus (DkQ)[-3]$ with $d = 0$

$D^b(B) = \{M \in D(B) \mid \sum \dim H^i(M) < \infty\}$ Full subcategory of $D(B)$.

$\text{per } D^b(B)$ = full triangulated subcategory of $D^b(B)$ stable under factors by B_B

$\mathcal{C}_Q \equiv D^b(B)/\text{per}(B)$.

7. RINGEL—DEGENERATIONS OF MODULES AND THE STRUCTURE OF PRÜFER MODULES

7.1. Degenerations of finite length modules. Defn: W' is a degeneration of W if there is a ses $0 \rightarrow X \rightarrow X \oplus W \rightarrow W' \rightarrow 0$. This is equivalent to the existence of a ses $0 \rightarrow W' \rightarrow Y \oplus W \rightarrow Y \rightarrow 0$.

If W and W' are degenerations of each other, then $W \cong W'$.

Bautista-Perres: Λ =an artin algebra over a commutative artinian ring k , module finite over k . Let W and W' be finite length Λ -modules with $top(W) \cong top(W')$ and $\Omega W \cong \Omega W'$. If $Ext^1(W, W) = Ext^1(W', W') = 0$, then $W \cong W'$.

What is only one of the Ext groups vanishes? If $Ext^1(W, W) = 0$ is W' a degeneration of W ?

Take monics

$$U_1 \begin{array}{c} \xrightarrow{w} \\ \xrightarrow{v} \end{array} U_2$$

Form a pushout

$$\begin{array}{ccccc} U_0 & \xrightarrow{w_0} & U_1 & \longrightarrow & W \\ v_0 \downarrow & & \downarrow v_1 & & \downarrow = \\ U_1 & \xrightarrow{w_1} & U_2 & \longrightarrow & W \\ \downarrow & & \downarrow & & \\ V & \longrightarrow & W & & \end{array}$$

Set $V = \text{coker } w$ and $V = \text{coker } v$. If $Ext^1(W, W) = 0$ then V is a degeneration of W .

Repeat this process.. obtain

$$\begin{array}{ccccccc} U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \longrightarrow & \dots \longrightarrow U_\infty = \bigcup U_i \\ v_0 \downarrow & & \downarrow v_1 & & \downarrow v_2 & & \downarrow v_\infty \\ U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \longrightarrow & \dots \longrightarrow U_\infty \end{array}$$

Then get

$$\begin{array}{ccccc} & & \varphi & & \\ & & \curvearrowright & & \\ U_\infty/U_0 & \xrightarrow{\cong} & U_\infty/U_1 & \xleftarrow{\text{projection}} & U_\infty/U_0 \end{array}$$

Properties of ϕ :

- (1) surjective
- (2) locally nilpotent
- (3) $\ker \phi = U_1/U_0$ has finite length

I stopped taking notes.....

8. BROWN—NOETHERIAN HOPF ALGEBRAS

8.1. Hopf background. He will insist that the antipode S is bijective.

If A is a finite dimensional Hopf algebra it is Frobenius, i.e., self-injective, so $\text{Hom}_k(A, k) \cong {}^\nu A^1$ as a bimodule, where ν is the Nakayama automorphism. Say that A is symmetric if $\nu = \text{id}_A$.

When $\dim_k A < \infty$, $\text{Hom}_A({}_A k, A) \cong k$ and call the unique copy of ${}_A k$ as a left submodule of A the left integral and denote it by $\int_A^\ell \trianglelefteq A$; can do this on the right too. Say A is unimodular if $\int_A^\ell = \int_A^r$

Theorem 8.1 (Oberst, Schneider, 1973). $\nu = \text{id}_A$ if and only if A is unimodular and $S^2 = 1$.

8.2. Infinite dimensional Hopf algebras. Is every noetherian Hopf algebra AS-Gorenstein?

Yes for $U(\mathfrak{g})$, $U_q(\mathfrak{g})$, $\mathcal{O}(G)$, $\mathcal{O}_q(G)$, $A = kG$ when G is polycyclic-by-finite, and when A is affine noetherian PI (Wu-Zhang, 2003).

One requirement for AS-Gorenstein is that $\text{Ext}_A^d(k, A) \cong k_A$ which is a d -dimensional analogue of the fact that $\text{Hom}_A({}_A k, A)$ is 1-dimensional in the finite dimensional case. Wu-Zhang define homological left and right integrals,

$$\int_A^\ell = \text{Ext}_A^d({}_A k, A) \quad \text{and} \quad \int_A^r = \text{Ext}_A^d(k_A, A).$$

8.3. Dualizing complexes. Let B be a k -algebra and define $B^e := B \otimes_k B^{\text{op}}$. A complex $R \in \text{D}^b(\text{mod } B^e)$ is a dualizing complex if

- (1) R_B and ${}_B R$ have finite injective dimension
- (2) R is homologically finite
- (3) $B \rightarrow \text{RHom}^\bullet_B(R_B, R_B)$ and $B \rightarrow \text{RHom}^\bullet_B({}_B R, {}_B R)$ are isomorphisms.

Theorem 8.2 (VdB). (1) If a rigid R exists it is unique up to unique isomorphism

- (2) Suppose $\text{injdim } B = d < \infty$ and \exists an invertible B -module U such that

$$\text{Ext}_{B^e}^i(B, B^e) = \begin{cases} U & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

Then B has a rigid R and $R = U^{-1}[d]$.

Theorem 8.3 (Brown-Zhang). Let A be an AS-Gorenstein Hopf algebra.

- (1) If a rigid R exists it is unique up to unique isomorphism
- (2)

$$\text{Ext}_{A^e}^i(A, A^e) = \begin{cases} {}^1 A^\nu & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

- (3) A has a rigid dualizing complex $R = {}^\nu A^1[d]$ where ν is an algebra automorphism of A unique up to inner automorphism (call ν “the” Nakayama automorphism)
- (4) $\nu = S^2 \tau_{\pi_0}^\ell$.

$\pi : A \rightarrow k$ an algebra homomorphism gives a left winding automorphism

$$\tau_\pi^\ell : A \rightarrow A, \quad a \mapsto \sum \pi(a_1) a_2.$$

Use $\pi_0 : A \rightarrow A/\text{rann}(f_A^\ell)$.

8.4. Hochschild (co)homology.

Corollary 8.4. *Suppose A is AS-regular, $\text{gldim } A = d$.*

- (1) *Then $HH^i(A, M) \cong \mathbb{H}_{d-i}(A, {}^1M^\nu)$ for all i and all bimodules M .*
- (2) *A has a rigid $R = {}^\nu A^1[d]$.*
- (3) *$(A, \Delta^{\text{op}}, S^{-1}, \varepsilon)$ is Hopf and has Nakayama automorphism $\nu' = S^{-2}\tau_{\pi_0}^r$ so equal to the previous one and $S^4 = \gamma(\tau_{\pi_0}^r)(\tau_{\pi_0}^\ell)^{-1}$ for some inner γ .*

Note this recovers the Hochschild-Kostant-Rosenberg result.

9. STROPPEL—SERRE FUNCTORS, SYMMETRIC ALGEBRAS, AND TQFT

9.1. Serre functors. Let \mathcal{C} is a \mathbb{C} -linear category with $\dim \text{Hom}(-, -) < \infty$. A functor $S : \mathcal{C} \rightarrow \mathcal{C}$ is a Serre functor if

- (1) it is an equivalence and
- (2) there are functorial natural isomorphisms $\text{Hom}(M, N) \cong \text{Hom}(N, SM)^*$.

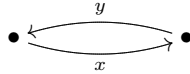
Let A be a finite dimensional \mathbb{C} -algebra and \mathcal{C} =free A -modules. Then S is a Serre functor if and only if $\text{Hom}_A(A, A) \cong \text{Hom}(A, SA)^*$, i.e., $A \cong (SA)^*$ as left modules and, the requirement that we have a *natural* isomorphism of bifunctors is equivalent to the requirement that $A \cong (SA)^*$ as bimodules. Hence $S \cong \text{id}_{\mathcal{C}}$ if and only if A is symmetric (i.e., Calabi-Yau of dimension 0!).

What is the Serre functor for $D^b(A)$? Kapranov: $D^b(A)$ has a Serre functor if and only if $\text{gldim } A < \infty$, and in this case $S = \mathcal{L}(A^* \otimes_A -)$.

Let $G = \text{SL}(n, \mathbb{C})$ and let B be the Borel subgroup. What is S for $D^b(\text{Perv}_G(G/B))$? We know this category is equivalent to $D^b(A)$ with $\dim A < \infty$, but don't know A .

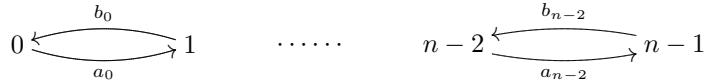
What is $D^b(\mathcal{O}_{\text{triv}}(\mathfrak{sl}_n))$?

For $n = 2$, $\text{Perv}(G/B) \cong \text{mod } A$ where A is the path algebra of the quiver



with relations $xy = 0$. $A_1 = \text{End}(\text{proj} - \text{inj}) = P(1) \oplus P(s)$, $P(s) = \text{proj-inj}$, $\text{End } P(s) \cong \mathbb{C}[x]/(x^2)$

For $\text{Perv}(G/P)$ with P maximal parabolic



with relations $\rightarrow \rightarrow = 0 = \leftarrow \leftarrow$ and $a_0 b_0 = 0$ and at each vertex

$$\begin{array}{c} \curvearrowright \bullet \curvearrowleft \\ \curvearrowleft \bullet \curvearrowright \end{array} = 0$$

$$X_i = Ae_i \otimes e_i A \otimes_A -$$

Theorem 9.1 (Stroppel-Mazorchuk). $G = \text{SL}(n, \mathbb{C})$? Let $\mathcal{P} = \text{Perv}(G/B)$. For any simple reflection $s \in S_n$ there is a right exact functor $T_s : \mathcal{P} \rightarrow \mathcal{P}$ such that

- (1) $\mathcal{L}T_s : D^b(\mathcal{P}) \rightarrow D^b(\mathcal{P})$ is an equivalence
- (2) they satisfy the braid relations,
- (3) $w_0 = s_{i_1} \cdots s_{i_r} \in S_n$ the longest element, the Serre functor is $(\mathcal{L}T_{s_{i_1}} \cdots \mathcal{L}T_{s_{i_1}})^2 = (\gamma \mathcal{L}T_{w_0})^2$.
- (4) $S = \text{partial co-approx wrt all simples that are not in head(proj-inj)}$.

9.2. Applications. Proof of Kapranov's conjecture (pved independently by Bezrukavnikov-Beilinson-Mirkovic).

$T = \oplus \text{indec. proj.}$ in $\text{Perv}(G/P)$, $G = \text{SL}(n, \mathbb{C})$, then $\text{End } T$ is symmetric. Big question: what is $\text{End } T$? It depends only on the partition of n

10. TQFT

Let A_n be the algebra such that $\text{mod } A_n \cong \text{Perv}(G/P)$ for $G = \text{SL}(2n, \mathbb{C})$ and $P \leftrightarrow [n, n]$ $\mathcal{L}T_{s_i} : D^b(\text{mod } A_n) \rightarrow D^b(\text{mod } A_n)$ corresponds to a simple crossing

of adjacent strands at i , and $(\mathcal{L}T_{s_i})^-$ corresponds to the other simple crossing of adjacent strands at i (the inverse crossing).

Theorem 10.1. *There is a functor*

$$\Phi : \{\text{oriented tangles}\} \longrightarrow \text{“my category”} =$$

Therefore get a functorial tangle invariant.

Objects	$2\mathbb{N}$	$D^b(A_n - \text{mod})$
Morphisms	tangles	functors
2-morphisms	cobordisms	natural transformations

The category of 2-dimensional TQFTs is equivalent to the category of finite dimensional Frobenius algebra $(A, m, \Delta, \varepsilon, \eta)$. Topologically a TQFT is a functor on the category of 2-cobordisms

$$2 - Cob \rightarrow \text{mod}k$$

The objects in $2 - Cob$ are oriented compact 1-manifolds and morphisms are cobordisms

10.1.

10.2.**11. MARTINO—SYMPLECTIC REFLECTION ALGEBRAS AND DEFORMED PRE-PROJECTIVE ALGEBRAS**

11.1. Poisson varieties. Work over \mathbb{C} . Let A be a Poisson algebra. For each $f \in A$, call $\Theta_f := \{f, -\}$ a Hamiltonian vector field.

Example: Let (V, ω, G) be a symplectic vector space with a finite group $G \subset \mathrm{Sp}(V)$. Then $(SV)^G$ is Poisson so V/G is a Poisson variety.

Let $x \in X$. The symplectic leaf through x , denoted $S(x)$, is obtained by taking the integral curves with respect to the Hamiltonian vector fields, and close up under this process. Each $S(x)$ is a symplectic manifold. Then X is a disjoint union of symplectic leaves. Note the leaves need not be algebraic subvarieties!

11.2. SRAs at $t = 0$. Take data (V, ω, G) as above and assume the data is indecomposable. Define $H_{t,c}$ a deformation of $SV \rtimes G$... Define $Z_c = Z(H_{0,c})$. This gets a Poisson structure by quantization with respect to t .

11.3. Deformed pre-projective algebras. Let Q be a finite quiver and \bar{Q} its double. Fix a dimension vector α . Then $\mathrm{Rep}(\bar{Q}, \alpha) = T^*\mathrm{Rep}(Q, \alpha)$, so is symplectic, and $G = G(\alpha)$ acts on it. There is a moment map

$$\mu : \mathrm{Rep}(\bar{Q}, \alpha) \rightarrow (\mathrm{Lie } G)^*$$

Let λ be a fixed point in $(\mathrm{Lie } G)^*$; then $\mathcal{N}(\lambda, \alpha) := \mu^{-1}(\lambda)//G$ is a Poisson variety and is the moduli space of semisimple representations of dimension α of the pre-projective algebra $\Pi_\lambda(Q)$.

Theorem 11.1. (1) *The symplectic leaves of $\mathcal{N}(\lambda, \alpha)$ are the representation type strata (i.e., according to the isotypic components)*
 (2) *For $V = (\mathbb{C}^2)^{\oplus n}$ and $G = S_n \wr \Gamma$ where $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$, $\mathrm{Spec } Z_c \cong \mathcal{N}(\lambda(c), \alpha)$ for a quiver Q_G and this identifies symplectic leaves.*
 (3)

Q_G is McKay quiver with an extending vertex ∞

12. I. REITEN—TILTING MODULES FOR CALABI-YAU ALGEBRAS

Joint with Iyama.

R commutative, noetherian ring, Λ an R -algebra, finitely generated as an R -module. Assume $R \subset \Lambda$.

12.1. Calabi-Yau algebras. Λ is n -Calabi-Yau if there are functorial isomorphisms

$$\text{Hom}(X, Y[n]) \cong D \text{Hom}(Y, X)$$

for all $X, Y \in D^b(\text{finite length } \Lambda\text{-modules})$.

Λ is n -Calabi-Yau⁻ if there are functorial isomorphisms

$$\text{Hom}(X, Y[n]) \cong D \text{Hom}(Y, X)$$

for all $X \in D^b(\text{finite length } \Lambda\text{-modules})$ and $Y \in K^b(\text{per } \Lambda)$.

Rickard: $n\text{-CY} \implies n\text{-CY}^-$.

Theorem 12.1. R local Gorenstein of dimension d

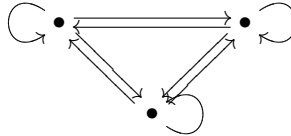
- (1) If Λ is $n\text{-CY}^-$, then $n = d$
- (2) Λ is $d\text{-CY} \iff \Lambda$ is a symmetric R -order, i.e., $\text{Hom}_R(\Lambda, R) \cong \Lambda$ as R - R -bimodule and $\Lambda \in \text{MCM}(R)$
- (3) Λ is $d\text{-CY} \iff \Lambda$ is a symmetric R -module and $\text{gldim } \Lambda = d$

Λ local commutative and $d\text{-CY}^-$ (resp., $d\text{-CY}$), then Λ is Gorenstein (resp., regular).

E.G., char $k = 0$ and $S = k[[x, y, z]]^G$ where $G \subset \text{SL}(3)$ is finite, $R = S^G$ and $\Lambda := S \rtimes G$ is 3-CY.

E.G., $G = \mu_3$ the quiver is (6-1)

$$\text{E.G., } G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \zeta = e^{2\pi i/3} \text{ Then } Q \text{ is}$$



12.2. Tilting theory.

12.2.1. R =complete local Gorenstein domain. A f.gend. Λ -module T is tilting if $\text{pdim}_\Lambda T \leq 1$, $\text{Ext}_\Lambda^1(T, T) = 0$, and there is a ses $0 \rightarrow \Lambda \rightarrow T_0 \rightarrow T_1 \rightarrow 0$ with $T_i \in \text{add } T$.

Deal mainly with T a reflexive R -module; if $d = 3$, then T reflexive implies $\text{pdim } T \leq 1$.

12.2.2. Let $\Lambda = P_1 \oplus \dots \oplus P_n$ where each P_i is indec. proj., Assume $\text{Ext}_\Lambda^1(S, S) = 0$ for all simple S . Let $d = 3$. Then Λ is a tilting module.

Fact: if Λ is 3-CY, then for each j , there exists a unique $P_j^* \not\cong P_j$ such that $T = \Lambda/P_j \oplus P_j^*$ is a tilting module. Note $P_j^* = \Omega^2 S_j$. Since P_j^* is reflexive, so is T . Then $\Lambda' := \text{End } T$ is derived equiv to Λ , so is also 3-CY.

Let $\mu_j(\Lambda) := \Lambda'$. Then $\mu_j(\mu_j(\Lambda)) = \Lambda$.

Can repeat this to get a sequence $\Lambda, \Lambda', \Lambda'', \dots \Lambda^{(n)}$.

Theorem 12.2. *Let Λ be 3-CY. If T_1 and T_2 are reflexive tilting modules, and $\Gamma_i := \text{End } T_i$, then $M := \text{Hom}_\Lambda(T_1, T_2)$ is a reflexive tilting Γ_1 -module and $\text{End}_{\Gamma_1} M \cong \Gamma_2$.*

12.2.3. Let Λ be 3-CY and $T = T_1 \oplus \cdots \oplus T_n$ a tilting module. For each j , there exists a unique $T_j^* \not\cong T_j$ such that $T' := T/T_j \oplus T_j^*$ is tilting.

Define $\mu_j(T) = T'$.

Theorem 12.3. *Let Λ be 3-CY. If T_1 and T_2 are reflexive tilting modules, and $\Gamma_i := \text{End } T_i$, then $M := \text{Hom}_\Lambda(T_1, T_2)$ is a reflexive tilting Γ_1 -module and $\text{End}_{\Gamma_1} M \cong \Gamma_2$.*

12.3. Connections with Van den Bergh's ncCR. R =normal Gorenstein domain, $\dim R = 3$, Λ an R -algebra

Definition 12.4. A Λ -module M with $\Gamma := \text{End}_\Lambda M$, gives a non-commutative crepant resolution (ncCR) if

- $M \cong M^{\vee\vee}$ and is a height one generator, i.e., $M_{\mathfrak{p}}$ is a $\Lambda_{\mathfrak{p}}$ -generator for all height one primes \mathfrak{p}
- $\Gamma_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$ -order with $\text{gldim } \Gamma_{\mathfrak{p}} = \text{ht } \mathfrak{p}$ for all maximal \mathfrak{p}

Conjecture (VdB): If M_1 and M_2 give ncCRs, then Γ_1 and Γ_2 are derived equivalent. ($\Gamma_i = \text{End}_\Lambda M_i$)

Theorem 12.5. *Let Λ be 3-CY. If M_1 and M_2 give ncCRs, then $M = \text{Hom}_\Lambda(M_1, M_2)$ is a reflexive tilting Γ_1 -module and $\text{End}_{\Gamma_1} M \cong \Gamma_2$, so Γ_1 and Γ_2 are derived equivalent.*

Lemma 12.6 (Key). *If Λ is an isolated sing, R local, $\text{depth Hom}(M, N) \geq 3$, then $\text{Ext}_\Lambda^1(M, N) = 0$.*

Theorem 12.7. *Let Λ be 3-CY. Then $\{M \mid M \text{ gives a ncCR}\} = \{\text{reflexive tilting modules}\}$.*

12.4. Connections with cluster algebras. Q a quiver with vertices $1, \dots, n$ with no loops or cycles of length 2.

Let R be a complete local Gorenstein ring and write $k = R/\mathfrak{m}$ is algebraically closed.

$\mathcal{A}(Q) \subset \mathbb{Q}(x_1, \dots, x_n)$

Associate to a 3-CY Λ a quiver Q_Λ .

13. J. BELL—PRIMITIVITY AND TWISTED HOMOGENEOUS COORDINATE RINGS

(with Dan Rogalski)

k =algebraically closed field, X =projective variety, $\sigma \in \text{Aut } X$, $\mathcal{L}=\sigma$ -ample invertible \mathcal{O}_X -module.

When is $B(X, \sigma, \mathcal{L})$ primitive? Want answer in terms of centers and height one primes. Idea is like Dixmier's result: $U(\mathfrak{g})/\mathfrak{p}$ is primitive if and only if it has finitely many height one primes and $Z(\text{Fract } U/\mathfrak{p}) = k$.

Theorem 13.1. *Suppose k is uncountable. The following are equivalent:*

- (1) $B(X, \sigma, \mathcal{L})$ is primitive
- (2) $\exists x \in X$ such that $\{\sigma^n(x) \mid n \in \mathbb{Z}\}$ is dense
- (3) $B(X, \sigma, \mathcal{L})$ has finitely many height one primes
- (4) $B(X, \sigma, \mathcal{L})$ has countably many height one primes
- (5) $Z(\text{Fract}_0 B(X, \sigma, \mathcal{L})) = Z(k(X)[t, t^{-1}; \sigma]) = k$

14. IAIN GORDON— q -SCHUR ALGEBRAS, CHEREDNIK ALGEBRAS AND QUIVER VARIETIES

14.1. q -Schur algebras. G = complex reflection group, \mathfrak{h} = reflection representation, i.e., only one non-trivial eigenvalue (any root of 1 is allowed), \mathcal{S} = generating set of reflections.

Let $\gamma : \mathcal{S}/\sim \rightarrow \mathbb{Q}$ be a class function; set $\underline{q} := e^{2\pi i \gamma} \in \mathbb{C}^\times$. Let $\mathcal{H}_G(\underline{q})$ = the cyclotomic Hecke algebra; it has two kinds of relations, braid relations (analogues of $T_1 T_2 T_1 = T_2 T_1 T_2$), and Hecke relations (like $(T - q)(T + 1) = 0$). At $\underline{q} = 1$, $\mathcal{H}_G(\underline{q}) = \mathbb{C}G$. When $G = S_n$ there is only one conjugacy class of reflections so only one q and this is the usual Iwahori-Hecke algebra $H_{S_n}(q)$. When $G = S_n \times (\mu_\ell \times \cdots \times \mu_\ell)$ get Ariki-Koike algebras.

For generic q , $\mathcal{H}_G(\underline{q})$ is semisimple but most interested in the non-semisimple ones. Want a quasi-hereditary algebra \mathcal{A} together with a covering $F : \mathcal{A} - \text{mod} \rightarrow \mathcal{H}_G(\underline{q}) - \text{mod}$ where $F = \text{Hom}_{\mathcal{A}}(P, -)$ where P is projective and $\text{End}_{\mathcal{A}} P \cong \mathcal{H}_G(\underline{q})$ and $\text{End}_{\mathcal{H}_G(\underline{q})} P \cong \mathcal{A}$.

When $G = S_n$, take $\mathcal{A} = S_q(n) = q$ -Schur algebra.

When $G = S_n \times (\mu_\ell \times \cdots \times \mu_\ell)$, \mathcal{A} = cyclotomic q -Schur algebra.

Theorem 14.1. (*Ginzburg-Guay-Opdam-Rouquier*) For $\underline{\gamma}$, there exists $\mathcal{A}_{\underline{\gamma}}$, a quasi-hereditary cover of $\mathcal{H}_G(\underline{q})$.

14.2. Construction. The rational Cherednik algebra of type G is $H_{\underline{\gamma}}$ generated by $\mathbb{C}[\mathfrak{h}]$, $\mathbb{C}G$, $\mathbb{C}[\mathfrak{h}^*]$ with relations

$$\begin{aligned} g x g^{-1} &= {}^g x, \\ g y g^{-1} &= {}^g y, \\ y x - x y &= \langle x, y \rangle - \sum_{s \in \mathcal{S}} \underline{\gamma}(s) \frac{\langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle}{\langle \alpha_s, \alpha_s^\vee \rangle} s. \end{aligned}$$

for $x \in \mathfrak{h}$, $y \in \mathfrak{h}^*$, $g \in G$, where α_s and α_s^\vee define the reflecting hyperplanes for s in \mathfrak{h} and \mathfrak{h}^* .

Theorem 14.2. (*Cherednik, Etingof-Ginzburg*) $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes G \cong \text{gr } H_{\underline{\gamma}}$ so there is a vector space isomorphism $\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}G \otimes \mathbb{C}[\mathfrak{h}^*] \cong H_{\underline{\gamma}}$

Is there a geometric picture for this?

Define $\mathcal{O}(\underline{\gamma})$ full subcat of $H_{\underline{\gamma}} - \text{mod}$ such that $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently. Verma module $\Delta_{\underline{\gamma}}(E) := H_{\underline{\gamma}} \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes G} E$ has a unique simple quotient $L_{\underline{\gamma}}(E)$, and as E varies over the irreducible reps of G this gives all simple modules in $\mathcal{O}(\underline{\gamma})$.

Lusztig's c -function

$$c_E(E) := \sum_{s \in \mathcal{S}} \underline{\gamma}(s) \frac{\text{Tr}(s|_E)}{\dim E}.$$

Define $E \succ F$ if $c_E - c_F \in \mathbb{Z}_{>0}$. Then

$$KZ : \mathcal{O}(\underline{\gamma}) \cong \mathcal{A}_{\underline{\gamma}} - \text{mod} \rightarrow \mathcal{H}_G(\underline{q}) - \text{mod}$$

is a cover.

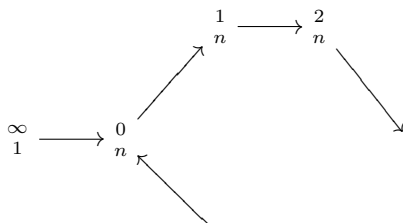
Theorem 14.3. (*Rouquier*) If $\underline{\gamma} - \underline{\gamma}' \in \mathbb{Z}^{\mathcal{S}/\sim}$ and $\underline{\gamma}$ and $\underline{\gamma}'$ induce the same ordering on \hat{G} , then $\mathcal{O}(\underline{\gamma}) \cong \mathcal{O}(\underline{\gamma}')$.

Define $E > F$ if $c_E - c_F \in \mathbb{Q}_{>0}$.

E.G., $G = S_n \ltimes (\mu_\ell \times \cdots \times \mu_\ell)$, $\mathcal{S} = \{s_{ij}, g_i g_j^{-1}\} \ g \in \mu_\ell$

$\widehat{G} \leftrightarrow \ell$ -multipartitions of n .

14.3. Quiver Varieties. $G = S_n \ltimes (\mu_\ell \times \cdots \times \mu_\ell) = S_n \wr \mu_\ell$ McKay quiver



$$\mu^{-1}(0) \subset \text{Rep}(\bar{Q}_\infty, n\delta + \varepsilon_\infty)$$

$$\begin{array}{ccc} \mathcal{M}_\theta(n) & = & \mu^{-1}(0) //_\theta \text{GL}(n\delta) \\ \downarrow \pi & & \\ \mathfrak{h} \oplus \mathfrak{h}^*/G & & \end{array}$$

θ moves in chambers to provide resolutions of singularities

14.4. Connection. $U_\gamma = eH_\gamma e$, $e \in \mathbb{C}G$ trivial idempotent

$\text{gr } U_\gamma \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^G$.

But very often U_γ is Morita equivalent to H_γ

Theorem 14.4. (Gordon-Stafford, Musson, Boyarchenko, Vale) For most $\underline{\gamma}$, there is a \mathbb{Z} -algebra Z_γ such that

$$\begin{array}{ccc} \text{coh } Z_\gamma & \xrightarrow{\text{gr}} & \mathcal{M}_\theta(n) \\ \downarrow \cong & & \downarrow \\ U_\gamma\text{-mod} & \xrightarrow{\text{gr}} & \mathfrak{h} \oplus \mathfrak{h}^*/G \end{array}$$

where $\theta_\gamma = (\gamma + \gamma_1 + \cdots + \gamma_{\ell-1}, -\gamma_1, \dots, -\gamma_{\ell-1})$.

14.5. Main Theorem.

Theorem 14.5. Suppose $\theta_\gamma = (\gamma + \gamma_1 + \cdots + \gamma_{\ell-1}, -\gamma_1, \dots, -\gamma_{\ell-1})$ and assume $\underline{\gamma}$ is in a Rouquier chamber. Then

- (1) $\mathcal{M}_{\theta_\gamma}(n)$ is a crepant resolution
- (2) $\mathcal{L}_{\theta_\gamma} := \pi^{-1}(\mathfrak{h}^* = 0)$ is Lagrangian and its components are naturally labelled by the ℓ -multipartitions of n
- (3) there is a natural partial ordering on the components and so on $\text{Irr}(S_n \wr \mu_\ell)$ which is refined by Rouquier's ordering
- (4) This ordering comes from combinatorics of higher level Fock spaces of Uglov.

Main idea of proof is that Lusztig's c -function is a Morse function for $\mathcal{M}_\theta(n)$

15. M. REINIEKE—(NON)SINGULAR QUIVER MODULI

15.1. Motivation. Interplay between $\mathbb{C}Q$ and $\mathcal{M}_\theta(Q)_d$ =moduli of d -diml reps of Q

Ask questions about the global geometry of the moduli space, compute χ =Euler characteristic and $\dim H^i(-, \mathbb{Q})$ =Betti numbers, explicit formulas

15.2. Notation. Fix (Q, d, θ) where $Q = (Q_0, Q_1)$, $d \in \mathbb{N}Q_0$, $\theta \in (\mathbb{Q}Q_0)^*$ a stability condition.

The slope of $X \in \text{Rep}_{\mathbb{C}}Q$ is $\mu(X) := \frac{\theta(\dim X)}{\dim X}$, and X is *stable* if for all proper submodules $Y \subset X$, $\mu(Y) < \mu(X)$

Theorem 15.1. (*King*) *There exist*

$$M_d^{st}(Q) = \{\text{isoclasses of stable reps of } \underline{\dim} X\}$$

and

$$M_d^{sst}(Q) = \{\text{isoclasses of polystable}^8 \text{ reps of } \underline{\dim} X\}$$

Always $M_d^{st}(Q)$ is smooth open subvariety of $M_d^{sst}(Q)$ and the latter is projective over the affine variety $M_d^{simp}(Q)$.

$M_d^{sst}(Q)$ is projective if Q has no oriented cycles.

Suppose d is coprime for θ , i.e., if $0 \neq e < d$, then $\mu(e) \neq \mu(d)$ ($\Rightarrow \gcd\{d_i\} = 1$). Then $M_d^{st}(Q) = M_d^{sst}(Q)$ so is smooth

16. ZHANG—FOUR DIMENSIONAL REGULAR ALGEBRAS

Problem: Classify these. Lower dimensional ones are known. Requires new ideas.

16.1. Background. k =field, algebraically closed, characteristic zero. A denotes an \mathbb{N} -graded connected k -algebra generated by A_1 and $\dim_k A_1 < \infty$.

A is AS-regular if

- (1) $\text{gldim } A = d < \infty$
- (2) $\text{Ext}_A^i(k, A) = \begin{cases} 0 & i \neq d, \\ k(\ell) & i = d. \end{cases}$
- (3) $\text{GKdim } A < \infty$

Write $g = \dim A_1$.

Problem: Classify those with $d = 4$. If $g = 4$, A is Koszul. If $g = 3$, there are 4 relations. If $g = 2$, there are 2 relations.

Example due to Caines, $g = 4$, not an Ore extension, $\text{Aut}_{gr} A = k^\times$, point scheme contains 13 points.

16.2. A_∞ -algebra methods.

Theorem 16.1 (Keller). *Let $A = k\langle A_1 \rangle / (R)$ with $R = R_{\geq 2}$ minimal. Set $E = \text{Ext}_A^*(k, k)$ with its A_∞ structure. The higher multiplications of E are determined by $m_n : (E_1)^{\otimes n} \cong (A_1^*)^{\otimes n} \rightarrow E^2$ is the composition of the dual of $i_n : R_n \rightarrow (A_1)^{\otimes n}$ with the inclusion of R_n^* in E^2 .*

Simple examples, $A = k[x]/(x^p)$, $p \geq 3$, $E = k[a, b]/(b^2)$ where $a = x^*$, $b = r^*$, where $r = x^p$ and (E, m_2, m_p) with $m_p(a^{\otimes p}) = b$.

When $d = 4$ and $g = 2, 3, 4$ we have $(E, m_2, m_3, \dots, m_{6-g})$. When $g = 2$, can work out all (E, m_2) so try to compute all possible m_3, m_4 , then use Keller's theorem to recover A .

Ex: Three other classes and $C = k\langle x, y \rangle$ with

$$xy^2 + pyxy + p^2y^2x = x^3y + jp^3yx^3 = 0$$

where $j^2 - j + 1 = 0$. These are all the \mathbb{Z}^2 -graded ones.

16.3. Extensions. Let A and B be connected graded. Call C an extension of $A|B$ if there is an "exact sequence" $0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0$ menaing

- (1) A is a subalgebra of C and the map $C \rightarrow B$ is an algebra homomorphism,
- (2) C is a free left and right A -module with basis given by a basis for B
- (3) $A_{\geq 1}C = CA_{\geq 1}$, and $C/(A_{\geq 1}) \cong B$.

Examples: tensor product $A \otimes_k B$

NE=normal extension, $A = k[x]$, $\deg x = 1$, C is an extension $k[x]|B$ if and only if x is normal and regular in C and $B = C/(x)$.

OE=Ore extensions, C is an extension of $A|k[x]$ if and only if $C = A[x; \sigma, \delta]$.

- (1) C need not be noetherian even if A and B are.
- (2) If C is NE of B or OE of A , then C is noetherian (resp., regular) iff A is, and then $\text{gldim } C = 1 + \text{gldim } A$.
- (3) If A and B are noetherian or regular, then $\text{gldim } C \leq \text{gldim } A + \text{gldim } B$.

16.4. Double extensions. An extension C is called a **double extension (DE)** of $A|B$ if B is regular of dimension 2.

My remarks. This is like a fiber product definition, especially if one has a \mathbb{Z}^2 -grading on C coming from a \mathbb{Z} -grading on each of A and B . Get a map $\text{Proj}_{nc} C \rightarrow \text{Proj}_{nc} A$ and $\text{Proj}_{nc} B \rightarrow \text{Proj}_{nc} C$.

Theorem 16.2. *If A is regular so is every double extension of it and $\text{gldim } C = \text{gldim } A + 2$.*

If A is noetherian is every DE of A noetherian?

Theorem 16.3. *Every 4-diml regular algebra that is a DE of a 2-diml regular algebra is universally noetherian, CM, and Auslander regular.*

DE data: Write $A_p[y_1, y_2; \sigma, \delta, w]$ for a DE of A with $p = \{p_{12}, p_{11}\}$ a parameter set, $s : A \rightarrow M_2(A)$ an algebra homomorphism, $\delta : A \rightarrow A^{\oplus 2}$ a σ -derivation, $w = \{t_0, t_1, t_2\} \subset A$, the tail.

C is generated by A and y_1 and y_2 with relations

$$y_2 y_1 = p_{12} y_1 y_2 + p_{11} y_1^2 + t_1 y_1 + t_2 y_2 + t_0$$

and

$$\begin{pmatrix} y_1 a \\ y_2 a \end{pmatrix} = \sigma(a) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \delta(a)$$

and some more relations involving $\{p, \sigma, \delta, w\}$.

(Does he mean every DE is of this form? and then use this classification to prove the theorem).

Every \mathbb{Z}^2 -graded $d = g = 4$ -dimensional regular algebra is either DE or OE or NE.

17. ROGALSKI—

Theorem 17.1 (with Stafford). *Let A be a connected \mathbb{N} -graded k -algebra such that*

- (1) $A = k[A_1]$, and A is a noetherian domain
- (2) A is a birationally commutative surface, i.e., $\text{Fract}_{gr} A = K[t, t^{-1}; \sigma]$ where K is a field with $\text{trdeg } K/k = 2$.
- (3) σ is geometric, i.e., there is a projective surface Y with $K = k(Y)$ and a $\tau \in \text{Aut } Y$ that produces σ by pullback of functions

Then A is isomorphic in big degree to a naive blowup⁹ $R(X, \mathcal{L}, \sigma, Z)$ where X is a projective surface such that $k(X) = K$, \mathcal{L} is an invertible \mathcal{O}_X -module, $\sigma \in \text{Aut } X$, and Z is a 0-dimensional subscheme of X and $\langle \sigma \rangle.z$ is dense in X for all $z \in Z$.

17.1. Definition of $R(X, \sigma, \mathcal{L}, Z)$. Define

$$\mathcal{L}_n = \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \dots \otimes (\sigma^*)^{n-1} \mathcal{L}.$$

Let $\mathcal{I} :=$ the ideal sheaf of Z and set

$$\mathcal{I}_n := \mathcal{I} \cdot \sigma^* \mathcal{I} \cdot \dots \cdot (\sigma^*)^{n-1} \mathcal{I}.$$

and

$$R_n := H^0(X, \mathcal{L}_n \otimes \mathcal{I}_n) \subset H^0(X, \mathcal{L}_n).$$

Typically this ring is not strongly noetherian, and these are the most interesting one.

17.2. An example. $B = k[x_1, x_2, x_3]$ with relations $x_i x_j = p_{ij} x_j x_i$ is birationally commutative if and only if $p_{12} p_{23} = p_{13}$. In other words, $B = B(\mathbb{P}^2, \mathcal{O}(1), \sigma)$. If $p \in \mathbb{P}^2$, then $R = R(\mathbb{P}^2, \mathcal{O}(1), \sigma, p)$ is the subalgebra of B generated by the 2-dimensional subspace of B_1 vanishing at p .

17.3. Some proof. Study point modules, but there is no scheme representing the point module functor.

17.4. On condition (3). A non-geometric example. Take $K = k(u, v)$ and $\sigma \in \text{Aut } K$ defined as follows: let $X = \mathbb{P}^2$ and $\sigma = \tau_2 \tau_1 : X \dashrightarrow X$ a birational map where $\tau_1 =$ Cremona transformation and $\tau_2 \in \text{PGL}_2(k)$. If σ were geometric there would be an upper bound on the number of fundamental points of σ^n ...but can check that this doesn't happen because τ_2 will move the fundamental points away from the earlier ones...here the number of fundamental points of σ^n is $3n$.

See also paper of Diller and Favre "Dynamics of bimeromorphic maps of surfaces" Using their results:

Theorem 17.2. *Let A be a connected graded \mathbb{N} -algebra birationally commutative domain with finite GK-dim and $Q(A) = K[t^{\pm 1}; \sigma]$ If σ is geometric, then $\text{GKdim } A \in \{3, 5\}$. If σ is not geometric then $\text{GKdim } A = 4$.*

Can also get $\text{GKdim } A = \infty$.

⁹The idea is that the ring models a blowup of X at Z and σ twists this.