Abstract. These are the notes I typed during the talks. I haven’t spent much time after the lectures cleaning them up so there are probably lots of typos and even some more serious errors. The hardest thing was getting the quiver diagrams done in real time. I often had to give up! Anyway, there may be some value to this so I’m placing it in the public domain. Feedback is welcome.

1. King—Moduli of sheaves from moduli of Kronecker modules with Alvarez-Consul

1.1. Basics. Let $X$ be a projective scheme over $k = ar{k}$. Important features of $\text{coh} X$: monoidal category with identity $O$, simple objects $O_x$, $x \in X$, and automorphisms $(n) : E \mapsto E(n)$ given by tensoring $n$ times with an ample invertible sheaf $O(1)$.

Notation:
- $H = H(n_0, n_1) := H^0(O(n_1 - n_0)) = \text{Hom}(O(-n_1), O(-n_0));$ typically $n_1 \geq n_0$.
- $\mathcal{M}_X(P) := \text{moduli space of semistable sheaves with Hilbert polynomial } P$. It is a projective scheme.

Motivating question: why is $\mathcal{M}_X(P)$ projective? Classical answer: because it is constructed by GIT. Not very helpful answer!

1.2. Modern answer. First, why is $X$ projective? Because when $n_1 - n_0 \gg 0$ (i.e., $\exists N \forall n_0, n_1$ with $n_1 - n_0 \geq N ...$) the map

$$x \mapsto \phi_x : H^0(O_x(n_0)) \otimes H \to H^0(O_x(n_1))$$

is an embedding $X \to \mathbb{P}(H^*) = \mathcal{M}_H(1, 1) =$ the moduli space of strictly non-zero $H$-Kronecker modules!\(^1\)

Theorem 1.1. $\mathcal{M}_X(P)$ is projective because for $n_1 \gg n_0 \gg 0$ (i.e., $\exists N_0 \forall n_0 \geq N_0 \exists N_1 \forall, n_1 \geq N_1 ...$) the functor

$$E \mapsto \phi_E : H^0(E(n_0)) \otimes H \to H^0(E(n_1))$$

gives an embedding $\mathcal{M}_X(P) \to \mathcal{M}_H(P(n_0), P(n_1)) =$ the moduli space of semistable $H$-Kronecker modules which is obviously (!) projective.

Remarks:
- (1) The proof gives a “new” construction of the $\mathcal{M}_X(P)$ (cf. Simpson).
- (2) the embedding is scheme-theoretic, except possibly at strictly semistable points when char $k = p$;

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\(^1\)Representations of dimension $(n_0, n_1) = (1, 1)$ of the Kronecker quiver with dim $H$ arrows.
(3) functoriality implies there should be an embedding of “non-commutative moduli”, whatever that means.

(4) how do we put the “right” non-commutative structure on $\mathcal{M}(P(n_0), P(n_1))$ and thus on $\mathcal{M}(P)$ (non-commutative necessary when dimensions are not co-prime)?

(5) if $\text{End } \mathcal{O} = k$ and $T = \mathcal{O}(n_0) \oplus \mathcal{O}(n_1)$, then

$$A := \text{End } T = \begin{pmatrix} k & 0 \\ H & k \end{pmatrix}$$

and obtain

$$\Phi := \text{Hom}(T^\vee, -) : \text{coh } X \rightarrow \text{mod } A$$

with adjoint $\Phi^\vee : V \mapsto T^\vee \otimes_A V$.

1.3. **Regularity.** $E$ has Hilbert polynomial

$$P(n) := \sum (-1)^i \dim H^i(E(n)) = rn^s + \text{lower degree terms}.$$  

where $s = \dim \text{Supp } E$ and $r$ is like rank $E$. Note $P(n) = \dim H^0(E(n))$ for $n \gg 0$ because $\mathcal{O}(1)$ is ample. (in fact, get = when $E$ is $n$-regular\(^2\). Regularity plays a key role.

**Theorem 1.2.** If $\mathcal{O}(n_1n_0)$ is 0-regular, then $\Phi$ is fully faithful on all $n_0$-regular $E$, i.e., the evaluation

$$\varepsilon_E : T^\vee \otimes_A \text{Hom}(T^\vee, E) \rightarrow E$$

is an isomorphism.

Does this mean there is some sort of embedding of the stack of $n$-regular sheaves?

1.4. **Stability.** An $A$-module $V$ (i.e., a representation of the $H$-Kronecker quiver $V_0 \otimes H \rightarrow V_1$) is semistable if for all $W \subset V$,

$$\frac{\dim W_0}{\dim W_1} \leq \frac{\dim V_0}{\dim V_1}$$

Note this is essentially the only (sensible) stability condition because the rank of the Grothendieck group is 2.

A sheaf $E$ is semistable if for all $E' \subsetneq E$ such that $s_{E'} = s_E$,

$$\frac{P_{E'}(n)}{r_{E'}} \leq \frac{P_E(n)}{r_E}$$

Note: when $\dim X = 1$ this reduces to

$$\frac{\deg_{E'}}{r_{E'}} \leq \frac{\deg_E}{r_E}.$$  

Boundedness: given $P$, for all $n \gg 0$ if $E$ is semistable for $P$, then $E$ is $n$-regular.

Set of Kronecker modules with fixed dimension vector is “bounded”. What does he mean?\(^2\)

Can write stability condition in a better way:

\(^2E is n-regular if $H^i(E(n - i)) = 0$ for all $i > 0$; if $E$ is $n$-regular it is $m$-regular for all $m \geq n$.\)
Lemma 1.3 (Rudakov). The stability condition is equivalent to the condition that
\[
\frac{P_E(n_0)}{P_E(n_1)} \leq \frac{P_E'(n_0)}{P_E'(n_1)}
\]
for \( n_1 \gg n_0 \gg 0 \). In other words, \( \Phi_E' \) does not destabilize \( \Phi(E) \) for large \( n \).

Theorem 1.4. Given \( P \), for \( n_1 \gg n_0 \gg 0 \), for any \( E \) with Hilbert polynomial \( P \), \( E \) is semistable if and only if \( E \) is \( n_0 \)-regular and pure (i.e., support of all submodules of \( E \) is same as that for \( E \)) and \( \Phi(E) \) is semistable.

Corollary 1.5. There is an embedding \( \mathfrak{M}_X(P) \to M_H(P(n_0), P(n_1)) \) for \( n_1 \gg n_0 \gg 0 \).

1.5. Theta functions. Why is \( M_H(P(n_0), P(n_1)) \) projective?

Theorem 1.6 (Schofield, Van den Bergh, Derksen-Weyman). An \( A \)-module \( V \) is semi-stable if and only if there is a \( \gamma : P_{k^1}^1 	o P_{k^0}^0 \) such that \( \text{Hom}(\gamma, V) : V_{k^0}^0 \to V_{k^1}^1 \) is an isomorphism, i.e., \( \theta_\gamma(E) := \det \text{Hom}(\gamma, E) \) is non-zero. Further, the \( \theta_\gamma \) embeds \( \mathfrak{M}_A \).

Adjunction \( \text{Hom}(\gamma, \Phi(E)) = \text{Hom}(\Phi^\vee \gamma, E) \)

i.e., \( \Phi(E) \) is semistable if and only if there is a map \( \delta : O(-n_1)^{k_1} \to O(-n_0)^{k_0} \) such that \( \theta_\delta = \det \text{Hom}(\delta, E) \neq 0 \).
2. LUNTS—DG-DEFORMATION

\( X \mapsto D^b(\text{coh}\, X) \) is a 2-functor. How might one try to reconstruct \( X \) from \( \text{coh}\, X \)? Moduli space of point-like objects in \( D^b(X) \) But needed a good theory of moduli spaces for families in derived categories.

2.1. Review of deformation functor. Fix a field \( k \). Let \( \mathcal{C} := \) the category of local commutative artin algebras \( (R, m) \) with \( R/m = k \)

\( X/k = \) algebra over \( k \), or a compact complex manifold, or a principal \( G \)-bundle, a repon of a group \( \Gamma \) etc.

\[
\text{Def}(X) : \mathcal{C} \rightarrow \text{Gpd}
\]

Objects in \( \text{Def}_R(X) \) are pairs \( (\mathcal{X}, \sigma) \) where \( \mathcal{X} \) is a flat family of objects such that

\[
\xymatrix{
\mathcal{X} \ar[d]_{\sigma} \ar[r] & X \\
\text{Spec } R \ar[r]_{i} & \text{Spec } k
}
\]

is a Cartesian square. Morphisms are maps \( f : \mathcal{X} \rightarrow \mathcal{Y} \) such that

\[
\xymatrix{
\mathcal{X} \ar[dr]_{\sigma_X} \ar[rr]^f & & \mathcal{Y} \\
\text{Spec } R \ar[ur]_{\sigma_Y} & &
}
\]

commutes.

\( \text{Def}_R(X) \) is pro-representable by a complete local \( \hat{R} \) if

\[
\text{Hom}_{\mathcal{C}}(\hat{R}, R) = \pi_0 \text{Def}_R(X).
\]

functorially in \( R \).

2.2. Universal approach: DG-Lie algebras. (Grothendieck, Drinfeld, Deligne)

\( \text{Def}(X) \) is controlled by a DG-Lie algebra \( g = g(X) = \) the Lie algebra of infinitesimal derived automorphisms of \( X \), i.e., deformations of \( X = \) inner deformations of the differential in \( g \).

The Deligne or Maurer-Cartan groupoid \( \text{Def}(g) : \mathcal{C} \rightarrow \text{Gpd} \). Since \( R \) is commutative, \( g \otimes m \) is also a DG-Lie algebra (nilpotent because \( m \) is a nilpotent ideal).

\[
\text{Ab} \text{Def}_R(g) = MC(g \otimes m) = \{ \alpha \in g \otimes m \mid da + \frac{1}{2} [\alpha, \alpha] = 0 \}
\]

a super-bracket. The MC condition is equivalent to the condition that \((d + \alpha)^2 = 0\)

where \( d + \alpha := d + [\alpha, -] : g \otimes R \rightarrow g \otimes R \).

Only works in char \( k = 0 \)

Morphisms in \( MC(g \otimes m) \) is a pair \( \alpha, p \) \( \exp(g \otimes m)^p \) contains \( f(d + \alpha)f^{-1} = d + \beta \).

Example: \( X = \) compact complex manifold, \( \text{Def}_{\mathcal{C}^d/(\mathcal{U}^d)}(X) = H^1(X, T_X) \).

\( g(X) = \Gamma(X, \mathcal{C}_d(T_X)) \) and \( m = \mathbb{C}t \) and \( m \otimes g = t.g = g \)

Amnon says this won’t work because \( g \) not abelian??
(1) Invariance Theorem: $\varphi : \mathfrak{g} \to \mathfrak{h}$ s.t. $\varphi^* : \text{Def}(\mathfrak{g}) \to \text{Def}(\mathfrak{h})$ is an isomorphism of functors.

(2) Representability Theorem. Assume $\varphi^* = 0$ and $\dim \mathfrak{g}^1 < \infty$. Then $\text{Def}(\mathfrak{g})$ is prorepresentable by $H_0(\mathfrak{g})^*$ which is a cocommutative coalgebra.

(3)

2.3. Linear Deformation Theory. Want to deform modules. Given $E \in \mathcal{T}$ where $\mathcal{T}$ comes from a DG-category.

dgart=DG artinian algebras $(R/\mathfrak{m})$ such that $R/\mathfrak{m} = k$ dgart=DG artinian algebras living in non-positive degree.

$A$ a DG-category $E \in A^0\text{-mod.}$

$A_R := A \otimes_k R$

2.4. DG algebra.

2.5. Example.

2.6. Point-like objects. $E$ such that $\text{RHom}^*(E, E) \cong \Lambda \text{Ext}^1(E, E)$
3. **Bondal—Derived categories of toric varieties and mirror symmetry**

3.1. **Weighted \( \mathbb{P}^n \)'s and toric stacks.** Let \( X \) be a projective toric stack over \( \mathbb{C} \) with fan \( \Sigma_X \) and dense torus \( T \cong (\mathbb{C}^\times)^n \).

If \( X \) is smooth, then \( \Sigma_X \) is simplicial i.e., all polytopes are simplices. We should also allow \( X \) to be a stack—then \( \Sigma_X \) is still simplicial but can have more general simplices.

Write \( \text{Pic}_T X \) for the \( T \)-equivariant Picard group. The forgetful functor \( \text{coh}_T X \rightarrow \text{coh} X \) on invertible sheaves gives a homomorphism between the two Picard groups:

\[
0 \rightarrow M \rightarrow \text{Pic}_T X \rightarrow \text{Pic} X \rightarrow 0
\]

and the kernel \( M \) is a lattice naturally isomorphic to the character group \( T^\vee \). Write \( \mathbb{N} = \text{Hom}(M, \mathbb{Z}) \) for the dual lattice

**Theorem 3.1** (Bondal-Ruan). **There is a fully faithful functor**

\[
\text{D}^b_{\text{coh}}(X) \rightarrow \text{D}^b(T, \Sigma)
\]

where the RHS is the category of constructible sheaves on a real torus \( T = (S^1)^n \) with respect to a stratification \( \Sigma \). This is an equivalence under some conditions (*) (which imply \( X \) is Fano); the conditions hold in \( \dim X = 2 \) when \( X \) is del Pezzo. In \( \dim 3 \) there are 18 Fanos, and the conditions hold for 16 of them.

Should think of \( \text{D}^b(T, \Sigma) \) as mirror

\[
\text{D}^b_{\text{coh}}(X) \equiv \text{D}^b(T, \Sigma) \equiv \text{Fuk}(Y, \omega).
\]

Mirror symmetry is a relation between algebraic geometry and symplectic geometry

\[
\text{D}^b_{\text{coh}}(X) \equiv \text{Fuk}(Y, \omega)
\]

e.g., when \( (X, J) \) is Calabi-Yau variety there is a \( (Y, \omega) \) on the RHS

Fano varieties are NOT Calabi-Yau. For Fano varieties we use a Landau-Ginzburg potential \( W \) and associate to \( (Y, \omega, W) \) a category \( \text{Fuk}(Y, \omega, W) \) that is a replacement for the Fukaya category.

**Theorem 3.2** (Kawamata). **If \( X \) is a toric stack, then \( \text{D}^b(X) \) has a full exceptional collection of coherent sheaves.**

Conjecture: there is a full exceptional collection of line bundles.

Exceptional collection corresponds to critical points for \( W \) but sometimes \( Y \) is affine so some critical points may be missing so don’t get line bundles? ? ? ?

3.2. Define the real torus \( T := M_\mathbb{R}/M \).

For each \( \ell \in \mathbb{N} \), define \( G_\ell \) to be the kernel in the exact sequence

\[
0 \rightarrow G_\ell \rightarrow T \xrightarrow{F_\ell} T \rightarrow 1
\]

where \( F_\ell(t_1, \ldots, t_n) = (t_1^\ell, \ldots, t_n^\ell) \). Since \( G_\ell \) is the \( \ell \)-torsion subgroup it is isomorphic to \( (\mu_\ell)^n \). Applying \( \text{Hom}(-, \mathbb{C}^\times) \) to this gives an exact sequence

\[
0 \rightarrow M \xrightarrow{\ell} M \rightarrow G_\ell^\vee \cong M/\ell M \rightarrow 0
\]

in which \( G_\ell^\vee \) is the character group of \( G_\ell \). The last isomorphism is natural so composing with the multiplication map by \( \ell^{-1} \), namely

\[
\frac{1}{\ell} : \frac{M}{\ell M} \rightarrow \frac{M \otimes \mathbb{R}}{M} \cong \mathbb{T}
\]
we get a natural inclusion \( G_\ell \to T \) with image the set of rational points in \( T \) with denominator \( \ell \).

Now want to define a map \( G_\ell \to \text{Pic} X \).

The map \( F_\ell : T \to T \) extends to \( F_\ell : X \to X \). There is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{F_\ell} & X \\
\uparrow & & \uparrow \\
T & \xrightarrow{F_\ell} & T
\end{array}
\]

We can decompose \((F_\ell)_* \mathcal{O}_X = \bigoplus_{\chi \in G_\ell} \mathcal{L}_\chi\) as a direct sum of line bundles. The rule \( \chi \mapsto \mathcal{L}_\chi \) gives us \( G_\ell \to \text{Pic} X \).

Taking the dual of \( G_\ell \to T \) gives a map \( G_\ell \to T^\vee = M \to M_R \to T \). There is unique \( \Phi \) making

\[
\begin{array}{ccc}
G_\ell & \xrightarrow{\Phi} & \text{Pic} X \\
\downarrow & & \downarrow \\
T & \rightarrow & \downarrow
\end{array}
\]

commute.

Let \( D_1, \ldots, D_N \) be \( T \)-invariant prime divisors giving a basis for \( \text{Pic}_T X \). If \( D = \sum a_i D_i \in M_R \) then

\[
\Phi(D) := \sum [a_i] D_i \in \text{Pic} X
\]

where \([a_i]\) is the integer part. Clearly, \( B = \text{Im} \Phi \) is a finite set. Define the strata of \( T \) to be the level sets of \( \Phi \). Associate to each stratum the line bundle that is its image.

(\text{My comment: all this seems to depend on choice of } \ell \ldots??) Bondal says independent of \( \ell \) and under some conditions \( B \) provides a complete strong exceptional collection.

Now associate a quiver to the stratification: pick a point \( x_S \) in each stratum \( S \), and draw one arrow \( S \to S' \) for each homotopy class of oriented paths from \( x_S \) to \( x_{S'} \) that are compatible with \( \Sigma \) in the sense that a path can pass through a point on the boundary of a stratum only on the way inside the stratum.

3.3. The case of \( \mathbb{P}^1 \). For \( X = \mathbb{P}^1 \), \( T = S^1 \). Then \( \Sigma \) is the stratification of \( S^1 \) with two strata, a point \( p \) and its complement \( S^1 - \{p\} \). Say \( p = 0 \) and pick \( \infty \in S^1 - \{p\} \). The quiver has vertices \( 0 \) and \( \infty \) and two arrows from \( \infty \) to \( 0 \), one arrow for each of the two paths from \( \infty \) to \( 0 \).

Associate \( \mathcal{O} \) to the point 0 and \( \mathcal{O}(-1) \) to the other stratum. Two \( T \)-invariant divisors on \( \mathbb{P}^1 \), 0 and \( \infty \) and \( \Phi(a_0 D_0 + a_\infty D_\infty) = \mathcal{O}(a_0 + a_\infty) \).

Let \( i : \{p\} \to S^1 \) and \( j : S^1 - \{p\} \to S^1 \) be the inclusions. Let \( \mathcal{F} \) be a constructible sheaf of \( \mathbb{C} \)-vector spaces on \( S^1 \) with respect to \( \Sigma \). Then \( j^* \mathcal{F} \) is a sheaf on the contractible space \( S^1 - \{p\} \) so is constant, say \( V_1 \). Let \( V_0 = i^* \mathcal{F} \).

\(^3 \text{cf. the Frobenius map}

\(^4 \text{A sheaf on a stratified space is constructible if its restriction to each stratum is locally constant—e.g., the solution complex of a holonomic } \mathcal{D} \text{-module.} \)
For an open set $U \subset S^1$, 
\[ \mathcal{F}(U) \cong \begin{cases} V_0 & \text{if } p \in U, \\ V_1 & \text{if } p \notin U. \end{cases} \]

Take an open set $U$ containing $p$, and open sets $U_1, U_2 \subset U$ that do not contain $p$, one to the right of $p$, the other to the left. There are restriction maps $\mathcal{F}(U) \to \mathcal{F}(U_i)$ that then give two maps
\[ V_0 \to V_1 \]

In this way we get a functor from $D^b(T, \Sigma)$ to the derived category of representations of the Kronecker quiver, but the latter is equivalent to $D^b(P^1)$.

Which sheaves are sent to the indecomposable projectives? Maybe $i_! C_p$ and $j_* C_{S^1 - p}$?

More generally, if $i : S \to T$ is the inclusion of a stratum I think $i_! C_S$ is injectives and $i_* C_S$ is projective.

3.4. The case of $\mathbb{P}^2$. Now $T = S^1 \times S^1$, the 2-torus, and he draws it as $\mathbb{R}^2 / \mathbb{Z}^2$.

There are three strata.

The four bullets are identified to a single point in $T$ forming a stratum, $S_0$. The lower left triangle, without its vertices but with its edges and interior forms a stratum $S_1$. The interior of the top right triangle forms a stratum, $S_2$.

The quiver is
\[ S_2 \quad \rightarrow \quad S_1 \quad \rightarrow \quad S_0 \]

How to get this? No oriented paths leaving $S_0$ are compatible with the stratification. Picking $x_1$ in the interior of $S_1$, there are three homotopy classes from $x_1$ to the three vertices of that triangle; since the edges of $S_1$ belong to $S_1$, not to $S_2$, there can be no paths leaving $S_1$. There was lots here that I did not type!

3.5. Vanishing Cycles. Let $f : M \to S$ be a proper holomorphic map from an $n$-dimensional complex manifold with boundary to a Riemann surface. Suppose $f$ has no critical points on the boundary of $M$ and only non-degenerate critical points on the interior with distinct critical values. Let $\gamma : [0, 1] \to S$ be a path such that $\gamma(0)$ is a critical value of $f$ and $\gamma(t)$ is a regular value for all $t \in (0, 1]$ For each $U \subset [0, 1]$ define
\[ M_U := \{(x, t) \in M \times U \mid f(x) = \gamma(t)\}. \]

Then $H_0(M_{[0,1]}, M_1) \cong \mathbb{Z}$. An $n$-chain $\Delta$ on $M_{[0,1]}$ is called a Lefschetz thimble if it generates this group, and its boundary $\partial \Delta$ is then called a vanishing cycle. It is uniquely determined by $\gamma$ up to sign.

3.6. The LG-potential for $\mathbb{P}^2$. The fan for $\mathbb{P}^2$ has rays spanned by $(0, 1), (1, 0), (-1, -1)$. Define a LG-potential\(^5\)
\[ W = x + y + \frac{1}{xy}. \]

\(^5\)Notice $W$ encodes the rays of the fan.
lots here i didn’t type
Fuk depends only on \([\omega] \in H^2(Y)\).
4. Berest—Cherednik algebras, Calogero-Moser spaces, ...

Work in progress with O. Chalykh

4.1. Calogero-Moser spaces. Fix $n \in \mathbb{N}$ and define

$$\widetilde{CM}_n := \left\{ (\bar{X}, \bar{Y}) \in M_n(\mathbb{C})^2 \mid \text{rank}([\bar{X}, \bar{Y}] + \text{id}) = \begin{cases} 0 & n=0, \\ 1 & \text{otherwise}. \end{cases} \right\}$$

Then $GL(n, \mathbb{C})$ acts on $\widetilde{CM}_n$ by simultaneous conjugation—it is a free action so can form the quotient

$$CM_n := \widetilde{CM}_n // GL(n, \mathbb{C}).$$

Facts: [See Wilson 1998]

1. $CM_n$ is a smooth connected affine variety of dimension $2n$
2. it has a natural symplectic structure
3. $\cong \text{Hilb}_n(\mathbb{C}^2)$ as a real $C^\infty$ manifold
4. can interpret $CM_n$ as completed phase space of the rational Calogero-Moser system of $n$ particles on $\mathbb{C}$ with Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 - \sum_{i<j} \frac{1}{(x_i - x_j)^2}.$$ 

5. can interpret $\sqcup_{i=1}^{\infty} CM_n$ as the space of rational solutions (decaying at $\infty$ in $x$) of the KP hierarchy of non-linear PDEs, the first of which is

$$\frac{3}{4} u_{yy} = (u_t - \frac{1}{4}(u_{xxx} + 6uu_x))_x$$

4.2. $\Pi_\lambda(Q)$ and the Weyl algebra. For any finite quiver $Q$ with vertex set $I$ and any $\lambda : I \rightarrow \mathbb{C}$ define the deformed pre-projective algebra

$$\Pi_\lambda(Q) := \frac{\mathbb{C}Q}{(\sum_a [a, a^*] - \sum_{i \in I} \lambda_i e_i)}.$$

Let $Q$ be the quiver

$$\begin{array}{c}
\text{e}_0 \\
\downarrow i \\
\text{e}_1
\end{array}$$

So its double $\bar{Q}$ is

$$\begin{array}{c}
\text{e}_0 \\
\downarrow j \\
x
\text{e}_1 \\
\downarrow y
\end{array}$$

For $\lambda = (\lambda_0, \lambda_1)$, $\Pi_\lambda(Q)$ is the path algebra of $\bar{Q}$ modulo the relation

$$[x, y] + ij - \lambda_1 e_1 = ji + \lambda_0 e_0.$$ 

Proposition 4.1. The Weyl algebra is $A = A_1(\mathbb{C}) \cong \mathbb{C}[X, Y]$ with relation $XY - YX = 1$. Let $\Pi = \Pi_\lambda(Q)$ be the deformed pre-projective algebra above. Then

$$\frac{\Pi}{\Pi \text{e}_0 \Pi} \cong A_1(\mathbb{C}).$$
4.3. A Rational Cherednik algebra.

\[ W = S_n, \quad \mathfrak{h} = \mathbb{C}^n, \quad t = 0, \quad t = 1. \]

is \( H = H_{0,1}(S_n) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n][s_{ij}] \) with relations
\[ [y_i, x_j] = s_{ij} \quad (i \neq j), \quad [y_k, x_k] = \sum_{i \neq k} s_{ik}. \]

Define
\[ CM := \bigcup_{i=0}^{\infty} CM_n \]

\[ \begin{array}{c}
\text{isoclasses of right ideals of } A_1 \\
\text{isoclasses of simple } \Pi_\lambda(Q)-\text{modules, } \lambda = (-n, 1) \\
\text{of dimension } \alpha = (1, n)
\end{array} \]

\[ \begin{array}{c}
\text{isoclasses of simple } H_{0,1}(S_n)-\text{modules, } n \geq 0
\end{array} \]

Question: explain the mysterious bijection between simple \( H_{0,1}(S_n) \)-modules and left ideals of \( A_1 \). More generally, explain the relation between simple modules over \( H_{0,1}(S_n) \) and \( \Pi_\lambda(Q) \) and left ideals of \( A_1 \).

4.4. Functors. There are adjoint triples \((i^*, i_*, i'_*)\) and \((j_!, j^*, j_*\) of functors

The first three functors are induced by the homomorphism \( \Pi_\lambda(Q) \rightarrow A_1 \) in the obvious way. The \( \Pi\mathcal{e}_0\mathcal{e}_0 \)-bimodule \( \Pi\mathcal{e}_0 \) gives an exact functor \( j^* = - \otimes_{\Pi} \mathcal{e}_0 \) which has right adjoint \( j_* \mathcal{Hom}_{\mathcal{e}_0\mathcal{e}_0}(\mathcal{e}_0, -) \). If \( X \) is a right \( \Pi \)-module, then
\[ j^*X = X \otimes_{\Pi} \mathcal{e}_0 = Xe_0 \cong \mathcal{Hom}_{\Pi}(e_0\Pi, X). \]

So \( j^* \) also has a left adjoint \( j! = - \otimes_{e_0\Pi e_0} e_0\Pi \).

**Theorem 4.2.** There is an injection

\[ \begin{array}{c}
isoclasses of simple \\
\Pi_\lambda(Q)-\text{modules, } \lambda = (-n, 1) \\
of dimension } \alpha = (1, n)
\end{array} \stackrel{F = i^!j^*}{\rightarrow} \begin{array}{c}
isoclasses of right ideals of } A_1
\end{array} \]
4.5. **First step of the proof.** Let $X \in \text{Mod}\Pi_3(Q)$ be simple. There is a natural transformation $j_! j^* \to \text{id}_{\text{Mod}_{e_0}\Pi e_0}$ and the associated map

$$\phi : j_! j^* X \to X$$

is non-zero, so surjective because $X$ is simple. Since $\dim X < \infty$ and $A_1$ has no non-zero finite dimensional modules, $i_! X = 0$. Hence if we apply $i_!$, which is left exact, to the exact sequence

$$0 \to M \to j_! j^* X \to X \to 0$$

we get $M \cong i_! j_! j^* X$.

More to do here!

4.6. **Relation to $H_{0,1}(S_n)$.**

**Theorem 4.3.** There is an algebra isomorphism $\theta : e_0\Pi_3(Q)e_0 \to eHe$ given by

$$ja(X,Y)i \to -nea(x_1,y_1)e.$$ 

The composition functor $\text{Mod}H \to \text{Mod}(eHe) \to \text{Mod}B \to \text{Mod}A$ maps simple $H$-modules injectively to isoclasses of right ideals in $A_1$. 
5. HILLE—Strong exceptional sequences of line bundles on toric varieties

5.1. Motivation. Work over $\mathbb{C}$.

Theorem 5.1 (Beilinson). Define

$$
\Phi = \text{Hom}_{\mathbb{P}^n}(\bigoplus_{i=0}^n \mathcal{O}(i), -) : \text{coh} \mathbb{P}^n \to \text{mod} A.
$$

Then $R\Phi : \text{D}^b(\mathbb{P}^n) \to \text{D}^b(A)$ is an equivalence, and where $A = \text{End}(\bigoplus_{i=0}^n \mathcal{O}(i))$.

Want to extend this to smooth projective toric varieties.

5.2. Let $E = (L_1, \ldots, L_n)$ be a sequence of line bundles on $X$.

We say $E$ is

1. complete if $n = \text{rank} K_0(X)$;
2. full if $\langle L_1, \ldots, L_n \rangle = \text{D}^b(X)$; full $\Rightarrow$ complete
3. strongly exceptional if $\text{Ext}^k(L_i, L_j) = 0$ for all $k \neq 0$; in that case one can change the ordering so that $\text{Hom}(L_j, L_i) = 0 \forall j > i$
4. exceptional if $\text{Ext}^k(L_j, L_i) = 0 \forall j > i$.

5.3. Conjectures. Let $X$ be a smooth projective toric variety from now on!

1. (A. King) (unpubl paper on his home page) Every such $X$ has a full, strongly exceptional sequence.

Kawamata proved $X$ always has a full exceptional sequences of sheaves.

2. On any Fano $X$ there exists a helix: i.e., if $E = (L_1, \ldots, L_n)$ is a strong exc. seq., so is each length $n$ subsequence of $S(E) = (\ldots, L_{n-1} \otimes \omega, L_n \otimes \omega, L_1, \ldots, L_n, L_1 \otimes \omega^{-1}, \ldots)$.

3. On $X$ a full exceptional sequence of line bundles exists.

5.4. Toric surfaces. Write $\Sigma_X$ for the fan of the toric variety $X$.

Write $n = \text{rank} K_0(X)$, $d = \text{dim} X$ and $t = \text{rank}$ of the equivariant Picard group. Then $\text{rank} \text{Pic}(X) = t - d$ and there are $T$-invariant prime divisors $D_1, \ldots, D_t$ that provide a basis for the equivariant Picard group.

$$
\omega^{-1} = \mathcal{O}(\sum_{i=1}^t D_i).
$$

Draws a picture of a fan for a toric surface...the cones correspond to the orbits... 0 corresponds to the dense orbit, the rays correspond to the $T$-invariant divisors, and the big cones correspond to the fixed points. He labels the rays $\mathbb{R}_{\geq 0} v_i, i = 1, 2, \ldots$, moving counterclockwise round the origin.

Write $P(\Sigma) = \cup_{i} \text{convexhull}(v_i, v_{i+1}, 0)$. Then $X$ is Fano if and only if $P(\Sigma)$ is strictly convex. In that case, one has a polytope from which one can recover the fan.

Define a graph $\Gamma_X = (\Sigma \cap S^{d-1})$ with vertices the $D_i$s, and an edge between $i$ and $j$ if $\text{codim}_{D_i} D_j \cap D_i = 1$. When $\text{dim} X = 2$, this graph is $\tilde{A}_{t-1}$ (and $t = n$).

**Example:** The Hirzebruch surface $\mathbb{F}_n$: rays through $(0,1), (1,0), (0,-1), (-1,0)$...etc

---

6 Can do this because if $\mathcal{L}$ and $\mathcal{L}'$ are non-isomorphic bundles only one of $\text{Hom}(\mathcal{L}, \mathcal{L}')$ and $\text{Hom}(\mathcal{L}', \mathcal{L})$ can be non-zero—enough to prove this when $\mathcal{L}' = \mathcal{O}$; think about $\mathcal{O} \to \mathcal{L} \to \mathcal{O}$
5.5. Constructions. For simplicity will focus on line bundles of the form $\mathcal{L} = \mathcal{O}(\sum a_i D_i)$ $a_i \in \{0, \pm 1\}$.

**Theorem 5.2.** Let $\mathcal{E} = (\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \ldots, \mathcal{O}(D_1 + \cdots + D_{t-1}))$.

1. $\mathcal{E}$ exceptional $\Rightarrow v_1, \ldots, v_t, v_1$ is a Hamiltonian cycle in $\Gamma_X$
2. $\dim X \leq 3$ then iff in (1)
3. $\mathcal{E}$ is complete $\Leftrightarrow \mathcal{E}$ is full $\Leftrightarrow t = n$ $\Leftrightarrow X = \mathbb{P}^n$ or $\dim X = 2$
4. If $X$ Fano and $\mathcal{E}$ exceptional, then $\mathcal{E}$ is strongly exceptional.

**Theorem 5.3.** Let $\dim X = 2$ and $\mathcal{E} = (\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \ldots, \mathcal{O}(D_1 + \cdots + D_{t-1})). \omega^{-1} = \mathcal{O}(\sum_{i=1}^t D_i)$.

1. Then $\mathcal{E}$ is full exceptional if we take the cyclic orientation on the divisors
2. $\mathcal{E}$ is strongly exceptional iff $D_i^2 \geq -1$ for all $i = 1, \ldots, t - 1$
3. $S(\mathcal{E})$ is a helix iff $D_i^2 \geq -1$ for all $i$

Take $X$ equal to $\mathbb{F}_2$ blown up at 3 points, three additional rays through $(1, -1)$, $(2, -1), (3, -1)$ has no strongly exceptional sequence of length 7.

**Theorem 5.4.** Suppose $X$ is Fano and $\dim X = 3$. Then strongly exceptional sequence exists (complete) and can be completed to a helix.

strongly exceptional sequence exists (complete) on the maximal Fanos. Craw
5.6. B. Keller—Acyclic Calabi-Yau categories are cluster categories
(Joint with I. Reiten.)

6.1. Cluster Categories. Fix field \( k \), \( Q \) a finite quiver without oriented cycles. \( kQ \)=path algebra, \( \dim kQ < \infty \), \( \text{mod} kQ \)=category of finite dimensional right \( kQ \)-modules. \( D^b(kQ) \)=bounded derived category\(^7\) of \( \text{mod} kQ \). \( S \)=suspension, \( M \mapsto M[1] \). \( \Sigma \)=Serre functor=the unique auto-equivalence such that

\[
D \text{Hom}(X, Y) \cong \text{Hom}(Y, \Sigma X)
\]

where \( D = \text{Hom}_k(-, k) \).

\( C_Q \)=the cluster category = the orbit category of \( D^b(kQ)/ (\Sigma^{-1} \circ [2]) \). Objects in \( C_Q \) are same as those in \( D^b(kQ) \) but

\[
\text{Hom}_{C_Q}(X, Y) := \oplus \text{Hom}_{D^b}(X, (\Sigma^{-1} \circ [2])^n, Y)
\]

\( C_Q \) was invented by Buan-Marsh-Reineke-Reiten-Todorov in the general case, and by Caldero-Chapoton-Schiffer for \( Q \) of type \( A_n \). Their motivation was to categorify the cluster algebras of Fomin-Zelevinsky (2000)

Example. Let \( Q \) be the quiver

\[
1 \rightarrow 2
\]

The AR-quiver for \( \text{mod} kQ \) is

\[
\begin{array}{c}
\text{there are infinitely many } \mathbb{Z}A_\infty \\
\text{components in between}
\end{array}
\]

\[
\tau^{-1} P_1
\]

The picture of \( D^b(\text{mod} kQ) \) should go here

\[
\begin{array}{c}
\text{ }\\
\text{ }\\
\text{ }\\
\text{ }\\
\text{ }\\
\text{ }\\
\text{ }\\
\end{array}
\]

\[
F = \Sigma^{-1} \circ [2] = \tau^{-1} \circ [1]
\]

The picture of \( C_Q \) should go here.

\( ^7 \)Says that formally inverting all quisms gives the same category whether or not one first mods out homotopies.
6.2. Properties of $C_Q$.

1. $C_Q$ is triangulated, even algebraic triangulated, i.e., $C_Q \equiv E$ the stable category for some Frobenius category $E$.

2. Hom-finite

3. 2-Calabi-Yau, i.e., it has a Serre functor induced by $\Sigma$, still call it $\Sigma$, so on $C_Q$ we have an isomorphism of triangle functors $\Sigma \cong [2]$ as triangle functors.

4. $T_Q = \text{image of } kQ \text{ under } \pi : D^b(kQ) \to C_Q$ is a cluster-tilting object, i.e.,
   - $\text{Hom}(T_Q, T_Q[1]) = 0$
   - $\text{Hom}(T_Q, X[1]) = 0$ implies $X \in \text{add } T_Q$

5. $\text{End}_{C_Q}(T_Q) \cong kQ$. Hence the quiver of $\text{End}(T_Q)$ has no oriented cycles.

6.3. Main Theorem.

**Theorem 6.1.** Suppose $k$ is algebraically closed. If $C$ is an algebraic 2-CY-category having a cluster-tilting object $T$ such that the quiver of $\text{End} T$ has no oriented cycles, then there is a triangle equivalence $C_Q \to C$ such that $T_Q \mapsto T$.

Let $G = \mu_3$ act on $S = k[[x, y, z]]$ via multiplication on $x, y, z$. Then $R = S^G$ is an isolated singularity and Gorenstein, so $\mathcal{E} := \text{MCM}(R)$ is Frobenius and $C = E$ is 2-CY (Auslander) $T = S$ is cluster-tilting (Iyama), $\text{End}_R T = S \times G$

\[ (6-1) \]

\[ \begin{array}{ccc}
   S^G & \longrightarrow & T_1 \\
   \downarrow & & \downarrow \\
   T_2 & \longrightarrow & T_2 \\
\end{array} \]

\[ \text{End}_C(T) = \text{End}_R(T), T_1 \longrightarrow T_2 \text{ Main Theorem implies } C \equiv C_Q \text{ where } Q \text{ is the quiver } 1 \longrightarrow 2 \]

Terminology: An acyclic 2-CY-category is a category $C$ as in the theorem.

Conjecture: If $C$ is a 2-CY-category with a cluster tilting object $T'$ having $\leq 3$ non-isomorphic indecomposable summands (and assume the quiver of $\text{End} T'$ has no loops and no 2-cycles), then $C$ is an acyclic 2-CY-category.

6.4. The proof. Uses DG-algebra $B = kQ \oplus (DkQ)[-3]$ with $d = 0$

\[ D^b(B) = \{ M \in D(B) \mid \sum \dim H^i(M) < \infty \} \text{ Full subcategory of } D(B), \]

per $D^b(B) = \text{full triangulated subcategory of } D^b(B)$ stable under factors by $B_B$

\[ C_Q \equiv D^b(B)/\text{per}(B). \]
7. Ringel—Degenerations of modules and the structure of Prüfer modules

7.1. Degenerations of finite length modules. Defn: $W'$ is a degeneration of $W$ if there is a ses $0 \to X \to X \oplus W \to W' \to 0$. This is equivalent to the existence of a ses $0 \to W' \to Y \oplus W \to Y \to 0$.

If $W$ and $W'$ are degenerations of each other, then $W \cong W'$.

Bautista-Perres: $\Lambda$ an artin algebra over a commutative artinian ring $k$, module finite over $k$. Let $W$ and $W'$ be finite length $\Lambda$-modules with $\text{top}(W) \cong \text{top}(W')$ and $\Omega W \cong \Omega W'$. If $\text{Ext}^1(W, W) = \text{Ext}^1(W', W') = 0$, then $W \cong W'$.

What is only one of the Ext groups vanishes? If $\text{Ext}^1(W, W) = 0$ is $W'$ a degeneration of $W$?

Take monics

$\begin{array}{c}
\xymatrix{
U_1 & & U_2 \\
\downarrow_{v_0} & & \\
U_1 & & U_2 \\
\downarrow & & \\
V & & W
}\end{array}$

Form a pushout

$\begin{array}{c}
\xymatrix{
U_0 & U_1 & U_2 & \cdots & U_\infty = \bigcup U_i \\
\downarrow_{v_0} & \downarrow_{u_1} & \downarrow & \cdots & \downarrow_{v_\infty} \\
U_1 & U_2 & U_3 & \cdots & U_\infty
}\end{array}$

Set $V = \text{coker } w$ and $V = \text{coker } v$. If $\text{Ext}^1(W, W) = 0$ then $V$ is a degeneration of $W$.

Repeat this process. obtain

$\begin{array}{c}
\xymatrix{
U_\infty/U_0 & U_\infty/U_1 & \cdots & U_\infty/U_0 \\
\downarrow^{v_\infty} & \downarrow & \cdots & \downarrow
}\end{array}$

Then get

Properties of $\varphi$:
(1) surjective
(2) locally nilpotent
(3) $\ker \varphi = U_1/U_0$ has finite length

I stopped taking notes.....
8. **Brown—Noetherian Hopf Algebras**

### 8.1. Hopf background

He will insist that the antipode $S$ is bijective.

If $A$ is a finite dimensional Hopf algebra it is Frobenius, i.e., self-injective, so $\text{Hom}_k(A, k) \cong {}^*A^1$ as a bimodule, where $\nu$ is the Nakayama automorphism. Say that $A$ is symmetric if $\nu = \text{id}_A$.

When $\dim_k A < \infty$, $\text{Hom}_A(A k, A) \cong k$ and call the unique copy of $A k$ as a left submodule of $A$ the left integral and denote it by $f_A^L \leq A$; can do this on the right too. Say $A$ is unimodular if $f_A^L = f_A^R$.

**Theorem 8.1** (Oberst, Schneider, 1973). \( \nu = \text{id}_A \) if and only if $A$ is unimodular and $S^2 = 1$.

### 8.2. Infinite dimensional Hopf algebras

Is every noetherian Hopf algebra $A S$-Gorenstein? Yes for $U(g), U_q(g), O(G), O_q(G), A = kG$ when $G$ is polycyclic-by-finite, and when $A$ is affine noetherian PI (Wu-Zhang, 2003).

One requirement for $A S$-Gorenstein is that $\text{Ext}^d_A(k, A) \cong k A$ which is a $d$-dimensional analogue of the fact that $\text{Hom}_A(A k, A)$ is 1-dimensional in the finite dimensional case. Wu-Zhang define homological left and right integrals, $\int^L_A = \text{Ext}^d_A(k, A)$ and $\int^R_A = \text{Ext}^d_A(k A, A)$.

### 8.3. Dualizing complexes

Let $B$ be a $k$-algebra and define $B^e := B \otimes_k B^{op}$. A complex $R \in D^b(\text{mod} B^e)$ is a dualizing complex if

1. $R_B$ and $R_B$ have finite injective dimension
2. $R$ is homologically finite
3. $B \to R\text{Hom}^*_B(R_B, R_B)$ and $B \to R\text{Hom}^*_B(B R_B, B R_B)$ are isomorphisms.

**Theorem 8.2** (VdB). (1) If a rigid $R$ exists it is unique up to unique isomorphism

2. Suppose $\text{injdim} B = d < \infty$ and $\exists$ an invertible $B$-module $U$ such that $\text{Ext}^i_{B^e}(B, B^e) = \begin{cases} U & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$

Then $B$ has a rigid $R$ and $R = U^{-1}[d]$.

**Theorem 8.3** (Brown-Zhang). Let $A$ be an AS-Gorenstein Hopf algebra.

1. If a rigid $R$ exists it is unique up to unique isomorphism

2. $\text{Ext}^i_{A^e}(A, A^e) = \begin{cases} {}^1A^\nu & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$

3. $A$ has a rigid dualizing complex $R = {}^*A^1[d]$ where $\nu$ is an algebra automorphism of $A$ unique up to inner automorphism (call $\nu$ “the” Nakayama automorphism)

4. $\nu = S^2 \pi_{\pi_0}$.

$\pi : A \to k$ an algebra homomorphism gives a left winding automorphism $\tau^\ell_\pi : A \to A, \quad a \mapsto \sum \pi(a_1) a_2$.
Use $\pi_0 : A \to A/rann(f^A)$.

8.4. Hochschild (co)homology.

Corollary 8.4. Suppose $A$ is AS-regular, $\text{gldim } A = d$.

1. Then $HH^i(A, M) \cong \Cohom_{d-i}(A, 1^A M^\nu)$ for all $i$ and all bimodules $M$.
2. $A$ has a rigid $R = ^\nu A^1[d]$.
3. $(A, \Delta^\text{op}, S^{-1}, \epsilon)$ is Hopf and has Nakayama automorphism $\nu' = S^{-2}\tau^\epsilon_{\pi_0}$ so equal to the previous one and $S^4 = \gamma(\tau^\epsilon_{\pi_0})(\tau^\epsilon_{\pi_0})^{-1}$ for some inner $\gamma$.

Note this recovers the Hochschild-Kostant-Rosenberg result.
9. Serre—Serre functors, symmetric algebras, and TQFT

9.1. Serre functors. Let $\mathcal{C}$ be a $\mathbb{C}$-linear category with $\dim \text{Hom}(-,-) < \infty$. A functor $S : \mathcal{C} \rightarrow \mathcal{C}$ is a Serre functor if

1. it is an equivalence and
2. there are functorial natural isomorphisms $\text{Hom}(M,N) \cong \text{Hom}(N,SM)^*$.

Let $A$ be a finite dimensional $\mathbb{C}$-algebra and $C$-free $A$-modules. Then $S$ is a Serre functor if and only if $\text{Hom}_A(A,A) \cong \text{Hom}(A,SA)^*$, i.e., $A \cong (SA)^*$ as left modules and the requirement that we have a natural isomorphism of bifunctors is equivalent to the requirement that $A \cong (SA)^*$ as bimodules. Hence $S \cong \text{id}_C$ if and only if $A$ is symmetric (i.e., Calabi-Yau of dimension 0).

What is the Serre functor for $D^b(A)$? Kapranov: $D^b(A)$ has a Serre functor if and only if $\text{gl.dim} A < \infty$, and in this case $S = \mathcal{L}(A^* \otimes_A -)$.

Let $G = \text{SL}(n, \mathbb{C})$ and let $B$ be the Borel subgroup. What is $S$ for $D^b(\text{Perv}_G(G/B))$?

We know this category is equivalent to $D^b(A)$ with $A < \infty$, but don’t know $A$.

What is $D^b(\mathcal{O}_{\text{triv}}(\mathfrak{sl}_n))$?

For $n = 2$, $\text{Perv}(G/B) \equiv \text{mod}A$ where $A$ is the path algebra of the quiver

$$
\bullet \xrightarrow{y} \bullet
$$

with relations $xy = 0$. $A_1 = \text{End}(\text{proj} - \text{inj}) = P(1) \oplus P(s)$, $P(s) = \text{proj-inj}$, $\text{End} P(s) \cong \mathbb{C}[x]/(x^2)$.

For $\text{Perv}(G/P)$ with $P$ maximal parabolic

$$
0 \xrightarrow{b_0} 1 \xrightarrow{\cdots} n-2 \xrightarrow{b_{n-2}} n-1
$$

with relations $\cdots = 0 = \cdots$ and $a_0 b_0 = 0$ and at each vertex

$$
\begin{array}{cc}
\circ \\
\end{array}
\begin{array}{cc}
\circ \\
\end{array}
= 0
$$

$X_i = A e_i \otimes e_i A \otimes_A -$

Theorem 9.1 (Stroppel—Mazorchuk). $G = \text{SL}(n, \mathbb{C})$? Let $\mathcal{P} = \text{Perv}(G/B)$. For any simple reflection $s \in S_n$ there is a right exact functor $T_s : \mathcal{P} \rightarrow \mathcal{P}$ such that

1. $\mathcal{L}T_s : D^b(\mathcal{P}) \rightarrow D^b(\mathcal{P})$ is an equivalence
2. they satisfy the braid relations,
3. $w_0 = s_{i_1} \cdots s_{i_r} \in S_n$ the longest element, the Serre functor is $(\mathcal{L}T_{s_{i_1}} \cdots \mathcal{L}T_{s_{i_r}})^2 = (\gamma \mathcal{L}T_{w_0})^2$.
4. $S$ = partial co-approx wrt all simples that are not in head(\text{proj-inj}).


$T = \oplus \text{inde}$. proj. in $\text{Perv}(G/P)$, $G = \text{SL}(n, \mathbb{C})$, then $\text{End} T$ is symmetric. Big question: what is $\text{End} T$? It depends only on the partition of $n$.

10. TQFT

Let $A_n$ be the algebra such that $\text{mod}A_n \equiv \text{Perv}(G/P)$ for $G = \text{SL}(2n, \mathbb{C})$ and $P \equiv [n,n]$. $\mathcal{L}T_{s_i} : D^b(\text{mod}A_n) \rightarrow D^b(\text{mod}A_n)$ corresponds to a simple crossing.
of adjacent strands at \( i \), and \((CT_n)^-\) corresponds to the other simple crossing of adjacent strands at \( i \) (the inverse crossing).

**Theorem 10.1.** There is a functor

\[
\Phi : \{\text{oriented tangles}\} \longrightarrow \text{“my category”} = 
\]

Therefore get a functorial tangle invariant.

<table>
<thead>
<tr>
<th>Objects</th>
<th>Morphisms</th>
<th>2-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{N} )</td>
<td>tangles</td>
<td>( \text{D}^b(A_n - \text{mod}) )</td>
</tr>
<tr>
<td>( \text{tangles} )</td>
<td>cobordisms</td>
<td>functors</td>
</tr>
<tr>
<td>( \text{natural transformations} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The category of 2-dimensional TQFTs is equivalent to the category of finite dimensional Frobenius algebra \((A, m, \Delta, \varepsilon, \eta)\). Topologically a TQFT is a functor on the category of 2-cobordisms

\[
2 - \text{Cob} \rightarrow \text{mod} \kappa 
\]

The objects in \( 2 - \text{Cob} \) are oriented compact 1-manifolds and morphisms are cobordisms

10.1.
10.2. Martino—Symplectic reflection algebras and deformed pre-projective algebras

11.1. Poisson varieties. Work over $\mathbb{C}$. Let $A$ be a Poisson algebra. For each $f \in A$, call $\Theta_f := \{ f, \}^{-1}$ a Hamiltonian vector field.

Example: Let $(V, \omega, G)$ be a symplectic vector space with a finite group $G \subset \text{Sp}(V)$. Then $(SV)^G$ is Poisson so $V/G$ is a Poisson variety.

Let $x \in X$. The symplectic leaf through $x$, denoted $S(x)$, is obtained by taking the integral curves with respect to the Hamiltonian vector fields, and close up under this process. Each $S(x)$ is a symplectic manifold. Then $X$ is a disjoint union of symplectic leaves. Note the leaves need not be algebraic subvarieties!

11.2. SRAs at $t = 0$. Take data $(V, \omega, G)$ as above and assume the data is indecomposable. Define $H_{t,c}$ a deformation of $SV \rtimes G$. Define $Z_c = Z(H_{0,c})$. This gets a Poisson structure by quantization with respect to $t$.

11.3. Deformed pre-projective algebras. Let $Q$ be a finite quiver and $ar{Q}$ its double. Fix a dimension vector $\alpha$. Then $\text{Rep}(\bar{Q}, \alpha) = T^* \text{Rep}(Q, \alpha)$, so is symplectic, and $G = G(\alpha)$ acts on it. There is a moment map

$$\mu: \text{Rep}(Q, \alpha) \to (\text{Lie} G)^*$$

Let $\lambda$ be a fixed point in $(\text{Lie} G)^*$; then $\mathcal{N}(\lambda, \alpha) := \mu^{-1}(\lambda)/G$ is a Poisson variety and is the moduli space of semisimple representations of dimension $\alpha$ of the pre-projective algebra $\Pi(\alpha)$.

Theorem 11.1. (1) The symplectic leaves of $\mathcal{N}(\lambda, \alpha)$ are the representation type strata (i.e., according to the isotypic components)

(2) For $V = (\mathbb{C}^2)^{\oplus \alpha}$ and $G = S_n \wr \Gamma$ where $\Gamma \subset \text{SL}_2(\mathbb{C})$, $\text{Spec } Z_c \cong \mathcal{N}(\lambda(c), \alpha)$ for a quiver $Q_G$ and this identifies symplectic leaves.

(3) $Q_G$ is McKay quiver with an extending vertex $\infty$
12. I. Reiten—Tilting modules for Calabi-Yau algebras

Joint with Iyama.

R commutative, noetherian ring, Λ an R-algebra, finitely generated as an R-module. Assume R ⊂ Λ.

12.1. Calabi-Yau algebras. Λ is n-Calabi-Yau if there are functorial isomorphisms

\[ \text{Hom}(X, Y[n]) \cong D \text{Hom}(Y, X) \]

for all X, Y ∈ Db(finite length Λ-modules).

Λ is n-Calabi-Yau− if there are functorial isomorphisms

\[ \text{Hom}(X, Y[n]) \cong D \text{Hom}(Y, X) \]

for all X ∈ Db(finite length Λ-modules) and Y ∈ K^b(per Λ).

Rickard: n−CY ⇒ n−CY−.

Theorem 12.1. R local Gorenstein of dimension d

1. If Λ is n−CY−, then n = d
2. Λ is d-CY ⇔ Λ is a symmetric R-order, i.e., Hom_R(Λ, R) ∼= Λ as R-R-bimodule and Λ ∈ MCM(R)
3. Λ is d-CY ⇔ Λ is a symmetric R-module and gldim Λ = d

A local commutative and d-CY− (resp., d-CY) then Λ is Gorenstein (resp., regular).

E.G., char k = 0 and S = k[[x, y, z]]^G where G ⊂ SL(3) is finite, R = S^G and Λ := S ⋊ G is 3-CY.

E.G., G = µ_3 the quiver is (6-1)

E.G., G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \zeta = e^{2\pi i/3} Then Q is

12.2. Tilting theory.

12.2.1. R=complete local Gorenstein domain. A f.gend. Λ-module T is tilting if pdim_Λ T ≤ 1, Ext^1_Λ(T, T) = 0, and there is a ses 0 → Λ → T_0 → T_1 → 0 with T_i ∈ add T.

Deal mainly with T a reflexive R-module; if d = 3, then T reflexive implies pdim T ≤ 1.

12.2.2. Let \( \Lambda = P_1 \oplus \cdots \oplus P_n \) where each \( P_i \) is indec. proj., Assume Ext^1_Λ(S, S) = 0 for all simple S. Let d = 3. Then \( \Lambda \) is a tilting module.

Fact: if \( \Lambda \) is 3-CY, then for each \( j \), there exists a unique \( P_j^* \neq P_j \) such that \( T = \Lambda/P_j \oplus P_j^* \) is a tilting module. Note \( P_j^* = \Omega^2 S_j \). Since \( P_j^* \) is reflexive, so is \( T \).

Then \( \Lambda' := \text{End } T \) is derived equiv to \( \Lambda \), so is also 3-CY.

Let \( \mu_j(\Lambda) : = \Lambda' \). Then \( \mu_j(\mu_j(\Lambda)) = \Lambda \).

Can repeat this to get a sequence \( \Lambda, \Lambda', \Lambda'', \cdots \Lambda^{(n)} \).
Theorem 12.2. Let \( \Lambda \) be 3-CY. If \( T_1 \) and \( T_2 \) are reflexive tilting modules, and \( \Gamma_1 := \text{End}_\Lambda(T_1, T_2) \) is a reflexive \( \Gamma_1 \)-module and \( \text{End}_\Gamma M \cong \Gamma_2 \).

12.2.3. Let \( \Lambda \) be 3-CY and \( T = T_1 \oplus \cdots \oplus T_n \) a tilting module. For each \( j \), there exists a unique \( T_j^* \not\sim T_j \) such that \( T' := T_j / T_j \oplus T_j^* \) is tilting.

Define \( \mu_j(T) = T' \).

Theorem 12.3. Let \( \Lambda \) be 3-CY. If \( T_1 \) and \( T_2 \) are reflexive tilting modules, and \( \Gamma_i := \text{End}_\Lambda(T_i) \), then \( M := \text{Hom}_\Lambda(T_1, T_2) \) is a reflexive tilting \( \Gamma_1 \)-module and \( \text{End}_\Gamma M \cong \Gamma_2 \).

12.3. Connections with Van den Bergh’s ncCR. \( R \) = normal Gorenstein domain, \( \dim R = 3 \), \( \Lambda \) an \( R \)-algebra

Definition 12.4. A \( \Lambda \)-module \( M \) with \( \Gamma := \text{End}_\Lambda M \), gives a non-commutative crepant resolution (ncCR) if
- \( M \cong M \vee \vee \) and is a height one generator, i.e., \( M_p \) is a \( \Lambda_p \)-generator for all height one primes \( p \)
- \( \Gamma_p \) is an \( R_p \)-order with \( \text{gldim} \Gamma_p = \text{ht} p \) for all maximal \( p \)

Conjecture (VdB): If \( M_1 \) and \( M_2 \) give ncCRs, then \( \Gamma_1 \) and \( \Gamma_2 \) are derived equivalent. (\( \Gamma_i = \text{End}_\Lambda M_i \))

Theorem 12.5. Let \( \Lambda \) be 3-CY. If \( M_1 \) and \( M_2 \) give ncCRs, then \( M = \text{Hom}_\Lambda(M_1, M_2) \) is a reflexive tilting \( \Gamma_1 \)-module and \( \text{End}_\Gamma M \cong \Gamma_2 \), so \( \Gamma_1 \) and \( \Gamma_2 \) are derived equivalent.

Lemma 12.6 (Key). If \( \Lambda \) is an isolated sing, \( R \) local, \( \text{depth} \text{Hom}(M, N) \geq 3 \), then \( \text{Ext}_\Lambda^1(M, N) = 0 \).

Theorem 12.7. Let \( \Lambda \) be 3-CY. Then \( \{ M \mid M \text{ gives a ncCR} \} = \{ \text{reflexive tilting modules} \} \).

12.4. Connections with cluster algebras. \( Q \) a quiver with vertices \( 1, \ldots, n \) with no loops or cycles of length 2.

Let \( R \) be a complete local Gorenstein ring and write \( k = R/\mathfrak{m} \) is algebraically closed.

\( \mathcal{A}(Q) \subset \mathbb{Q}(x_1, \ldots, x_n) \)

Associate to a 3-CY \( \Lambda \) a quiver \( Q_{\Lambda} \).
13. J. BELL—PRIMITIVITY AND TWISTED HOMOGENEOUS COORDINATE RINGS

(with Dan Rogalski)
k = algebraically closed field, \( X \) = projective variety, \( \sigma \in \text{Aut } X \), \( \mathcal{L} = \sigma \)-ample invertible \( \mathcal{O}_X \)-module.

When is \( B(X, \sigma, \mathcal{L}) \) primitive? Want answer in terms of centers and height one primes. Idea is like Dixmier’s result: \( \mathcal{U}(g)/p \) is primitive if and only if it has finitely many height one primes and \( Z(\text{Fract } U/p) = k \).

Theorem 13.1. Suppose \( k \) is uncountable. The following are equivalent:

1. \( B(X, \sigma, \mathcal{L}) \) is primitive
2. \( \exists x \in X \text{ such that } \{ \sigma^n(x) \mid n \in \mathbb{Z} \} \text{ is dense} \)
3. \( B(X, \sigma, \mathcal{L}) \) has finitely many height one primes
4. \( B(X, \sigma, \mathcal{L}) \) has countably many height one primes
5. \( Z(\text{Fract}_0 B(X, \sigma, \mathcal{L}))/k = Z(k(X)[t, t^{-1}; \sigma]) = k \)
14. Iain Gordon—q-Schur algebras, Cherednik algebras and quiver varieties

14.1. q-Schur algebras. $G$ = complex reflection group, $\mathfrak{h}$ = reflection representation, i.e., only one non-trivial eigenvalue (any root of 1 is allowed), $S$ = generating set of reflections.

Let $\gamma : S/\sim \rightarrow \mathbb{Q}$ be a class function; set $q := e^{2\pi i \gamma} \in \mathbb{C}^\times$. Let $\mathcal{H}_G(q) =$ the cyclotomic Hecke algebra; it has two kinds of relations, braid relations (analogues of $T_1T_2T_1 = T_2T_1T_2$, and Hecke relations (like $(T-q)(T+1) = 0$). At $q = 1$, $\mathcal{H}_G(q) = CG$. When $G = S_n$ there is only one conjugacy class of reflections so only one $q$ and this is the usual Iwahori-Hecke algebra $H_{S_n}(q)$. When $G = S_n \ltimes (\mu_1 \times \cdots \times \mu_\ell)$ get Ariki-Koike algebras.

For generic $q$, $\mathcal{H}_G(q)$ is semisimple but most interested in the non-semisimple ones. Want a quasi-hereditary algebra $\mathcal{A}$ together with a covering $F : \mathcal{A} \rightarrow \mathcal{H}_G(q) - \text{mod}$ where $F = \text{Hom}_A(P, -)$ where $P$ is projective and $\text{End}_A P \cong \mathcal{H}_G(q)$ and $\text{End}_{\mathcal{H}_G(q)} P \cong \mathcal{A}$.

When $G = S_n$, take $\mathcal{A} = S_q(n) = q$-Schur algebra.

When $G = S_n \ltimes (\mu_1 \times \cdots \times \mu_\ell)$, $\mathcal{A} =$ cyclotomic $q$-Schur algebra.

**Theorem 14.1.** (Ginzburg-Guay-Opdam-Rouquier) For $\gamma$, there exists $A_\gamma$, a quasi-hereditary cover of $\mathcal{H}_G(q)$.

14.2. Construction. The rational Cherednik algebra of type $G$ is $H_\gamma$ generated by $\mathbb{C}[\mathfrak{h}]$, $CG$, $\mathbb{C}[\mathfrak{h}^*]$ with relations

$$gxg^{-1} = g x,$$

$$g y g^{-1} = g y,$$

$$x y - y x = \sum_{s \in S} \gamma(s) (\alpha_s, y)(\alpha_s, \alpha_s^\vee) s.$$

for $x \in \mathfrak{h}$, $y \in \mathfrak{h}^*$, $g \in G$, where $\alpha_s$ and $\alpha_s^\vee$ define the reflecting hyperplanes for $s$ in $\mathfrak{h}$ and $\mathfrak{h}^*$.

**Theorem 14.2.** (Cherednik, Etingof-Ginzburg) $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes G \cong \text{gr} H_\gamma$ so there is a vector space isomorphism $\mathbb{C}[\mathfrak{h}] \otimes CG \otimes \mathbb{C}[\mathfrak{h}^*] \cong H_\gamma$

Is there a geometric picture for this?

Define $O(\gamma)$ full subcat of $H_\gamma - \text{mod}$ such that $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently. Verma module $\Delta_{\gamma}(E) := H_{\gamma} \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes G} E$ has a unique simple quotient $L_\gamma(E)$, and as $E$ varies over the irreducible reps of $G$ this gives all simple modules in $O(\gamma)$.

Lusztig’s c-function

$$c_E(E) := \sum_{s \in S} \gamma(s) \frac{\text{Tr}(s|E)}{\text{dim} E}.$$

Define $E \succ F$ if $c_E - c_F \in \mathbb{Z}_{>0}$. Then

$$KZ : O(\gamma) \equiv A_\gamma - \text{mod} \rightarrow \mathcal{H}_G(q) - \text{mod}$$

is a cover.

**Theorem 14.3.** (Rouquier) If $\gamma - \gamma' \in \mathbb{Z}^{S/\sim}$ and $\gamma$ and $\gamma'$ induce the same ordering on $\mathcal{G}$, then $O(\gamma) \equiv O(\gamma')$. 
Define $E > F$ if $c_E - c_F \in \mathbb{Q}_{>0}$.

E.G., $G = S_n \ltimes (\mu_\ell \times \cdots \times \mu_\ell)$, $S = \{s_{ij}, g_i g_j^{-1} \mid g \in \mu_\ell\}$

$\tilde{G} \leftrightarrow \ell$-multipartitions of $n$.

14.3. Quiver Varieties. $G = S_n \ltimes (\mu_\ell \times \cdots \times \mu_\ell) = S_\ell \, \mu_\ell$ McKay quiver

$$
\begin{array}{c}
1 \quad n \\
\downarrow \\
0 \quad n
\end{array}
\quad
\begin{array}{c}
1 \quad n \\
\downarrow \\
0 \quad n
\end{array}
\quad
\begin{array}{c}
\infty \quad 1 \\
\downarrow \\
1 \quad n
\end{array}
$$

$$
\mu^{-1}(0) \subset \text{Rep}(\bar{Q}_\infty, n\delta + \varepsilon_\infty)
$$

$$
\begin{array}{c}
\mathcal{M}_\theta(n) \\
\downarrow \pi
\end{array}
\quad
\begin{array}{c}
\mathfrak{h} \oplus \mathfrak{h}^*/G
\end{array}
$$

$\theta$ moves in chambers to provide resolutions of singularities

14.4. Connection. $U_\gamma = eH_\gamma e$, $e \in CG$ trivial idempotent

$\text{gr } U_\gamma \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*/G]$.

But very often $U_\gamma$ is Morita equivalent to $H_\gamma$

Theorem 14.4. (Gordon-Stafford, Musson, Boyarchenko, Vale) For most $\gamma$, there is a $\mathbb{Z}$-algebra $Z_\gamma$ such that

$$
\begin{array}{c}
\text{coh } Z_\gamma \cong \text{gr } M_\theta(n) \\
\downarrow \cong
\end{array}
\quad
\begin{array}{c}
U_\gamma \text{ mod } \cong \text{gr } \mathfrak{h} \oplus \mathfrak{h}^*/G
\end{array}
$$

where $\theta_\gamma = (\gamma + \gamma_1 + \cdots + \gamma_{\ell-1}, -\gamma_1, \ldots, -\gamma_{\ell-1})$.

14.5. Main Theorem.

Theorem 14.5. Suppose $\theta_\gamma = (\gamma + \gamma_1 + \cdots + \gamma_{\ell-1}, -\gamma_1, \ldots, -\gamma_{\ell-1})$ and assume $\gamma$ is in a Rouquier chamber. Then

1. $\mathcal{M}_{\theta_\gamma}(n)$ is a crepant resolution
2. $\mathcal{L}_{\theta_\gamma} := \pi^{-1}(\mathfrak{h}^* = 0)$ is Lagrangian and its components are naturally labelled by the \ell-multipartitions of $n$
3. there is a natural partial ordering on the components and so on $\text{Irr}(S_\ell \, \mu_\ell)$ which is refined by Rouquier’s ordering
4. This ordering comes from combinatorics of higher level Fock spaces of Uglov.

Main idea of proof is that Lusztig’s $c$-function is a Morse function for $\mathcal{M}_\theta(n)$
15. M. Reinieke—(Non)singular quiver moduli

15.1. Motivation. Interplay between $\mathbb{C}Q$ and $M_\theta(Q)_d$=moduli of $d$-diml repns of $Q$
 Ask questions about the global geometry of the moduli space, compute $\chi$=Euler characteristic and $\dim H^i(-, \mathbb{Q})$=Betti numbers, explicit formulas

15.2. Notation. Fix $(Q, d, \theta)$ where $Q = (Q_0, Q_1)$, $d \in \mathbb{N}Q_0$, $\theta \in (\mathbb{Q}Q_0)^*$ a stability condition.
 The slope of $X \in \text{Rep}_{\mathbb{C}}Q$ is $\mu(X) := \frac{\theta(\dim X)}{\dim X}$, and $X$ is stable if for all proper submodules $Y \subset X$, $\mu(Y) < \mu(X)$

Theorem 15.1. (King) There exist

$M_{d}^{st}(Q) = \{\text{isoclasses of stable reps of } \dim X\}$

and

$M_{d}^{sst}(Q) = \{\text{isoclasses of polystable reps of } \dim X\}$

Always $M_{d}^{st}(Q)$ is smooth open subvariety of $M_{d}^{sst}(Q)$ and the latter is projective over the affine variety $M_{d}^{ssimp}(Q)$.

$M_{d}^{st}(Q)$ is projective if $Q$ has no oriented cycles.

Suppose $d$ is coprime for $\theta$, i.e., if $0 \neq e < d$, then $\mu(e) \neq \mu(d)$ ($\Rightarrow \gcd\{d_i\} = 1$).

Then $M_{d}^{st}(Q) = M_{d}^{sst}(Q)$ so is smooth
16. ZHANG—Four dimensional regular algebras

Problem: Classify these. Lower dimensional ones are known. Requires new ideas.

16.1. Background. \( k \) = field, algebraically closed, characteristic zero. \( A \) denotes an \( \mathbb{N} \)-graded connected \( k \)-algebra generated by \( A_1 \) and \( \text{dim}_k A_1 < \infty \).

\( A \) is AS-regular if

1. \( \text{gldim} A = d < \infty \)
2. \( \text{Ext}^i_A (k, A) = 0 \quad i \neq d \), \( \text{Ext}^d_A (k, A) = k \)
3. \( \text{GKdim} A < \infty \)

Write \( g = \text{dim} A_1 \).

Problem: Classify those with \( d = 4 \). If \( g = 4 \), \( A \) is Koszul. If \( g = 3 \), there are 4 relations. If \( g = 2 \), there are 2 relations.

Example due to Caines, \( g = 4 \), not an Ore extension, \( \text{Aut}_{gr} A = k^\times \), point scheme contains 13 points.

16.2. \( A_\infty \)-algebra methods.

Theorem 16.1 (Keller). Let \( A = k[A_1]/(R) \) with \( R = R_{\geq 2} \) minimal. Set \( E = \text{Ext}^*_A (k, k) \) with its \( A_\infty \) structure. The higher multiplications of \( E \) are determined by \( m_n : (E_1)^{\otimes n} \cong (A_1)^{\otimes n} \rightarrow E^2 \) is the composition of the dual of \( i_n : R_n \rightarrow (A_1)^{\otimes n} \) with the inclusion of \( R_n^* \) in \( E^2 \).

Simple examples, \( A = k[x]/(x^p) \), \( p \geq 3 \), \( E = k[a, b]/(b^2) \) where \( a = x^*, b = r^* \), where \( r = x^p \) and \( (E, m_2, m_3, \ldots, m_6, g) \). When \( g = 2 \), can work out all \( (E, m_2) \) so try to compute all possible \( m_3, m_4 \), then use Keller’s theorem to recover \( A \).

Ex: Three other classes and \( C = k[x, y] \) with

\[
xy^2 + pyxy + p^2y^2x = x^3 + jy^3 + x^3 = 0
\]

where \( j^2 - j + 1 = 0 \). These are all the \( \mathbb{Z}^2 \)-graded ones.

16.3. Extensions. Let \( A \) and \( B \) be connected graded. Call \( C \) an extension of \( A|B \) if there is an “exact sequence” \( 0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0 \) meaning

1. \( A \) is a subalgebra of \( C \) and the map \( C \rightarrow B \) is an algebra homomorphism,
2. \( C \) is a free left and right \( A \)-module with basis given by a basis for \( B \)
3. \( A_{\geq 1}C = CA_{\geq 1} \), and \( C/(A_{\geq 1}) \cong B \).

Examples: tensor product \( A \otimes_k B \)

NE=normal extension, \( A = k[x], \text{deg} x = 1 \), \( C \) is an extension \( k[x]|B \) if and only if \( x \) is normal and regular in \( C \) and \( B = C/(x) \).

OE=Ore extensions, \( C \) is an extension of \( A[k[x]] \) if and only if \( C = A[x; \sigma, \delta] \).

1. \( C \) need not be noetherian even if \( A \) and \( B \) are.
2. If \( C \) is NE of \( B \) or OE of \( A \), then \( C \) is noetherian (resp., regular) iff \( A \) is, and then \( \text{gldim} C = 1 + \text{gldim} A \).
3. If \( A \) and \( B \) are noetherian or regular, then \( \text{gldim} C \leq \text{gldim} A + \text{gldim} B \).
16.4. **Double extensions.** An extension $C$ is called a **double extension** (DE) of $A|B$ if $B$ is regular of dimension 2.

**My remarks.** This is like a fiber product definition, especially if one has a $\mathbb{Z}^2$-grading on $C$ coming from a $\mathbb{Z}$-grading on each of $A$ and $B$. Get a map Proj$_{nc} C \rightarrow$ Proj$_{nc} A$ and Proj$_{nc} B \rightarrow$ Proj$_{nc} C$.

**Theorem 16.2.** If $A$ is regular so is every double extension of it and $\text{gldim } C = \text{gldim } A + 2$.

If $A$ is noetherian is every DE of $A$ noetherian?

**Theorem 16.3.** Every 4-diml regular algebra that is a DE of a 2-diml regular algebra is universally noetherian, CM, and Auslander regular.

**DE data:** Write $A_p[y_1, y_2; \sigma, \delta, w]$ for a DE of $A$ with $p = \{p_{12}, p_{11}\}$ a parameter set, $s : A \rightarrow M_2(A)$ an algebra homomorphism, $\delta : A \rightarrow A^{\otimes 2}$ a $\sigma$-derivation, $w = \{t_0, t_1, t_2\} \subset A$, the tail.

$C$ is generated by $A$ and $y_1$ and $y_2$ with relations

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + t_1y_1 + t_2y_2 + t_0$$

and

$$\begin{pmatrix} y_1a \\ y_2a \end{pmatrix} = \sigma(a) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \delta(a)$$

and some more relations involving $\{p, \sigma, \delta, w\}$.

(Does he mean every DE is of this form? and then use this classification to prove the theorem).

Every $\mathbb{Z}^2$-graded $d = g = 4$-dimensional regular algebra is either DE or OE or NE.
17. Rogalski—

**Theorem 17.1** (with Stafford). Let $A$ be a connected $\mathbb{N}$-graded $k$-algebra such that

1. $A = k[A_1]$, and $A$ is a noetherian domain
2. $A$ is a birationally commutative surface, i.e., $\text{Fract}_{gr} A = K[t, t^{-1}; \sigma]$ where $K$ is a field with $\text{trdeg} K/k = 2$.
3. $\sigma$ is geometric, i.e., there is a projective surface $Y$ with $K = k(Y)$ and a $\tau \in \text{Aut} Y$ that produces $\sigma$ by pullback of functions.

Then $A$ is isomorphic in big degree to a naive blowup $^9 R(X, L, \sigma, Z)$ where $X$ is a projective surface such that $k(X) = K$, $L$ is an invertible $O_X$-module, $\sigma \in \text{Aut} X$, and $Z$ is a 0-dimensional subscheme of $X$ and $\langle \sigma \rangle z$ is dense in $X$ for all $z \in Z$.

17.1. **Definition of $R(X, \sigma, L, Z)$**. Define

$$L_n = L \otimes \sigma^* L \otimes \cdots \otimes (\sigma^*)^{n-1} L.$$ 

Let $\mathcal{I} := $ the ideal sheaf of $Z$ and set

$$\mathcal{I}_n := \mathcal{I} \sigma^* \mathcal{I} \cdots (\sigma^*)^{n-1} \mathcal{I}.$$ 

and

$$R_n := H^0(X, L_n \otimes \mathcal{I}_n) \subset H^0(X, L_n).$$

Typically this ring is not strongly noetherian, and these are the most interesting one.

17.2. **An example**. $B = k[x_1, x_2, x_3]$ with relations $x_i x_j = p_{ij} x_j x_i$ is birationally commutative if and only if $p_{12} p_{23} = p_{13}$. In other words, $B = B(\mathbb{P}^2, O(1), \sigma)$. If $p \in \mathbb{P}^2$, then $R = R(\mathbb{P}^2, O(1), \sigma, p)$ is the subalgebra of $B$ generated by the 2-dimensional subspace of $B_1$ vanishing at $p$.

17.3. **Some proof**. Study point modules, but there is no scheme representing the point module functor.

17.4. **On condition (3)**. A non-geometric example. Take $K = k(u, v)$ and $\sigma \in \text{Aut} K$ defined as follows: let $X = \mathbb{P}^2$ and $\sigma = \tau_2 \tau_1 : X \dashrightarrow X$ a birational map where $\tau_1 = $ Cremona transformation and $\tau_2 \in \text{PGL}_2(k)$. If $\sigma$ were geometric there would be an upper bound on the number of fundamental points of $\sigma^n$...but can check that this doesn’t happen because $\tau_2$ will move the fundamental points away from the earlier ones...here the number of fundamental points of $\sigma^n$ is $3n$.

See also paper of Diller and Favre “Dynamics of bimeromorphic maps of surfaces” Using their results:

**Theorem 17.2**. Let $A$ be a connected graded $\mathbb{N}$-algebra birationally commutative domain with finite GK-dim and $Q(A) = K[t^{\pm 1}; \sigma]$ If $\sigma$ is geometric, then $\text{GKdim} A \in \{3, 5\}$. If $\sigma$ is not geometric then $\text{GKdim} A = 4$.

Can also get $\text{GKdim} A = \infty$.

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^9The idea is that the ring models a blowup of $X$ at $Z$ and $\sigma$ twists this.