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An Example of a Ring Morita Equivalent to the Weyl Algebra A_1

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1. INTRODUCTION

If R is a prime, principal right ideal ring, then Goldie [2] shows that R is an $n \times n$ matrix ring over a right Ore domain D for some $n \geqslant 1$. Jategaonkar [3, p. 137] asks whether the right Ore domain D so determined is necessarily unique. The example produced below shows that the domain is not uniquely determined. The example is a noetherian domain S which is not isomorphic to the Weyl algebra A_1, but satisfies $M_2(S) \cong M_2(A_1)$. It is, of course, well known that $M_2(A_1)$ is a principal right and left ideal ring.

Kaplansky asked the more general question: if A and B are right noetherian rings satisfying $M_n(A) \cong M_n(B)$, are A and B necessarily isomorphic? This more general question was answered in the negative by Jategaonkar [3, p. 44] and Knus [4, Example 4.1].

If I is a non-zero right ideal of A_1, then by Webber [6] $I \oplus I = A_1 \oplus A_1$ and so $M_2(\mathrm{End}_{A_1}(I)) \cong M_2(A_1)$. It is natural to ask whether A_1 and $\mathrm{End}_{A_1}(I)$ are necessarily isomorphic. The ring S, produced below, is in fact isomorphic to $\mathrm{End}_{A_1}(I)$ for a right ideal I of A_1. Thus this question also has a negative answer.

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2. THE EXAMPLE

Let k be an algebraically closed field of characteristic zero. Let $A = A_1(k)$ denote the Weyl algebra over the field k with two generators p and q subject to the relation $pq - qp = 1$.

Put $I = p^2A + (pq + 1)A$. Then I is a maximal, non-principal right ideal of A (see Rinehart [5]). As A is a prime, noetherian ring the endomorphism

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ring of I is isomorphic to $S = \{x \in Q \mid xI \subseteq I\}$, where Q is the quotient division ring of A.

For $t \in \mathbb{Z}$, put $D(t) = \{x \in A \mid [pq, x] = tx\}$, where $[a, b]$ denotes $ab - ba$. Then $A = \bigoplus_{-\infty}^{\infty} D(t)$ and

$$\begin{aligned} D(t) &= q^t k[pq], && t \geqslant 0 \\ &= p^{-t} k[pq], && t < 0. \end{aligned}$$

The subspaces $D(t)$ give A the structure of a graded ring because $D(s)\,D(t) \subseteq D(s+t)$. As I is generated by homogeneous elements, $I = \bigoplus_{-\infty}^{\infty} (I \cap D(t))$. It is easy to check that

$$\begin{aligned} I \cap D(t) &= (pq+1)\, q^t\, k[pq], && t \geqslant 0 \\ &= (pq+1)p\, k[pq], && t = -1 \\ &= p^{-t} k[pq], && t \leqslant -2. \end{aligned}$$

PROPOSITION 1. *The ring S is given by, $S = \bigoplus_{-\infty}^{\infty} E(t)$ where*

$$\begin{aligned} E(t) &= (pq+1)\, q^t\, k[pq](pq+1)^{-1}, && t \geqslant 2 \\ &= (pq+1)q\, k[pq], && t = 1 \\ &= k[pq], && t = 0 \\ &= (pq+1)p\, k[pq], && t = -1 \\ &= p^{-t}\, k[pq], && t \leqslant -2. \end{aligned}$$

Proof. Notice that $x \in S$ if and only if $xp^2 \in I$ and $x(pq+1) \in I$. Thus $S = Ip^{-2} \cap I(pq+1)^{-1}$, and because p^2 and $(pq+1)$ are homogeneous elements $S = \bigoplus_{-\infty}^{\infty} E(t)$ where $E(t) = (I \cap D(t-2))\,p^{-2} \cap (I \cap D(t))(pq+1)^{-1}$. We now show that $E(t)$ is as claimed. First, observe that $p^{-1} = q(pq)^{-1}$ and $p^{-2} = q^2(pq)^{-1}\,(pq+1)^{-1}$.

(i) $t \geqslant 2$

$$\begin{aligned} E(t) &= (pq+1)\, q^{t-2}\, k[pq]\, p^{-2} \cap (pq+1)\, q^t\, k[qp](pq+1)^{+1} \\ &= (pq+1)\, q^t\, k[pq](pq)^{-1}\,(pq+1)^{-1} \cap (pq+1)\, q^t\, k[pq](pq+1)^{-1} \\ &= (pq+1)\, q^t\, k[pq](pq+1)^{-1}; \end{aligned}$$

(ii) $t = 1$

$$\begin{aligned} E(1) &= (pq+1)p\, k[pq]\, p^{-2} \cap (pq+1)q\, k[pq](pq+1)^{-1} \\ &= (pq+1)q\, k[pq](pq)^{-1} \cap (pq+1)q\, k[pq](pq+1)^{-1} \\ &= (pq+1)q\, k[pq]; \end{aligned}$$

(iii) $t=0$

$$E(0)=p^2k[pq]p^{-2}\cap(pq+1)q\,k[pq](pq+1)^{-1}$$
$$=k[pq];$$

(iv) $t=-1$

$$E(-1)=p^3k[pq]p^{-2}\cap(pq+1)\,pk\,[pq](pq+1)^{-1}$$
$$=(pq+1)p\,k[pq];$$

(v) $t\leqslant -2$

$$E(t)=p^{-t+2}\,k[pq]p^{-2}\cap p^{-t}\,k[pq](pq+1)^{-1}$$
$$=p^{-t}\,k[pq].$$

Q.E.D.

Look at pq as an element of S. It is clear, for all $t\in\mathbb{Z}$, that

$$E(t)=\{x\in S\mid[pq,x]=tx\}.$$

It is now easy to see that pq is a strictly semi-simple element of S in the sense of Dixmier [1].

PROPOSITION 2. *The rings S and A are not isomorphic.*

Proof. Suppose $\theta\colon A\to S$ is an isomorphism. Then S may be considered as the Weyl algebra over k generated by the elements $\bar{p}=\theta(p)$ and $\bar{q}=\theta(q)$ subject to the relation $pq-qp=1$. Now pq is a strictly semi-simple element of S and so by [1, Théorème 9.2] (under the additional hypothesis that k is algebraically closed) there exists an automorphism Φ of S such that $\Phi(pq)=\bar{p}\bar{q}+\beta$ for some scalars α,β with $\alpha\neq 0$. Let $\psi=\Phi^{-1}\theta$ and consider the new isomorphism $\psi\colon A\to S$.

Let $a=\psi(p)$, $b=\psi(q)$. From the above

$$ab=\psi(pq)=\Phi^{-1}(\bar{p}\bar{q})=\alpha^{-1}(pq-\beta)\in E(0).$$

Considering $a=\sum a_t$, $b=\sum b_s$ with $a_t\in E(t)$ and $b_s\in E(s)$ it can be seen that the only way ab can be element of $E(0)$ is if $a\in E(t)$ and $b\in E(-t)$ for some $t\in\mathbb{Z}$. If $a\in E(t)$ and $b\in E(-t)$ with either $t\geqslant 1$ or $t\leqslant -1$, look at ab as a polynomial in pq. By rearranging the terms in the expression for ab (using identities such as $(pq)p=p(pq-1)$) it can indeed be made into a polynomial in pq. Moreover, the degree of ab as a polynomial in pq will be at least two—consequently $ab\neq\alpha^{-1}(pq-\beta)$. However, if $a,b\in E(0)$, then $[a,b]=0$ contradicting the fact that $[a,b]=\psi([p,q])=1$. We conclude that no such $a,b\in S$ can exist. Q.E.D.

It seems likely that infinitely many different such endomorphism rings exist—it would be interesting to know whether this is in fact the case.

References

1. J. Dixmier, Sur les Algebres de Weyl, *Bull. Soc. Math. France* **96** (1968), 209–242.
2. A. W. Goldie, Non-commutative principal ideal rings, *Arch. Math.* **13** (1962), 214–221.
3. A. V. Jategaonkar, "Left Principal Ideal Rings," Lecture Notes in Mathematics No. 123, Springer-Verlag, Berlin/Heidelberg/New York, 1970.
4. M. A. Knus, Algebres d'Azumaya et modules projectifs, *Comm. Math. Helv.* **45** (1970), 372–383.
5. G. S. Rinehart, Note on the global dimension of a certain ring, *Proc. Amer. Math. Soc.* **13** (1962), 341–346.
6. D. B. Webber, Ideals and modules of simple hereditary rings, *J. Algebra* **16** (1970), 239–242.