An Example of a Ring Morita Equivalent to the Weyl Algebra $A_1$

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1. Introduction

If $R$ is a prime, principal right ideal ring, then Goldie [2] shows that $R$ is an $n \times n$ matrix ring over a right Ore domain $D$ for some $n \geq 1$. Jategaonkar [3, p. 137] asks whether the right Ore domain $D$ so determined is necessarily unique. The example produced below shows that the domain is not uniquely determined. The example is a noetherian domain $S$ which is not isomorphic to the Weyl algebra $A_1$, but satisfies $M_2(S) \cong M_2(A_1)$. It is, of course, well known that $M_2(A_1)$ is a principal right and left ideal ring.

Kaplansky asked the more general question: if $A$ and $B$ are right noetherian rings satisfying $M_n(A) \cong M_n(B)$, are $A$ and $B$ necessarily isomorphic? This more general question was answered in the negative by Jategaonkar [3, p. 44] and Knus [4, Example 4.1].

If $I$ is a non-zero right ideal of $A_1$, then by Webber [6] $I \oplus I = A_1 \oplus A_1$ and so $M_2(\text{End}_{A_1}(I)) \cong M_2(A_1)$. It is natural to ask whether $A_1$ and $\text{End}_{A_1}(I)$ are necessarily isomorphic. The ring $S$, produced below, is in fact isomorphic to $\text{End}_{A_1}(I)$ for a right ideal $I$ of $A_1$. Thus this question also has a negative answer.

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2. The Example

Let $k$ be an algebraically closed field of characteristic zero. Let $A = A_1(k)$ denote the Weyl algebra over the field $k$ with two generators $p$ and $q$ subject to the relation $pq - qp = 1$.

Put $I = p^2A + (pq + 1)A$. Then $I$ is a maximal, non-principal right ideal of $A$ (see Rinehart [5]). As $A$ is a prime, noetherian ring the endomorphism
ring of $I$ is isomorphic to $S = \{ x \in Q \mid xI \subseteq I \}$, where $Q$ is the quotient division ring of $A$.

For $t \in \mathbb{Z}$, put $D(t) = \{ x \in A \mid [pq, x] = tx \}$, where $[a, b]$ denotes $ab - ba$. Then $A = \bigoplus_{-\infty}^\infty D(t)$ and

\[
D(t) - q^tk[pq], \quad t \geq 0
\]
\[
= p^{-t}k[pq], \quad t < 0.
\]

The subspaces $D(t)$ give $A$ the structure of a graded ring because $D(s)D(t) \subseteq D(s + t)$. As $I$ is generated by homogeneous elements, $I = \bigoplus_{-\infty}^\infty (I \cap D(t))$. It is easy to check that

\[
I \cap D(t) = (pq + 1)q^tk[pq], \quad t \geq 0
\]
\[
= (pq + 1)p k[pq], \quad t = -1
\]
\[
= p^{-t}k[pq], \quad t \leq -2.
\]

**Proposition 1.** The ring $S$ is given by, $S = \bigoplus_{-\infty}^\infty E(t)$ where

\[
E(t) = (pq + 1)q^tk[pq](pq + 1)^{-1}, \quad t \geq 2
\]
\[
= (pq + 1)q k[pq], \quad t = 1
\]
\[
= k[pq], \quad t = 0
\]
\[
= (pq + 1)p k[pq], \quad t = -1
\]
\[
= p^{-t}k[pq], \quad t \leq -2.
\]

**Proof.** Notice that $x \in S$ if and only if $xp^2 \in I$ and $x(pq + 1) \in I$. Thus $S = I(pq + 1)^{-1}$, and because $p^2$ and $(pq + 1)$ are homogeneous elements $S = \bigoplus_{-\infty}^\infty E(t)$ where $E(t) = (I \cap D(t - 2))p^{-2} \cap (I \cap D(t))(pq + 1)^{-1}$. We now show that $E(t)$ is as claimed. First, observe that $p^{-1} = q(pq)^{-1}$ and $p^{-2} = q^2(pq)^{-1} (pq + 1)^{-1}$.

(i) \quad $t \geq 2$

\[
E(t) = (pq + 1)q^{-2}k[pq]p^{-2} \cap (pq + 1)q^tk[ap](pq + 1)^{+1}
\]
\[
= (pq + 1)q^tk[pq](pq)^{-1}(pq + 1)^{-1} \cap (pq + 1)q^tk[pq](pq + 1)^{-1}
\]
\[
= (pq + 1)q^tk[pq](pq + 1)^{-1};
\]

(ii) \quad $t = 1$

\[
E(1) = (pq + 1)p k[pq]p^{-2} \cap (pq + 1)q k[pq](pq + 1)^{-1}
\]
\[
= (pq + 1)q k[pq](pq)^{-1} \cap (pq + 1)q k[pq](pq + 1)^{-1}
\]
\[
= (pq + 1)q k[pq];
\]
(iii) \( t = 0 \)
\[
E(0) = p^2k[pq]p^{-2} \cap (pq + 1)k[pq](pq + 1)^{-1} = k[pq];
\]

(iv) \( t = -1 \)
\[
E(-1) = p^3k[pq]p^{-2} \cap (pq + 1)pk[pq](pq + 1)^{-1} = (pq + 1)p[k[pq];
\]

(v) \( t \leq -2 \)
\[
E(t) = p^{-t+2}k[pq]p^{-2} \cap p^{-t}k[pq](pq + 1)^{-1} = p^{-t}k[pq].
\]

Q.E.D.

Look at \( pq \) as an element of \( S \). It is clear, for all \( t \in \mathbb{Z} \), that
\[
E(t) = \{ x \in S \mid [pq, x] = tx \}.
\]

It is now easy to see that \( pq \) is a strictly semi-simple element of \( S \) in the sense of Dixmier [1].

**PROPOSITION 2.** The rings \( S \) and \( A \) are not isomorphic.

**Proof.** Suppose \( \theta: A \rightarrow S \) is an isomorphism. Then \( S \) may be considered as the Weyl algebra over \( k \) generated by the elements \( \bar{p} = \theta(p) \) and \( \bar{q} = \theta(q) \) subject to the relation \( pq - qp = 1 \). Now \( pq \) is a strictly semi-simple element of \( S \) and so by [1, Théorème 9.2] (under the additional hypothesis that \( k \) is algebraically closed) there exists an automorphism \( \Phi \) of \( S \) such that \( \Phi(pq) = \bar{p}\bar{q} + \beta \) for some scalars \( \alpha, \beta \) with \( \alpha \neq 0 \). Let \( \psi = \Phi^{-1}\theta \) and consider the new isomorphism \( \psi: A \rightarrow S \).

Let \( a = \psi(p), b = \psi(q) \). From the above
\[
ab = \psi(pq) = \Phi^{-1}(\bar{p}\bar{q}) = \alpha^{-1}(pq - \beta) \in E(0).
\]

Considering \( a = \sum a_t, b = \sum b_t \) with \( a_t \in E(t) \) and \( b_t \in E(s) \) it can be seen that the only way \( ab \) can be element of \( E(0) \) is if \( a \in E(t) \) and \( b \in E(-t) \) for some \( t \in \mathbb{Z} \). If \( a \subseteq E(t) \) and \( b \subseteq E(-t) \) with either \( t \geq 1 \) or \( t \leq -1 \), look at \( ab \) as a polynomial in \( pq \). By rearranging the terms in the expression for \( ab \) (using identities such as \( (pq)p = p(pq - 1) \)) it can indeed be made into a polynomial in \( pq \). Moreover, the degree of \( ab \) as a polynomial in \( pq \) will be at least two—consequently \( ab \neq \alpha^{-1}(pq - \beta) \). However, if \( a, b \in E(0) \), then \( [a, b] = 0 \) contradicting the fact that \( [a, b] = \psi([p, q]) = 1 \). We conclude that no such \( a, b \in S \) can exist.

Q.E.D.
It seems likely that infinitely many different such endomorphism rings exist—it would be interesting to know whether this is in fact the case.

REFERENCES