The Global Homological Dimension of the Ring of Differential Operators on a Nonsingular Variety over a Field of Positive Characteristic

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1. Introduction

Let $k$ be a commutative ring, and $A$ a commutative $k$-algebra. In Section 2 we define $D(A)$ the ring of $k$-linear differential operators on $A$. If $k$ is a field of characteristic zero and $A$ is the coordinate ring of a nonsingular affine algebraic variety over $k$, then it is shown in [3, 5] that the global homological dimension of $D(A)$ (gl. dim $D(A)$), equals the dimension of the variety.

Here we prove that if $k$ is an algebraically closed field of characteristic $p > 0$, and $A$ is the coordinate ring of a nonsingular affine algebraic variety, $X$, over $k$ then $\text{gl. dim } D(A) = \text{dim } X$. In [5] it is shown that the weak global dimension of $D(A)$ ($w\text{-dim. } D(A)$), equals dim $X$.

As $D(A)$ is not noetherian there is no apriori reason for $w\text{-dim } D(A)$ and gl. dim $D(A)$ to be equal. It is a relatively straightforward matter to see that $D(A)$ is a union of subalgebras each of which has global dimension equal to dim $X$, and so a theorem of Berstein [2] gives gl. dim $D(A) \leq \text{dim } X + 1$. So the point is to show that in this particular situation Berstein's result can be improved to show $\text{gl. dim } D(A) \leq \text{dim } X$. That the global dimension is bounded below by dim $X$ is a consequence of the fact that $w\text{-dim } D(A) = \text{dim } X$.

2. Differential Operators

Let $k$ be a commutative ring, and $A$ a commutative $k$-algebra. Then $\text{End}_k A$ may be made into an $A \otimes_k A$-module by defining $((a \otimes b) \theta)(c) = \theta(b \cdot a \otimes c)$.

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$a\theta(bc)$ for $\theta \in \text{End}_k A$ and $a, b, c \in A$. We write $[a, \theta]$ for $(a \otimes 1 - 1 \otimes a)\theta$, so $[a, \theta](b) = a\theta(b) - \theta(ab)$.

**Definition 2.1.** The space of $k$-linear differential operators of order $\leq n$ on $A$ is defined inductively by $\text{Diff}^k_0 A = 0$, and for $n \geq 0$ $\text{Diff}^n_k A = \{ \theta \in \text{End}_k A \mid [a, \theta] \in \text{Diff}^{n-1}_k A \text{ for all } a \in A \}$. The ring of $k$-linear differential operators on $A$ is $D(A) = \bigcup_{n=0}^\infty \text{Diff}^n_k A$.

**Remark 2.2.** (1) $\text{Diff}^1_k A$ is an $A \otimes A$-submodule of $\text{End}_k A$.

(2) If $\theta \in \text{End}_k A$, then $\theta \in \text{Diff}^n_k A$, if and only if, for all $a_0, a_1, \ldots, a_n \in A$ one has $[a_0 \ldots [a_n, \theta] \ldots] = 0$.

(3) The reader is referred to [6, 7, 8] for an introduction to differential operators on commutative rings.

**Definition 2.3.** Denote by $\mu: A \otimes_k A \to A$ the multiplication map $\mu(a \otimes b) = ab$. This is a $k$-algebra map (also an $A$-module map for either the right or left $A$-module structure on $A \otimes A$). Thus $I = \ker \mu$ is an ideal of $A \otimes_k A$.

**Theorem 2.4** (Heynemann–Sweedler [7], Grothendieck [6]). Let $\theta \in \text{End}_k A$. Then $\theta \in \text{Diff}^n_k A$, if and only if, $\Gamma^{n+1} \cdot \theta = 0$.

From now on, char $k = p > 0$, and $A = k[t_1, \ldots, t_n]$ is a finitely generated commutative $k$-algebra. We also assume $k$ is contained in $A$.

**Definition 2.5.** For $r \geq 0$, define $A_\sim$ to be the subalgebra of $A$ generated by $k$ and all elements $a^p$ with $a \in A$. Clearly $A = A_0 \supseteq A_1 \supseteq \cdots$, and for $r > s$, $A_s$ is a finitely generated $A_r$-module.

**Lemma 2.6.** $A_\sim = k[t_1^p, \ldots, t_n^p]$.

**Proof.** By induction it suffices to prove the result for $r = 1$. Clearly $k[t_1^p, \ldots, t_n^p] \subset A_1$, so only the reverse inclusion must be established. Let $a \in A$, and write $a = \sum \lambda_j t^j$, where $\lambda_j \in k$ and $J = (j_1, \ldots, j_n)$ is a multi-index and $t^j = t_1^{j_1} \cdots t_n^{j_n}$ (there is not necessarily a unique such expression for $a$).

As char $k = p$, if $u, v \in A$ then $(u + v)^p = u^p + v^p$, so by induction (on the number of nonzero $\lambda_j$ occurring in the expression for $a$), $a^p = \sum \lambda_j^p t^j$. But $(t^j)^p = (t_1^p)^{j_1} \cdots (t_n^p)^{j_n} \in k[t_1^p, \ldots, t_n^p]$.

Hence $a^p \in k[t_1^p, \ldots, t_n^p]$.

This shows that we could actually define $A_\sim$ to be $k[t_1^p, \ldots, t_n^p]$ and that such a definition is independent of the choice of generators for $A$.

**Theorem 2.7.** $D(A) = \bigcup_{r=0}^\infty \text{End}_A A$.

**Proof [5, Lemma 3.3].** Let $\theta \in D(A)$ of order $< p^r$. As $(1 \otimes t_j - t_j \otimes 1) \in I$ for all $j$, $(1 \otimes t_j - t_j \otimes 1)^{p^r} = 1 \otimes t_j^{p^r} - t_j^{p^r} \otimes 1 \in I^{p^r}$. Hence $0 = (1 \otimes t_j^{p^r} - t_j^{p^r} \otimes 1) \cdot \theta = -[t_j^{p^r}, \theta]$. Thus the action of $\theta$ on $A$ commutes
with the action of \( A_r \) on \( A \) given by multiplication, as \( A_r = k[t_1', \ldots, t_n'] \). That is, \( \theta \in \text{End}_A A \). Hence \( D(A) \subseteq \bigcup_{r=0}^\infty \text{End}_A A \).

Conversely, let \( \theta \in \text{End}_A A \). Then certainly \( \theta \in \text{End}_A A \). We claim that \( \theta \) is a differential operator of order \( \leq s = np' - 1 \). To see this note that \( f^{r+1} \) is generated (as an ideal) by all \((1 \otimes t_1 - t_1 \otimes 1)^{h_1} \cdots (1 \otimes t_n - t_n \otimes 1)^{h_n}\) where \( j_1 + \cdots + j_n = np' \). This is because \( I \) is generated by \( 1 \otimes t_1 - t_1 \otimes 1, \ldots, 1 \otimes t_n - t_n \otimes 1 \). As \( j_1 + \cdots + j_n = np' \) some \( j_i \geq p' \), and thus \((1 \otimes t_i - t_i \otimes 1)^{p' - 1} = 1 \otimes t_i' - t_i' \otimes 1 \) divides \((1 \otimes t_1 - t_1 \otimes 1)^{h_1} \cdots (1 \otimes t_n - t_n \otimes 1)^{h_n}\). But \( 0 = -(t_{p'}', \theta) = (1 \otimes t_i - t_i \otimes 1)^{p'} \cdot \theta \), hence \((1 \otimes t_1 - t_1 \otimes 1)^{h_1} \cdots (1 \otimes t_n - t_n \otimes 1)^{h_n} \cdot \theta = 0 \) and \( f^{r+1} \cdot \theta = 0 \) as required.

**Notation.** Write \( D_r = \text{End}_A A \), so \( D(A) = \bigcup_{r=0}^\infty D_r \). Also write \( D = D(A) \).

Note that the action of \( A \) on itself by multiplication, enables us to consider \( A \) as a subalgebra of \( D \). In fact \( A = D_0 \), and \( D_0 \subseteq D_1 \subseteq \cdots \). Also notice that \( A_r \) is contained in the centre of \( D_r \). For each \( r, A \) is a finitely generated \( A_r \) module, generated by \( t_1^i \cdots t_n^i \) with \( 0 \leq j_i < p' \) for \( i = 1, \ldots, n \). Hence \( D_r \) is a finitely generated \( A_r \)-module. In particular \( D_r \) is a finitely generated module over its centre, so satisfies a polynomial identity. However, \( D \) will not in general satisfy a polynomial identity (when \( A \) is a polynomial ring over \( k \), \( D \) does not satisfy a polynomial identity).

The following is used in Section 3.

**Proposition 2.8.** Let \( M \) be a \( D \)-module with a chain of k-vector subspaces \( M_0 \subseteq M_1 \cdots \) such that

(a) \( M = \bigcup_{r=0}^\infty M_r \),

(b) each \( M_r \) is a \( D_r \)-module, and for large \( r \), length \( D_r M_r \), \( \leq l \). Then \( M \) is of finite length as a \( D \)-module, and length \( D M \leq l \).

**Proof.** Suppose \( l = 1 \). We must show \( M \) is a simple \( D \)-module. Pick \( 0 \neq m \in M, m' \in M \). For sufficiently large \( r, m, m' \in M_r \) and \( M_r \) is a simple \( D_r \)-module, so there exists \( d \in D_r \) with \( dm = m' \). Hence \( M \) is simple.

Suppose now \( l \geq 2 \), and the proposition is true for integers less than \( l \). If \( M \) is a simple \( D \)-module we are finished. So suppose \( M \) is not simple and pick a proper \( D \)-submodule, \( N \neq 0 \). Put \( N_r = M_r \cap N \), notice that \( N = \bigcup_{r=0}^\infty N_r \), and that each \( N_r \) is a \( D_r \)-module. We show that for large \( r \), length \( D_r N_r \leq l - 1 \). To see this pick \( m \in M \setminus N \). There exists \( s \) such that \( m \in M_r \) for all \( r \geq s \). But \( m \notin N_r \). Hence if \( r \geq s \), \( M_r \nsubseteq N_r \), so for large \( r \), length \( D_r N_r \leq l - 1 \). Applying the induction hypothesis length \( D N \leq l - 1 \).

We have shown that any proper submodule of \( M \) is of finite length \( \leq l - 1 \). Hence length \( D M \leq l \).
3. Global Dimension

Henceforth, \( k \) is an algebraically closed field with \( \text{char } k = p > 0 \), \( X \) is a nonsingular affine algebraic variety over \( k \), and \( A = \mathcal{O}(X) \) is the coordinate ring of \( X \).

Our immediate goal is statements (3) and (4) of Proposition 3.2. We begin with the following Lemma, parts (i) and (ii) of which are to be found in [5].

**Lemma 3.1.** Let \( R = \mathcal{O}(X) \) be the coordinate ring of a nonsingular affine variety \( X \) over an algebraically closed field \( k \) of characteristic \( p > 0 \). Let \( q = p' \). Define \( S \) to be the image of the map \( x \to x^q \) on \( R \). Then:

(i) \( S \cong R \) as rings,

(ii) \( R \) is a finitely generated projective \( S \)-module,

(iii) \( \text{Hom}_S(R, S) \) is a finitely generated projective \( R \)-module of rank 1.

*Proof.* The statements (i) and (ii) appear in [5] as Lemma 3.1 and Proposition 2.2. Because \( \text{Hom}_S(R, S) \) is a finitely generated projective \( S \)-module, it is also a finitely generated projective \( R \)-module by [5, Proposition 2.2]. It follows that the canonical \( S \)-algebra homomorphism \( R \to \text{End}_R(\text{Hom}_S(R, S)) \) is an isomorphism. Hence by [11, Proposition 7.5], \( \text{Hom}_S(R, S) \) is of rank 1 as an \( R \)-module. 

**Proposition 3.2.** (1) \( D_r \) is Morita equivalent to \( A_r \), the progenerator being the \( D_r - A_r \) bimodule \( A \).

(2) For \( s \geq r \), \( D_s \) is a finitely generated projective right \( D_r \)-module and a generator in \( \text{Mod-D}_r \).

(3) \( D \) is a flat right \( D_r \)-module.

(4) If \( M \) is a simple \( D_r \)-module then \( D \otimes_{D_r} M \) is a simple \( D \)-module.

(5) \( D \) is a projective right \( D_r \)-module, for all \( r \geq 0 \).

(6) The above statements are true if "right" is replaced by "left."

*Proof.* (1) appears in the proof of [5, Lemma 3.4]. It is a consequence of the definition of \( D_r \) and of Lemma 3.1 with \( R = A, S = A_r \).

In order to prove (2) we recall the following consequence of the Morita Theorems: Let \( R \) be a commutative ring, \( P \) a progenerator in \( \text{Mod-R} \), and set \( D = \text{End}_R(P) \). If \( M \) is a \( D \)-module, then \( M \) is a progenerator in \( \text{Mod-D} \) if and only if its is a progenerator in \( \text{Mod-R} \).

Let \( s \geq r \). By the previous paragraph, to show \( D_s \) is a progenerator in \( \text{Mod-D}_r \), it is enough to show it is a progenerator in \( \text{Mod-A}_r \). However,
(again by the previous paragraph) $D_s$ is a progenerator in Mod-$A_r$. By [5, Proposition 2.2] this ensures $D_s$ is a projective $A_r$-module, and a generator (since over a commutative ring any faithful projective module is a generator).

This establishes (2) and (3) is a consequence since $D = \lim_{s \rightarrow r} D_s$ is a direct limit of flat $D_r$-modules.

Let $M$ be a simple $D_r$-module and let $s \geq r$. Consider $A_r \subseteq A \subseteq A_r$. Clearly any simple $A$-module is a simple $A_r$-module (since $k$ is algebraically closed). A simple $A_r$-module is of the form $A_r/I$ for $I$ a maximal ideal. But $A$ is integral over $A_r$, so there is a maximal ideal $J$ of $A$ such that $I = J \cap A_r$. Hence $A_r/I \rightarrow A/J$ is an isomorphism of $A_r$-modules. Thus any simple $A_r$-module is also a simple $A$-module (although not uniquely). Now, if $N$ is a simple $A$-module then $D_r \otimes_A N \cong (A \otimes_A \text{Hom}_{A_r}(A, A_r)) \otimes_A N \cong A \otimes_A N$ (where the final isomorphism uses the fact that $\text{Hom}_{A_r}(A, A_r)$ is a rank 1 projective $A$-module, and faithful). By the Morita Theorems, $D_r \otimes_A N$ is a simple $D_r$-module and every simple $D_r$-module is of the form $D_r \otimes_A N$ for a suitable simple $A$-module $N$.

Finally, with $M$ as above, then $M \cong D_r \otimes_A N$ for some simple $A$-module $N$. Hence $D_r \otimes_D M \cong D_r \otimes_D(D_r \otimes_A N) \cong D_r \otimes_A N$ which is simple. Now we apply Proposition 2.8 to conclude that $D \otimes_D M$ is a simple $D$-module; for $s \geq r$ write $M_s$ for the $D_s$-submodule of $D \otimes_D M$ generated by $M$, then $M_s$ is certainly of length $\leq 1$, being a homomorphic image of $D_r \otimes_D M$. This completes the proof of (4).

To see that (3) can be improved to show $D$ is projective as a right $D_r$-module recall [1, Proposition 3]. As $D = \lim_{s \rightarrow r} D_s$, it is enough to show that each $D_{s+1}/D_s$ is a projective right $D_r$-module for $s \geq r$. As $D_s$ is a projective right $D_r$-module, it is enough to show that $D_{s+1}/D_s$ is a projective right $D_r$-module, or equivalently, that $D_{s+1} = D_r \oplus W$ for some right $D_r$-module $W$. Observe that in the proof of (4) we have shown that if $M$ is a simple $D_r$-module then $D_{s+1} \otimes_D M$ is a simple $D_{s+1}$-module, in particular nonzero. In the language of [12] this says that $D_{s+1}$ is a faithfully projective right $D_r$-module. The conclusion of the Theorem in [12] gives the existence of the required $W$, proving (5).

Finally to see that the above statements are true if “right” is replaced by “left” is routine. For example, if $s \geq r$, to show that $D_s$ is a projective left $D_r$-module, it is sufficient (by Morita equivalence) to show that $\text{Hom}_{A_r}(A, A_r) \otimes_D D_s$ is projective as a (left) $A_r$-module. But this is isomorphic to $A_r \otimes_A \text{Hom}_{A_r}(A, A_r)$ and this is projective as an $A_r$-module since $\text{Hom}_{A_r}(A, A_r)$ is a projective $A_r$-module. We leave the rest of the proof of (6) to the enthusiastic reader!

**Lemma 3.4.** Let $J$ be a left ideal of $D_r$. There exists a left ideal $J'$ of $D_r$, containing $J$ such that
(i) \( pd_{D_r}(D_r/J') < \dim X \)

(ii) \( J'/J \) is of finite length.

**Proof.** \( D_r \) is noetherian so define \( J' \) to be the largest left ideal containing \( J \) such that \( J'/J \) is of finite length. Then \( D_r/J' \) contains no artinian sub-modules. Hence by [4, Corollary 5] \( pd_{D_r}(D_r/J') \leq \text{gl. dim } D_r - 1 = n - 1 \).

Alternatively, one can use the fact that \( D_r \) is Morita equivalent to \( A_r \), and use the fact that the lemma is true for \( A_r \), noting that the statements are Morita invariant. 

**Lemma 3.5.** Let \( I \) be a left ideal of \( D_r \). Put \( I_r = I \cap D_r \). Then for each \( r \) there is a left ideal \( I'_r \) of \( D_r \) containing \( I_r \) such that, for all \( r \),

(i) \( I'_r/I_r \) is of finite length.

(ii) \( pd_{D_r}(D_r/I'_r) < \dim X \).

(iii) \( I'_r \subset I_r \).

**Proof.** After Lemma 3.4 we need only show we can choose the \( I'_r \) such that (iii) is satisfied. Suppose \( I_{r-1} \) has been chosen. Note that \( I_r + D_r I_{r-1}/I_r \cong D_r I_{r-1}/I_r \cap D_r I_{r-1} \), which is a homomorphic image of \( D_r I_{r-1}/D_r I_{r-1} \) (since \( I_{r-1} \subset I_r \)). But \( D_r I_{r-1}/D_r I_{r-1} \cong D_r \otimes_{D_{r-1}} (I_{r-1}/I_{r-1}) \) and hence \( I_r + D_r I_{r-1}/I_r \) is of finite length as a \( D_r \)-module. Now apply (3.4) to \( J = I_r + D_r I_{r-1} \), and put \( I'_r = J' \). Since \( J'/J \) is of finite length and \( J/I_r \) is of finite length, \( I'_r/I_r \) is of finite length.

In the next proof we will make frequent use of the fact that if \( 0 \to X \to Y \to Z \to 0 \) is a short exact sequence of modules over a ring \( R \) with \( pd_R(Y) < n \) and \( pd_R(Z) \leq n \) then \( pd_R(X) < n \). The truth of this can be seen from considering the long exact sequence for \( Ext \); in particular, 

\[ \cdots \to Ext^n(Y,-) \to Ext^n(X,-) \to Ext^{n+1}(Z,-) \to \cdots \]

**Theorem 3.6.** If \( I \) is a left ideal of \( D_r \), then \( pd_D(I) < \dim X \).

**Proof.** Suppose \( \dim X = n \). Put \( I_r = I \cap D_r \), and choose the \( I'_r \) as in the lemma above. Put \( T_r = I \cap DI'_r \). First note that \( T_r/DI'_r \) is a submodule of \( DI'_r/DI'_r \cong D \otimes_{D_r} (I'_r/I_r) \) and hence of finite length. In particular, \( T_r \) is finitely generated as a left ideal, since \( DI'_r \) is finitely generated. We also have for all \( r \) that, \( T_{r-1} \subset T_r \) and \( I = \bigcup_{r=0}^\infty DI'_r = \bigcup_{r=0}^\infty T_r \).

We will next show that \( pd_D(T_r/T_{r-1}) < n \) for all \( r \), and hence by [1, Proposition 3], \( pd_D(I_r) < n \).

As \( T_r \) and \( T_{r-1} \) are finitely generated, there exists \( m \) with \( T_r = D(T_r \cap D_m) \) and \( T_{r-1} = D(T_{r-1} \cap D_m) \). Hence \( T_r/T_{r-1} \cong D \otimes_{D_m} (T_r \cap D_m/T_{r-1} \cap D_m) \).
Because \( I/T_r = I/I \cap D^r \simeq I + D^r/I \cap D^r \subset D/D^r \), there is a short exact sequence of \( D_m \)-modules \( 0 \to I/T_r \to D/D^r \to Z \to 0 \). As \( D/D^r \simeq D \otimes_{D_r}(D_r/I_r) \) and \( D \) is flat as a right \( D_r \)-module, applying the functor \( D \otimes_{D_r} \) to a projective resolution of \( D_r/I_r \) (as a \( D_r \)-module) gives a projective resolution of \( D/D^r \) as a \( D \)-module. Hence \( pd_{D}(D/D^r) < n \) (as \( pd_{D_r}(D_r/I_r) < n \)). As \( D \) is projective as a left \( D_m \)-module, a \( D \)-projective resolution for \( D/D^r \) is also a \( D_m \)-projective resolution. Hence \( pd_{D_m}(D/D^r) < n \). But \( \text{gl. dim } D_m = n \), and hence \( pd_{D_m}(Z) \leq n \). Now we get \( pd_{D_m}(I/T_r) < n \).

Because \( T_r \cap D_m/T_r \cap D_m \simeq (T_r \cap D_m) + T_{r-1}/T_{r-1} \subset I/T_{r-1} \) there is a short sequence of \( D_m \)-modules

\[
0 \to T_r \cap D_m/T_r \cap D_m \to I/T_{r-1} \to Z \to 0.
\]

However, \( pd_{D_m}(Z) \leq n \) and \( pd_{D_m}(I/T_{r-1}) < n \), so \( pd_{D_m}(T_r \cap D_m/T_{r-1} \cap D_m) \leq n \). Applying the exact functor \( D \otimes_{D_m} \) gives \( pd_{D}(T_r/T_{r-1}) < n \).

**Theorem 3.7.** \( \text{gl. dim } D(A) = \dim X \).

**Proof.** Recall [9, Theorem 9.12] that \( \text{gl. dim } D = \sup \{ pd_{D}(D/I) \mid I \text{ is a left ideal of } D \} \). By (3.6), we get \( \text{gl. dim } D \leq \dim X \).

For the reverse inequality, observe that Chase [5] has already shown that \( \text{w-dim } D(A) \geq \dim X \).

**Remark.** The case of \( \text{gl. dim } D(k[t]) = 1 \) is proved in [10] using a slightly different argument to the above—the proof in [10] is somewhat cleaner than the above, and the comparison of the two proofs might be useful for the reader.

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**References**


