Provided for non-commercial research and education use.
Not for reproduction, distribution or commercial use.

This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

http://www.elsevier.com/copyright

Author's personal copy

Journal of Algebra 354 (2012) 95–109

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A non-commutative homogeneous coordinate ring for the degree six del Pezzo surface ☆

S. Paul Smith

Department of Mathematics, Box 354350, Univ. Washington, Seattle, WA 98195, United States

ARTICLE INFO

Article history:
Received 27 April 2011
Available online 16 January 2012
Communicated by Michel Van den Bergh

MSC:
14A22
16S38
16W50

Keywords:
Homogeneous coordinate ring
Noncommutative
del Pezzo surface
Graded algebra

ABSTRACT

Let R be the free $\mathbb{C}$-algebra on x and y modulo the relations $x^5 = yxy$ and $y^2 = xyx$ endowed with the $\mathbb{Z}$-grading $\deg x = 1$ and $\deg y = 2$. The ring R appears, in somewhat hidden guise, in a paper on quiver gauge theories. Let $\mathbb{B}_3$ denote the blow up of $\mathbb{CP}^2$ at three non-colinear points. The main result in this paper is that the category of quasi-coherent $\mathcal{O}_{\mathbb{B}_3}$-modules is equivalent to the quotient of the category of $\mathbb{Z}$-graded R-modules modulo the full subcategory of modules that are the sum of their finite dimensional submodules. This reduces almost all representation-theoretic questions about R to algebraic geometric questions about the del Pezzo surface $\mathbb{B}_3$. For example, the generic simple R-module has dimension six. Furthermore, the main result combined with results of Artin, Tate, Van den Bergh, and Stephenson implies that R is a noetherian domain of global dimension three.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

We will work over the field of complex numbers.

1.1. The surface obtained by blowing up $\mathbb{P}^2$ at three non-colinear points is, up to isomorphism, independent of the points. It is called the del Pezzo surface of degree six and we will denote it by $\mathbb{B}_3$.

1.2. Let R be the free $\mathbb{C}$-algebra on x and y modulo the relations

$$x^5 = yxy \quad \text{and} \quad y^2 = xyx. \tag{1.1}$$

☆ The author was supported by the National Science Foundation, Award No. 0602347.
E-mail address: smith@math.washington.edu.

0021-8693/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2011.11.030

Author's personal copy

Give R a $\mathbb{Z}$-grading by declaring that

$$\deg x = 1 \quad \text{and} \quad \deg y = 2.$$

The ring R arises, in somewhat hidden guise, in a paper about string theory [5] (see Section 1.5). The present paper concerns only the mathematical properties of R and its relation to the degree 6 del Pezzo surface.

1.3. The main result in this paper establishes the following surprising relationship between R and the degree six del Pezzo surface.

Theorem 1.1. *Let R be the non-commutative algebra $\mathbb{C}[x, y]$ defined by the relations* (1.1)*. Let $\mathsf{Gr}\,R$ be the category of $\mathbb{Z}$-graded left R-modules. There is an equivalence of categories*

$$\mathsf{Qcoh}\,\mathbb{B}_3 \equiv \frac{\mathsf{Gr}\,R}{\mathsf{Fdim}\,R}$$

where the left-hand side is the category of quasi-coherent $\mathcal{O}_{\mathbb{B}_3}$-modules and the right-hand side is the quotient category modulo the full subcategory $\mathsf{Fdim}\,R$ consisting of those modules that are the sum of their finite dimensional submodules.

Theorem 1.1 is a consequence of the following result.

Theorem 1.2. *Let R be the non-commutative algebra $\mathbb{C}[x, y]$ defined by the relations* (1.1)*. Let $\mathcal{L} = \mathcal{O}(-E)$ be the invertible $\mathcal{O}_{\mathbb{B}_3}$-module corresponding to a (-1)-curve E and σ an order 6 automorphism of $\mathbb{B}_3$ that cyclically permutes the six (-1)-curves on $\mathbb{B}_3$. Then R is isomorphic to the twisted homogeneous coordinate ring*

$$B(\mathbb{B}_3, \mathcal{L}, \sigma) := \bigoplus_{n \geqslant 0} H^0(\mathbb{B}_3, \mathcal{L}_n)$$

where

$$\mathcal{L}_n := \mathcal{L} \otimes (\sigma^*)\mathcal{L} \otimes \cdots \otimes (\sigma^*)^{n-1}\mathcal{L}.$$

In the terminology of Artin, Tate, and Van den Bergh [1] and Artin and Van den Bergh [3], $B(\mathbb{B}_3, \mathcal{L}, \sigma)$ is a *twisted homogeneous coordinate ring* of $\mathbb{B}_3$. Results of Artin, Tate, and Van den Bergh, and Stephenson [10] now imply that R is a 3-dimensional Artin–Schelter regular algebra and therefore has the following properties.

Corollary 1.3. *Let R be the non-commutative algebra $\mathbb{C}[x, y]$ defined by the relations* (1.1)*. Then*

(1) *R is a left and right noetherian domain*;
(2) *R has global homological dimension* 3;
(3) *R is Auslander–Gorenstein and Cohen–Macaulay in the non-commutative sense*;
(4) *the Hilbert series of R is the same as that of the weighted polynomial ring on three variables of weights* 1, 2, *and* 3;
(5) *R is a finitely generated module over its center* [9, *Corollary* 2.3];
(6) *$R^{(6)} := \bigoplus_{n=0}^{\infty} R_{6n}$ is isomorphic to $\bigoplus_{n=0}^{\infty} H^0(\mathbb{B}_3, \mathcal{O}(-nK))$ where $K = K_{\mathbb{B}_3}$ is the canonical divisor on $\mathbb{B}_3$*;
(7) $\operatorname{Spec} R^{(6)}$ *is the canonical cone over $\mathbb{B}_3$, i.e., the cone obtained by collapsing the zero section of the total space of the canonical bundle over $\mathbb{B}_3$.*

Author's personal copy

This close connection between R and $\mathbb{B}_3$ means that almost all aspects of the representation theory of R can be expressed in terms of the geometry of $\mathbb{B}_3$. We plan to address this question in another paper.

1.4. The justification for calling R a non-commutative homogeneous coordinate ring for $\mathbb{B}_3$ is the similarity between the equivalence of categories in Theorem 1.1 and following theorem of Serre [7]:

> if $X \subset \mathbb{P}^n$ is the scheme-theoretic zero locus of a graded ideal I in the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$ with its standard grading, and $A = S/I$, then there is an equivalence of categories
>
> $$\mathsf{Qcoh}\, X \equiv \frac{\mathsf{Gr}\, A}{\mathsf{Fdim}\, A} \tag{1.2}$$
>
> where the right-hand side is the quotient category of $\mathsf{Gr}\, A$, the category of graded A-modules, by the full subcategory $\mathsf{Fdim}\, A$ consisting of modules whose non-zero finitely generated submodules have support only at the origin.

1.5. Motivation The results in this paper are a prerequisite for some results in [8] where three superpotential algebras appearing in the string theory literature are investigated by relating them to twisted homogeneous coordinate rings. In [5], Beasley and Plesser study a superpotential algebra they dub the dP_3I path algebra. In [8], we will show that the dP_3I path algebra is isomorphic to $R \rtimes \mu_6$, the skew group ring for the 6th roots of unity acting on R by $\xi \cdot r = \xi^n r$ for $r \in R_n$; the isomorphism is established in [8]. An intimate understanding of R therefore leads to a detailed understanding of the dP_3I path algebra. The dP_3 in the notation dP_3I refers to the de Pezzo surface obtained by blowing up 3 non-colinear points in $\mathbb{P}^2$. The I in dP_3I is to distinguish this algebra from two other path algebras with relations that Beasley and Plesser associate to the degree-six del Pezzo surface.

2. $R = \mathbb{C}[x, y]$ with $x^5 = yxy$ and $y^2 = xyx$ is an iterated Ore extension

The following result is a straightforward calculation. The main point of it is to show that R has the same Hilbert series as the weighted polynomial ring on three variables of weights 1, 2, and 3.

Proposition 2.1. *(See Stephenson [10,11].) The ring $R := \mathbb{C}[x, y]$ with defining relations*

$$x^5 = yxy \quad \text{and} \quad y^2 = xyx$$

is an iterated Ore extension of the polynomial ring $\mathbb{C}[w]$. Explicitly, if ζ is a fixed primitive 6th root of unity, R has the following properties.

(1) *$R = \mathbb{C}[w][z; \sigma][x; \tau, \delta]$ where $\sigma \in \operatorname{Aut}\mathbb{C}[z]$, $\tau \in \operatorname{Aut}\mathbb{C}[w][z; \sigma]$, and δ is a τ-derivation defined as follows*

$$\begin{aligned} \sigma(w) &= \zeta w, & & \\ \tau(w) &= -\zeta^2 w, & \tau(z) &= \zeta z, \\ \delta(w) &= z, & \delta(z) &= -w^2. \end{aligned}$$

(2) *A set of defining relations of $R = \mathbb{C}[z, w, x]$ is given by*

$$zw = \zeta wz,$$
$$xw = -\zeta^2 wx + z,$$
$$xz = \zeta zx - w^2.$$

Author's personal copy

(3) *R has basis $\{w^i z^j x^k \mid i, j, k \geqslant 0\}$.*
(4) *R is a noetherian domain.*
(5) *The Hilbert series of R is $(1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1}$.*

Proof. Define the elements

$$\begin{aligned} w &:= y - x^2, \\ z &:= xw + \zeta^2 wx \\ &= xy + \zeta^2 yx - \zeta x^3 \end{aligned}$$

of R. Since y belongs to the subalgebra of R generated by x and w, $\mathbb{C}[x, y] = \mathbb{C}[x, w] = \mathbb{C}[x, w, z]$. It is easy to check that

$$zw = \zeta wz, \qquad xw = z - \zeta^2 wx, \qquad xz = \zeta zx - w^2. \tag{2.1}$$

Let R' be the free algebra $\mathbb{C}\langle w, x, z\rangle$ modulo the relations in (2.1). We will show R' is isomorphic to R. We already know there is a homomorphism $R' \to R$ and we will now exhibit a homomorphism $R \to R'$ by showing there are elements x and Y in R' that satisfy the defining relations for R. Define the element $Y := w + x^2$ in R'. A straightforward computation in R' gives

$$xwx - x^2 w = w^2 + wx^2$$

so

$$Y^2 = w^2 + x^2 w + wx^2 + x^4 = xwx + x^4 = xYx.$$

The next calculation uses the identity $1 - \xi + \xi^2 = 0$ repeatedly. Deep breath...

$$\begin{aligned} YxY &= (w + x^2)xw + wx^3 + x^5 \\ &= (w + x^2)(z - \zeta^2 wx) + [wx^3 + x^5] \\ &= x^2 z - \zeta^2 x^2 wx + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= x(\zeta zx - w^2) - \zeta^2 x(z - \zeta^2 wx)x + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= (\zeta - \zeta^2)xzx - xw^2 - \zeta xwx^2 + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= (\zeta - \zeta^2)(\zeta zx - w^2)x - (z - \zeta^2 wx)w - \zeta(z - \zeta^2 wx)x^2 + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= (\zeta^2 - \zeta^3)zx^2 - (\zeta - \zeta^2)w^2 x - zw + \zeta^2 wxw - \zeta zx^2 - wx^3 + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= (\zeta^2 - \zeta^3 - \zeta)zx^2 + \zeta^2 wxw + [(1 - \zeta)wz - \zeta w^2 x + x^5] \\ &= \zeta^2 wxw + [(1 - \zeta)wz - \zeta w^2 x + x^5] \\ &= \zeta^2 w(z - \zeta^2 wx) + [-\zeta^2 wz - \zeta w^2 x + x^5] \\ &= x^5. \end{aligned}$$

Since $YxY = x^5$, R is isomorphic to R'. Hence R is an iterated Ore extension as claimed. The other parts of the proposition follow easily. □

It is an immediate consequence of the relations that $x^6 = y^3$. Hence x^6 is in the center of R.

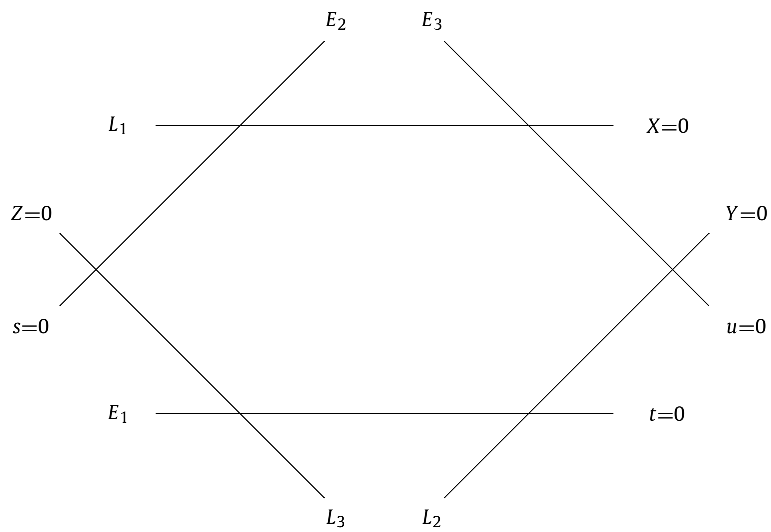
Author's personal copy

3. The del Pezzo surface $\mathbb{B}_3$

Let $\mathbb{B}_3$ be the surface obtained by blowing up the complex projective plane $\mathbb{P}^2$ at three non-colinear points. We will write

$$\pi : \mathbb{B}_3 \to \mathbb{P}^2$$

for the morphism that contracts the exceptional curves E_1, E_2, and E_3. The (-1)-curves on $\mathbb{B}_3$ lie in the following configuration



(3.1)

where L_1, L_2, and L_3 are the strict transforms of the lines in $\mathbb{P}^2$ spanned by the points that are blown up. (The labeling of the equations for the (-1)-curves is explained in Section 3.3.)

The union of the (-1)-curves is an anti-canonical divisor so we write

$$-K := L_1 + L_2 + L_3 + E_1 + E_2 + E_3$$

(K for canonical). This is, of course, an ample divisor.

3.1. The Picard group of $\mathbb{B}_3$ The morphism $\pi : \mathbb{B}_3 \to \mathbb{P}^2$ induces an injective group homomorphism $\pi^* : \operatorname{Pic} \mathbb{P}^2 \to \operatorname{Pic} \mathbb{B}_3$. We write $H = \pi^* L$ where L is a line in $\mathbb{P}^2$. Hence

$$\operatorname{Pic} \mathbb{B}_3 = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3.$$

We identify $\operatorname{Pic} \mathbb{B}_3$ with $\mathbb{Z}^4$ by using the ordered basis

$$H, \ -E_2, \ -E_1, \ -E_3.$$

Thus

$$H = (1, 0, 0, 0), \qquad E_1 = (0, 0, -1, 0), \qquad E_2 = (0, -1, 0, 0), \qquad E_3 = (0, 0, 0, -1).$$

In this basis the anti-canonical divisor is

$$-K = (3, 1, 1, 1).$$

Author's personal copy

The Picard group may be presented more symmetrically as

$$\operatorname{Pic}\mathbb{B}_3 = \frac{\bigoplus_{i=1}^{3}(\mathbb{Z}L_i \oplus \mathbb{Z}E_i)}{(E_i + L_j = E_j + L_i \mid 1 \leqslant i, j \leqslant 3)}.$$

It follows that

$$H = L_1 + E_2 + E_3 = L_2 + E_1 + E_3 = L_3 + E_1 + E_2$$

and

$$L_1 = (1,1,0,1), \qquad L_2 = (1,0,1,1), \qquad L_3 = (1,1,1,0).$$

3.2. Cox's homogeneous coordinate ring By definition, Cox's homogeneous coordinate ring [6] for a complete smooth toric variety X is

$$S := \bigoplus_{[\mathcal{L}]\in \operatorname{Pic} X} H^0(X, \mathcal{L}).$$

From now on, S denotes Cox's homogeneous coordinate ring for $\mathbb{B}_3$.

Let X, Y, Z, s, t, u be coordinate functions on $\mathbb{C}^6$. One can present $\mathbb{B}_3$ as a toric variety by defining it as the orbit space

$$\mathbb{B}_3 := \frac{\mathbb{C}^6 - W}{(\mathbb{C}^\times)^4}$$

where the irrelevant locus, W, is the union of nine codimension two subspaces, namely

$$\begin{array}{lll} X = t = 0, & X = Y = 0, & s = t = 0, \\ Y = s = 0, & Y = Z = 0, & u = t = 0, \\ Z = u = 0, & Z = X = 0, & s = u = 0 \end{array} \tag{3.2}$$

and $(\mathbb{C}^\times)^4$ acts with weights

$$\begin{array}{ccccc} X & 1 & 1 & 0 & 1 \\ Y & 1 & 0 & 1 & 1 \\ Z & 1 & 1 & 1 & 0 \\ s & 0 & -1 & 0 & 0 \\ t & 0 & 0 & -1 & 0 \\ u & 0 & 0 & 0 & -1\,. \end{array}$$

Therefore S is the $\mathbb{Z}^4$-graded polynomial ring

$$S = \mathbb{C}[X, Y, Z, s, t, u]$$

with the degrees of the generators given by their weights under the $(\mathbb{C}^\times)^4$ action, e.g., $\deg X = (1,1,0,1)$, $\deg u = (0,0,0,-1)$, etc. It follows from Cox's results [6, Section 3] that

$$\mathsf{Qcoh}\,\mathbb{B}_3 \equiv \frac{\mathsf{Gr}(S, \mathbb{Z}^4)}{\mathsf{T}}$$

Author's personal copy

where $\mathrm{Gr}(S, \mathbb{Z}^4)$ is the category of $\mathbb{Z}^4$-graded S-modules and T is the full subcategory consisting of the modules that are sums of finitely generated S-modules supported on W.

3.3. The labeling of the (-1)-curves in the diagram (3.1) is explained by the existence of the morphisms

$$\begin{array}{ccccc} & & \mathbb{B}_3 & & \\ & \overset{\pi}{\swarrow} & & \searrow & \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \end{array} \qquad \begin{array}{ccccc} & & (X, Y, Z, s, t, u) & & \\ & \overset{\pi}{\swarrow} & & \searrow & \\ (Xsu, Ytu, Zst) & & & & (YZt, XZs, XYu) \end{array}$$

that collapse the (-1)-curves: for example, π contracts the three divisors $t = 0$, $s = 0$, and $u = 0$, i.e., E_1, E_2, and E_3.

3.4. An order six automorphism σ of $\mathbb{B}_3$ The cyclic permutation of the six (-1)-curves on $\mathbb{B}_3$ extends to a global automorphism of $\mathbb{B}_3$ of order six. We now make this explicit.

The category of graded rings consists of pairs (A, Γ) consisting of an abelian group Γ and a Γ-graded ring A. A morphism $(f, \theta) : (A, \Gamma) \to (B, \Upsilon)$ consists of a ring homomorphism $f : A \to B$ and a group homomorphism $\theta : \Gamma \to \Upsilon$ such that $f(A_i) \subset B_{\theta(i)}$ for all $i \in \Gamma$.

Let $\tau : S \to S$ be the automorphism induced by the cyclic permutation

$$X \longrightarrow u \longrightarrow Y \xrightarrow[\tau]{} t \longrightarrow Z \longrightarrow s \quad (s \longrightarrow X) \tag{3.3}$$

and let $\theta : \mathbb{Z}^4 \to \mathbb{Z}^4$ be left multiplication by the matrix

$$\theta = \begin{pmatrix} 2 & -1 & -1 & -1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

Then $(\tau, \theta) : (S, \mathbb{Z}^4) \to (S, \mathbb{Z}^4)$ is an automorphism in the category of graded rings. The irrelevant locus (3.2) is stable under the action of τ, so τ induces an automorphism σ of $\mathbb{B}_3$. It follows from the definition of τ in (3.3) that σ cyclically permutes the six -1-curves.

Since $(\tau, \theta)^6 = \mathrm{id}_{(S, \mathbb{Z}^4)}$ the order of σ divides six. But the action of σ on the set of (-1)-curves has order six, so σ has order six as an automorphism of $\mathbb{B}_3$.

3.5. Fix a primitive cube root of unity ω. The left action of θ on $\mathbb{Z}^4 = \mathrm{Pic}\,\mathbb{B}^3$ has eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ \omega \\ \omega^2 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} 0 \\ 1 \\ \omega^2 \\ \omega \end{pmatrix},$$

with corresponding eigenvalues -1, $+1$, ω^2, ω.

Author's personal copy

4. A twisted hcr for $\mathbb{B}_3$

In this section we will prove the main theorem: R is isomorphic to a twisted homogeneous coordinate ring $B = B(\mathbb{B}_3, \mathcal{L}, \sigma)$ for $\mathbb{B}_3$. The degree-n homogeneous component of B is the global sections of an invertible $\mathcal{O}_{\mathbb{B}_3}$-module $\mathcal{L}_n$; i.e., $B_n = H^0(\mathbb{B}_3, \mathcal{L}_n)$. After defining $\mathcal{L}_n$ in Section 4.1 we prove some vanishing results for its cohomology that will be used later to prove that B is generated as a $\mathbb{C}$-algebra by B_1 and B_2. Since R is generated as a $\mathbb{C}$-algebra by $x \in R_1$ and $y \in R_2$ this will allow us to prove that the homomorphism $\Phi : R \to B$ defined in Proposition 4.5 is surjective. We also compute $\dim H^0(\mathbb{B}_3, \mathcal{L}_n) = \dim B_n$ and observe that this is the same as $\dim R_n$ which allows us to conclude that Φ is an isomorphism.

We write K for the canonical divisor on $\mathbb{B}_3$.

4.1. A sequence of line bundles on $\mathbb{B}_3$ We will blur the distinction between a divisor D and the class of the line bundle $\mathcal{O}(D)$ in $\operatorname{Pic}\mathbb{B}_3$.

We define a sequence of divisors: D_0 is zero; D_1 is the line L_1; for $n \geqslant 1$

$$D_n := \left(1 + \theta + \cdots + \theta^{n-1}\right)(D_1).$$

We will write $\mathcal{L}_n := \mathcal{O}(D_n)$. Therefore

$$\mathcal{L}_n = \mathcal{L}_1 \otimes \sigma^* \mathcal{L}_1 \otimes \cdots \otimes \left(\sigma^*\right)^{n-1} \mathcal{L}_1.$$

For example,

$$\begin{aligned}
\mathcal{O}(D_1) &= \mathcal{L}_1 = \mathcal{O}(1,1,0,1) = \mathcal{O}(L_1),\\
\mathcal{O}(D_2) &= \mathcal{L}_2 = \mathcal{O}(1,1,0,0) = \mathcal{O}(L_1 + E_3),\\
\mathcal{O}(D_3) &= \mathcal{L}_3 = \mathcal{O}(2,1,1,1) = \mathcal{O}(L_1 + E_3 + L_2),\\
\mathcal{O}(D_4) &= \mathcal{L}_4 = \mathcal{O}(2,1,0,1) = \mathcal{O}(L_1 + E_3 + L_2 + E_1),\\
\mathcal{O}(D_5) &= \mathcal{L}_5 = \mathcal{O}(3,2,1,1) = \mathcal{O}(L_1 + E_3 + L_2 + E_1 + L_3),\\
\mathcal{O}(D_6) &= \mathcal{L}_6 = \mathcal{O}(3,1,1,1) = \mathcal{O}(L_1 + E_3 + L_2 + E_1 + L_3 + E_2)\\
&= \mathcal{O}(-K).
\end{aligned}$$

Lemma 4.1. *Suppose $m \geqslant 0$ and $0 \leqslant r \leqslant 5$. Then*

$$D_{6m+r} = D_r - mK.$$

Proof. Since $\theta^6 = 1$,

$$\begin{aligned}
\sum_{i=0}^{6m+r-1} \theta^i &= \left(1 + \theta + \cdots + \theta^5\right) \sum_{j=0}^{m-1} \theta^{6j} + \theta^{6m}\left(1 + \theta + \cdots + \theta^{r-1}\right)\\
&= \left(1 + \theta + \cdots + \theta^{r-1}\right) + m\left(1 + \theta + \cdots + \theta^5\right)
\end{aligned}$$

where the sum $1 + \theta + \cdots + \theta^{r-1}$ is empty and therefore equal to zero when $r = 0$. Therefore $D_{6m+r} = D_r + mD_6 = D_r - mK$, as claimed. □

Author's personal copy

4.2. Vanishing results For a divisor D on a smooth surface X, we write

$$h^i(D) := \dim H^i\big(X, \mathcal{O}_X(D)\big).$$

We need to know that $h^1(D) = h^2(D) = 0$ for various divisors D on $\mathbb{B}_3$.

If $D - K$ is ample, then the Kodaira Vanishing Theorem implies that $h^0(K - D) = h^1(K - D) = 0$ and Serre duality then gives $h^2(D) = h^1(D) = 0$.

The notational conventions in Section 3.1 identify $\operatorname{Pic} \mathbb{B}_3$ with $\mathbb{Z}^4$ via

$$aH - cE_1 - bE_2 - dE_3 \equiv (a, b, c, d).$$

The intersection form on $\mathbb{B}_3$ is given by

$$H^2 = 1, \qquad E_i \cdot E_j = -\delta_{ij}, \qquad H \cdot E_i = 0,$$

so the induced intersection form on $\mathbb{Z}^4$ is

$$(a, b, c, d) \cdot \big(a', b', c', d'\big) = aa' - bb' - cc' - dd'.$$

Lemma 4.2. *Let $D = (a, b, c, d) \in \operatorname{Pic} \mathbb{B}_3 \equiv \mathbb{Z}^4$. Suppose that*

$$(a + 3)^2 > (b + 1)^2 + (c + 1)^2 + (d + 1)^2 \tag{4.1}$$

and

$$b, c, d > -1, \quad \text{and} \quad a + 1 > b + c, b + d, c + d. \tag{4.2}$$

Then $D - K$ is ample, whence $h^1(D) = h^2(D) = 0$.

Proof. The effective cone is generated by L_1, L_2, L_3, E_1, E_2, and E_3 so, by the Nakai–Moishezon criterion, $D - K$ is ample if and only if $(D - K)^2 > 0$ and $(D - K) \cdot L_i > 0$ and $(D - K) \cdot E_i > 0$ for all i. Now $D - K = (a + 3, b + 1, c + 1, d + 1)$, so $(D - K)^2 > 0$ if and only if (4.1) holds and $(D - K) \cdot D' > 0$ for all effective D' if and only if (4.2) holds.

Hence the hypothesis that (4.1) and (4.2) hold implies that $D - K$ is ample. The Kodaira Vanishing Theorem now implies that $h^0(K - D) = h^1(K - D) = 0$. Serre duality now implies that $h^2(D) = h^1(D) = 0$. □

Lemma 4.3. *For all $n \geqslant 0$, $h^1(D_n) = h^2(D_n) = 0$.*

Proof. The value of D_n for $0 \leqslant n \leqslant 6$ is given explicitly in Section 4.1. We also note that $D_7 = D_1 + D_6 = (4, 2, 1, 2)$. It is routine to check that conditions (4.1) and (4.2) hold for $D = D_n$ when $n = 0, 2, 3, 4, 5, 6, 7$. Hence $h^1(D_n) = h^2(D_n) = 0$ when $n = 0, 2, 3, 4, 5, 6, 7$.

We now consider D_1 which is the (-1)-curve $X = 0$. (Since $(D_1 - K) \cdot D_1 = 0$, $D_1 - K$ is not ample so we can't use Kodaira Vanishing as we did for the other small values of n.) It follows from the exact sequence $0 \to \mathcal{O}_{\mathbb{B}_3} \to \mathcal{O}_{\mathbb{B}_3}(D_1) \to \mathcal{O}_{D_1}(D_1) \to 0$ that $H^p(\mathbb{B}_3, \mathcal{O}_{\mathbb{B}_3}(D_1)) \cong H^p(\mathbb{B}_3, \mathcal{O}_{D_1}(D_1))$ for $p = 1, 2$. However, $D_1 \cong \mathbb{P}^1$, $\mathcal{O}_{D_1}(D_1)$ is the normal sheaf for $D_1 \subset \mathbb{B}_3$, and, since D_1 can be contracted to a smooth point on the degree 7 del Pezzo surface, $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{D_1}(-1)$. Therefore $H^p(\mathbb{B}_3, \mathcal{O}_{D_1}(D_1)) \cong H^p(\mathbb{P}^1, \mathcal{O}(-1))$ which is zero for $p = 1, 2$. It follows that $h^1(D_1) = h^2(D_1) = 0$.

Author's personal copy

Thus $h^1(D_n) = h^2(D_n) = 0$ when $0 \leqslant n \leqslant 7$. We have also shown that $D_n - K$ is ample when $2 \leqslant n \leqslant 7$. We now argue by induction. Suppose $n \geqslant 8$ and $D_{n-6} - K$ is ample. Now $D_n - K = D_{n-6} - K - K$. Since a sum of ample divisors is ample, $D_n - K$ is ample. It follows that $h^1(D_n) = h^2(D_n) = 0$. □

4.3. The twisted homogeneous coordinate ring $B(\mathbb{B}_3, \mathcal{L}, \sigma)$ We assume the reader is somewhat familiar with the notion of twisted homogeneous coordinate rings. Standard references for that material are [1–4].

The notion of a σ-ample line bundle [3] plays a key role in the study of twisted homogeneous coordinate rings. Because $\mathcal{L}_6$ is the anti-canonical bundle and therefore ample, $\mathcal{L}_1$ is σ-ample. This allows us to use the results of Artin and Van den Bergh in [3] to conclude that the twisted homogeneous coordinate ring

$$B = B(\mathbb{B}_3, \mathcal{L}, \sigma) = \bigoplus_{n=0}^{\infty} B_n = \bigoplus_{n=0}^{\infty} H^0(\mathbb{B}_3, \mathcal{L}_n) \tag{4.3}$$

is such that

$$\mathsf{Qcoh}\,\mathbb{B}_3 \equiv \frac{\mathsf{Gr}\,B}{\mathsf{Fdim}\,B} \tag{4.4}$$

where $\mathsf{Fdim}\,B$ is the full subcategory of $\mathsf{Gr}\,B$ consisting of those graded modules that are the sum of their finite dimensional submodules. Artin and Van den Bergh [3] show that the equivalence (4.4) implies that B has a host of good properties.

4.4. We will now compute the dimensions $h^0(D_n)$ of the homogeneous B_n of B. We will show that B has the same Hilbert series as the non-commutative ring R, i.e., the same Hilbert series as the weighted polynomial ring with weights 1, 2, and 3. The Hilbert series of R was computed in Proposition 2.1.

As usual we write $\chi(D) = h^0(D) - h^1(D) + h^2(D)$. The Riemann–Roch formula is

$$\chi\big(\mathcal{O}(D)\big) = \chi(\mathcal{O}) + \frac{1}{2} D \cdot (D - K).$$

We have $\chi(\mathcal{O}_{\mathbb{B}_3}) = 1$ and $K^2 = 6$.

Lemma 4.4. *Suppose* $0 \leqslant r \leqslant 5$. *Then*

$$h^0(D_{6m+r}) = \begin{cases} (m+1)(3m+r) & \text{if } r \neq 0, \\ 3m^2 + 3m + 1 & \text{if } r = 0 \end{cases}$$

and

$$\sum_{n=0}^{\infty} h^0(D_n) t^n = \frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

In particular, B and R have the same Hilbert series.

Author's personal copy

Proof. Computations for $1 \leqslant r \leqslant 5$ give $D_r^2 = r - 2$ and $D_r \cdot K = -r$. Hence

$$\begin{aligned}
\chi(D_{6m+r}) &= 1 + \frac{1}{2}(D_r - mK) \cdot \big(D_r - (m+1)K\big) \\
&= 1 + \frac{1}{2}\big(D_r^2 - (2m+1)D_r \cdot K + 6m(m+1)^2\big) \\
&= (3m+r)(m+1)
\end{aligned}$$

for $m \geqslant 0$ and $1 \leqslant r \leqslant 5$. When $r = 0$, $D_r = 0$ so

$$\chi(D_{6m}) = 3m^2 + 3m + 1.$$

By Lemma 4.3, $\chi(D_n) = h^0(D_n)$ for all $n \geqslant 0$ so it follows from the formula for $\chi(D_n)$ that

$$h^0(D_{n+6}) - h^0(D_n) = n + 6 \tag{4.5}$$

for all $n \geqslant 0$.

To complete the proof of the lemma, it suffices to show that $h^0(D_n)$ is the coefficient of t^n in the Taylor series expansion

$$f(t) := \frac{1}{(1-t)(1-t^2)(1-t^3)} = \sum_{n=0}^{\infty} a_n t^n.$$

Because

$$\big(1 - t^6\big) f(t) = \big(1 - t + t^2\big)(1-t)^{-2} = 1 + \sum_{n=1}^{\infty} n t^n,$$

it follows that

$$\begin{aligned}
\big(1 - t^6\big) f(t) &= a_0 + a_1 t + \cdots + a_5 t^5 + \sum_{n=0}^{\infty} (a_{n+6} - a_n) t^{n+6} \\
&= 1 + t + 2t + \cdots + 5t^5 + \sum_{n=6}^{\infty} n t^n \\
&= 1 + t + 2t + \cdots + 5t^5 + \sum_{n=0}^{\infty} (n+6) t^{n+6}.
\end{aligned}$$

In particular, if $0 \leqslant r \leqslant 5$, $a_r = h^0(D_r)$. We now complete the proof by induction. Suppose we have proved that $a_i = h^0(D_i)$ for $i \leqslant n + 5$. By comparing the expressions in the Taylor series we see that

$$a_{n+6} = a_n + (n+6) = h^0(D_n) + n + 6 = h^0(D_{n+6})$$

where the last equality is given by (4.5). □

Author's personal copy

4.4.1. Remark It wasn't necessary to compute $\chi(D_n)$ in the previous proof. The proof only used the fact that $\chi(D_{n+6}) - \chi(D_n) = n + 6$ which can be proved directly as follows

$$\begin{aligned}\chi(D_{n+6}) - \chi(D_n) &= \frac{1}{2}D_{n+6} \cdot (D_{n+6} - K) - \frac{1}{2}D_n \cdot (D_n - K)\\ &= \frac{1}{2}(D_{n+6} - D_n) \cdot (D_{n+6} + D_n - K)\\ &= -K \cdot \big(D_r - (m+1)K\big)\\ &= 6(m+1) - K \cdot D_r\\ &= n + 6.\end{aligned}$$

4.5. The isomorphism $R \to B(\mathbb{B}_3, \mathcal{L}, \sigma)$ By definition, $B_n = H^0(\mathbb{B}_3, \mathcal{L}_n)$. Since Cox's homogeneous coordinate ring, $S = \mathbb{C}[X, Y, Z, s, t, u]$, is the direct sum of $H^0(\mathbb{B}_3, \mathcal{L})$ as $[\mathcal{L}]$ ranges over $\operatorname{Pic}\mathbb{B}_3$, each B_n is a subspace of S. In particular, B itself is a subspace of S, but

the multiplication in B is not that in S.

The ring B has the following basis elements in the following degrees:

$\deg = n$	$\mathcal{L}_n$	basis for B_n				
1	$\mathcal{O}(1,1,0,1)$	X				
2	$\mathcal{O}(1,1,0,0)$	Xu	Zt			
3	$\mathcal{O}(2,1,1,1)$	XYu	YZt	XZs		
4	$\mathcal{O}(2,1,0,1)$	$XYtu$	YZt^2	$XZst$	X^2su	
5	$\mathcal{O}(3,2,1,1)$	$XYZtu$	YZ^2t^2	XZ^2st	X^2Zsu	X^2Yu^2
6	$\mathcal{O}(3,1,1,1)$	$XYZstu$	YZ^2st^2	XZ^2s^2t	X^2Zs^2u	X^2Ysu^2
					XY^2tu^2	$Y^2Zt^2u.$

The multiplication in B is Zhang's twisted multiplication [12] with respect to the automorphism τ defined in (3.3): the product in B of $a \in B_m$ and $b \in B_n$ is

$$a *_B b := a\tau^m(b). \tag{4.6}$$

To make it clear whether a product is being calculated in B or S we will write x for X considered as an element of B and y for Zt considered as an element of B. Therefore, for example,

$$\begin{aligned}x^5 &= X\tau(X)\tau^2(X)\tau^3(X)\tau^4(X)\tau^5(X)\\ &= XuYtZ\\ &= (Zt)Y(uX)\\ &= Zt\tau^2(X)\tau^3(Zt)\\ &= yxy\end{aligned}$$

Author's personal copy

and

$$y^2 = Zt\tau^2(Zt) = Zt(sX) = X(sZ)t = X\tau(zt)\tau^3(X) = xyx.$$

The following proposition is an immediate consequence of these two calculations.

Proposition 4.5. *Let R be the free algebra $\mathbb{C}\langle x, y\rangle$ modulo the relations $x^5 = yxy$ and $y^2 = xyx$. Then there is a $\mathbb{C}$-algebra homomorphism*

$$\Phi : R = \mathbb{C}[x, y] \to B(\mathbb{B}_3, \mathcal{L}, \sigma), \quad x \mapsto X, \ y \mapsto Zt.$$

Lemma 4.6. *The homomorphism in Proposition* 4.5 *is an isomorphism in degrees* $\leqslant 6$.[1]

Proof. By Proposition 2.1, R has Hilbert series $(1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1}$, so the dimension of R_n in degrees $1, 2, 3, 4, 5, 6$ is $1, 2, 3, 4, 5, 7$.

The nth row in the following table gives a basis for B_n, $1 \leqslant n \leqslant 6$. One proceeds down each column by multiplying on the right by x. There wasn't enough room on a single line for B_6 so we put the last two entries for B_6 on a new line.

$$\begin{array}{rclrclrclrclrcl}
x &=& X, &&&&&&&&&&&& \\
x^2 &=& Xu, & y &=& Zt, &&&&&&&&& \\
x^3 &=& XYu, & yx &=& YZt, & xy &=& XZs, &&&&&& \\
x^4 &=& XYtu, & yx^2 &=& YZt^2, & y^2 &=& XZst, & x^2y &=& X^2su, &&& \\
x^5 &=& XYZtu, & yx^3 &=& YZ^2t^2, & y^2x &=& XZ^2st, & xy^2 &=& X^2Zsu, & x^3y &=& X^2Yu^2, \\
x^6 &=& XYZstu, & yx^4 &=& YZ^2st^2, & y^2x^2 &=& XZ^2s^2t, & xy^2x &=& X^2Zs^2u, & x^2y^2 &=& X^2Ysu^2, \\
&&&&&&&&& yx^2y &=& Y^2Zt^2u, & x^4y &=& XY^2tu^2.
\end{array}$$

These calculations involving x and y are made by using the formula (4.6) in the same way it was used to show that $x^5 = yxy$. □

Lemma 4.7. *$\mathcal{L}_2$ is generated by its global sections.*

Proof. A line bundle on a variety is generated by its global sections if and only if for each point on the variety there is a section of the bundle that does not vanish at that point. In this case, $H^0(\mathbb{B}_3, \mathcal{L}_2)$ is spanned by Xu and Zt. One can see from the diagram (3.1) that the zero locus of Xu does not meet the zero locus of Zt, so the common zero locus of Xu and Zt is empty. □

Proposition 4.8. *As a $\mathbb{C}$-algebra, B is generated by B_1 and B_2.*

Proof. It follows from the explicit calculations in Lemma 4.6 that the subalgebra of B generated by B_1 and B_2 contains B_m for all $m \leqslant 6$. It therefore suffices to prove that the twisted multiplication map $B_2 \otimes B_n \to B_{n+2}$ is surjective for all $n \geqslant 5$.

By definition, $B_2 = H^0(\mathcal{L}_2)$ and this has dimension two so, by Lemma 4.7, there is an exact sequence $0 \to \mathcal{N} \to B_2 \otimes \mathcal{O}_{\mathbb{B}_3} \to \mathcal{L}_2 \to 0$ for some line bundle $\mathcal{N}$. In fact, $\mathcal{N} \cong \mathcal{L}_2^{-1}$.

By definition, $\mathcal{L}_{n+2} = \mathcal{L}_2 \otimes \mathcal{M}$ where $\mathcal{M} \cong \mathcal{O}(D_{n+2} - D_2)$, and the twisted multiplication map $B_2 \otimes B_n \to B_{n+2}$ is the ordinary multiplication map

$$B_2 \otimes H^0(\mathcal{M}) = H^0(\mathcal{L}_2) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L}_2 \otimes \mathcal{M}).$$

[1] We will eventually prove that Φ is an isomorphism in all degrees but the low degree cases need to be handled separately.

Author's personal copy

Applying $-\otimes\mathcal{M}$ to the exact sequence $0\to\mathcal{L}_2^{-1}\to B_2\otimes\mathcal{O}_{\mathbb{B}_3}\to\mathcal{L}_2\to 0$ and taking cohomology gives an exact sequence

$$0\to H^0\big(\mathcal{L}_2^{-1}\otimes\mathcal{M}\big)\to B_2\otimes H^0(\mathcal{M})\to H^0(\mathcal{L}_2\otimes\mathcal{M})\to H^1\big(\mathcal{L}_2^{-1}\otimes\mathcal{M}\big).$$

Hence, if $h^1(\mathcal{L}_2^{-1}\otimes\mathcal{M})=0$, then $B_2B_n=B_{n+2}$.

We will now show that $h^1(\mathcal{L}_2^{-1}\otimes\mathcal{M})=0$. Since $\mathcal{L}_2^{-1}\otimes\mathcal{M}\cong\mathcal{O}(-D_2+D_{n+2}-D_2)$ and $n+2\geqslant 7$,

$$\mathcal{L}_2^{-1}\otimes\mathcal{M}\cong\mathcal{O}(-D_{6m+r}-2D_2-K)$$

for suitable integers m and r such that $6m+r\geqslant 7$ and $0\leqslant r\leqslant 5$.

By Lemma 4.1, $D_{6m+r}-2D_2-K=D_r-2D_2-(m+1)K$. By Lemma 4.2, to show that $h^1(D_r-2D_2-(m+1)K)=0$ it suffices to show that conditions (4.1) and (4.2) hold for the divisors D in the following table:

		$D:=D_r-2D_2-(m+1)K\in\operatorname{Pic}\mathbb{B}_3$
$r=0$	$m\geqslant 2$	$(3m+1,m-1,m+1,m+1)\in\mathbb{Z}^4$
$r=1$	$m\geqslant 1$	$(3m+2,m,m+1,m+2)$
$r=2$	$m\geqslant 1$	$(3m+2,m,m+1,m+1)$
$r=3$	$m\geqslant 1$	$(3m+3,m,m+2,m+2)$
$r=4$	$m\geqslant 1$	$(3m+3,m,m+1,m+2)$
$r=5$	$m\geqslant 1$	$(3m+4,m+1,m+2,m+2)$.

This is a routine task. □

Theorem 4.9. *Let R be the free algebra $\mathbb{C}\langle x,y\rangle$ modulo the relations $x^5=yxy$ and $y^2=xyx$. The $\mathbb{C}$-algebra homomorphism*

$$\Phi:R=\mathbb{C}[x,y]\to B(\mathbb{B}_3,\mathcal{L},\sigma),\quad x\mapsto X,\ y\mapsto Zt,$$

is an isomorphism of graded algebras.

Proof. By Lemma 4.6, B_1 and B_2 are in the image of Φ. By Proposition 4.8, B is generated by B_1 and B_2. Hence Φ is surjective. But $\Phi(R_n)\subset B_n$, and R and B have the same Hilbert series, so Φ is also surjective. □

Consider $R^{(3)}\supset\mathbb{C}[x^3,xy,yx]$. Since $\dim R_6=7=(\dim R_3)^2-2$ there is a 2-dimensional space of quadratic relations among the elements x^3, xy, and yx. Hence $R^{(3)}$ is not a 3-dimensional Artin–Schelter regular algebra. The relations in the degree two component of $R^{(3)}$ are generated by

$$\big(x^3\big)^2=(xy)^2=(yx)^2.$$

Acknowledgments

The author is grateful to the following people: Amer Iqbal for bringing the paper [5] to his attention; Paul Hacking and Sándor Kovács for passing on some standard results about del Pezzo surfaces and algebraic geometry; Darin Stephenson for telling the author that R is an iterated Ore extension and for useful remarks about his papers [10] and [11]; Mark Blunk for pointing out an error in an earlier version of this paper.

Author's personal copy

References

[1] M. Artin, J. Tate, M. Van den Bergh, Some algebras related to automorphisms of elliptic curves, in: The Grothendieck Festschrift, vol. 1, Birkhäuser, Boston, 1990, pp. 33–85.
[2] M. Artin, J. Tate, M. Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. 106 (1991) 335–388.
[3] M. Artin, M. Van den Bergh, Twisted homogeneous coordinate rings, J. Algebra 133 (1990) 249–271.
[4] M. Artin, J.J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (1994) 228–287.
[5] C.E. Beasley, M.R. Plesser, Toric duality is Seiberg duality, J. High Energy Phys. 0112 (2001) 1–38, doi:10.1088/1126-6708/2001/12/001, arXiv:hep-th/0109053v1.
[6] D.A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995) 17–50.
[7] J.-P. Serre, Faisceaux algébriques cohérents, Ann. Math. 61 (1955) 197–278.
[8] S.P. Smith, Twisted homogeneous coordinate rings and three superpotential algebras, in preparation.
[9] S.P. Smith, J. Tate, The centers of the 3- and 4-dimensional Sklyanin algebras, K-Theory 8 (1994) 19–63.
[10] D.R. Stephenson, Artin–Schelter regular algebras of global dimension three, J. Algebra 183 (1996) 55–73.
[11] D.R. Stephenson, Algebras associated to elliptic curves, Trans. Amer. Math. Soc. 349 (1997) 2317–2340.
[12] J.J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. Lond. Math. Soc. 72 (1996) 281–311.