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A derived equivalence for a degree 6 del Pezzo surface over an arbitrary field

by

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Abstract

Let S be a degree six del Pezzo surface over an arbitrary field F. Motivated by the first author's classification of all such S up to isomorphism [3] in terms of a separable F-algebra $B \times Q \times F$, and by his K-theory isomorphism $K_n(S) \cong K_n(B \times Q \times F)$ for $n \geq 0$, we prove an equivalence of derived categories

$$\mathsf{D}^b(\mathsf{coh} S) \equiv \mathsf{D}^b(\mathsf{mod} A)$$

where A is an explicitly given finite dimensional F-algebra whose semisimple part is $B \times Q \times F$.

Key Words: Derived equivalence, del Pezzo surface, arbitrary field

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1. Introduction

We will work over an arbitrary field F.

Throughout S denotes a degree six del Pezzo surface over F. Equivalently, S is a smooth projective surface over F whose anti-canonical sheaf is ample and has self-intersection number 6.

Throughout $\bar{F}$ will denote a separable closure of F and we will write

$$\bar{S} = S_{\bar{F}} = S \times_{\operatorname{Spec} F} \operatorname{Spec} \bar{F}.$$

In [3], the first author classified such S up to isomorphism by associating to S a pair of separable F-algebras B and Q, both defined as endomorphism rings of certain locally free sheaves on S. Furthermore, it was shown there that the algebraic K-theory of S is isomorphic to that of the algebra $B \times Q \times F$.

Let $\mathsf{coh} S$ denote the category of coherent sheaves on S and let $\mathsf{mod} A$ denote the category of noetherian right A-modules. Let $\equiv$ denote equivalence of derived

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categories. Our main result (Theorem 4.5) establishes a derived equivalence

$$\mathsf{D}^b(\mathsf{coh}S) \equiv \mathsf{D}^b(\mathsf{mod}A) \tag{1-1}$$

where A is a finite dimensional F-algebra whose semi-simple quotient is $B \times Q \times F$. We prove this equivalence by constructing a tilting bundle $\mathcal{T}$ on S that has A as its endomorphism ring. (The definition of a tilting bundle is given in section 4.) The main novelty of our approach is that we do not make any assumptions on the base field F. Since the field F is arbitrary, we cannot assume that S is obtained by blowing up $\mathbb{P}^2_F$ (in fact S could be a minimal surface), nor can we use exceptional collections.

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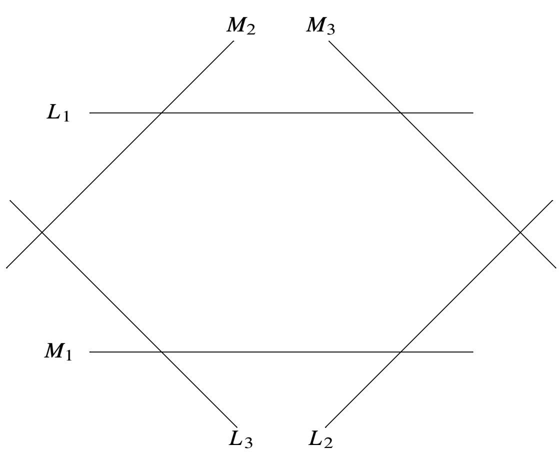
2. Basic facts about $\bar{S}$

In this section, we give basic facts about the degree 6 del Pezzo surface $\bar{S}$. Since all the results here are well-known, we do not give references.

There are six (-1)-curves on $\bar{S}$, which we may take to lie in the following configuration:



(2-1)

The Picard group is

$$\operatorname{Pic}\bar{S} \cong \frac{\bigoplus_{i=1}^{3}(\mathbb{Z}L_i \oplus \mathbb{Z}M_i)}{(M_i + L_j = M_j + L_i \mid 1 \le i, j \le 3)}.$$

Usually we only care about the class of a divisor in $\operatorname{Pic}\bar{S}$. We will write

$$D_1 \sim D_2$$

if D_1 and D_2 are linearly equivalent divisors.

As remarked in the discussion after Prop. 2.1 in [3], the group of connected components of the group $\operatorname{Aut}\bar{S}$ is $S_2 \times S_3$, which can be identified with the automorphism group of the hexagon of (-1)-curves on $\bar{S}$. In particular, there is an element $\sigma \in \operatorname{Aut}(\bar{S})$ that cyclically permutes the six exceptional lines. It is easy to see that $(1+\sigma)(1-\sigma^3)$ acts trivially on $\operatorname{Pic}\bar{S}$.

An anti-canonical divisor is

$$-K_{\bar{S}} := L_1 + L_2 + L_3 + M_1 + M_2 + M_3.$$

This is ample. We define two particular divisors

$$H \;:=\; L_1 + M_2 + M_3 \;\sim\; L_2 + M_1 + M_3 \;\sim\; L_3 + M_1 + M_2 \tag{2-2}$$

and

$$H' \;:=\; L_1 + L_2 + M_3 \;\sim\; L_2 + L_3 + M_1 \;\sim\; L_3 + L_1 + M_2 \tag{2-3}$$

on $\bar{S}$. Note that $\sigma(H) \sim H'$ and $\sigma^2(H) \sim H$.

We define the degree of a divisor C on $\bar{S}$ as $\deg C = -C \cdot K$. Each exceptional line has degree 1.

There are two morphisms $f, f' : \bar{S} \to \mathbb{P}^2_{\bar{F}}$, each of which realizes $\bar{S}$ as the blowup of $\mathbb{P}^2_{\bar{F}}$ at three non-collinear points. We choose these so that f contracts the lines L_1, L_2, and L_3 and f' contracts the lines M_1, M_2, and M_3. These two morphisms induce injective group homomorphisms $f^*, f'^* : \operatorname{Pic}\mathbb{P}^2 \to \operatorname{Pic}\bar{S}$. If ℓ is a line on $\mathbb{P}^2_{\bar{F}}$, then $f^*\ell = H$ and $f'^*\ell = H'$.

The action of $\operatorname{Gal}(\bar{F}/F)$ on the exceptional lines on $\bar{S}$ induces actions of $\operatorname{Gal}(\bar{F}/F)$ on

$$\overline{\mathcal{I}} := \bigoplus_{i=0}^{5} \mathcal{O}_{\bar{S}}(\sigma^i H)$$

and

$$\overline{\mathcal{J}} := \bigoplus_{i=0}^{5} \mathcal{O}_{\bar{S}}(\sigma^i (L_1 + M_2))$$

that are compatible with its action on $\bar{S}$. In particular, $\overline{\mathcal{I}}$ and $\overline{\mathcal{J}}$ are $\mathrm{Gal}(\bar{F}/F)$-invariant. It follows that the locally free sheaves $\overline{\mathcal{I}}$ and $\overline{\mathcal{J}}$ descend to locally free sheaves $\mathcal{I}$ and $\mathcal{J}$ on S.

Define

$$\overline{\mathcal{T}} := \overline{\mathcal{I}} \oplus \overline{\mathcal{J}} \oplus \mathcal{O}_{\bar{S}}, \qquad \mathcal{T} := \mathcal{I} \oplus \mathcal{J} \oplus \mathcal{O}_S,$$

and

$$B := \mathrm{End}_S \mathcal{I}, \qquad Q := \mathrm{End}_S \mathcal{J}, \qquad A := \mathrm{End}_S \mathcal{T}.$$

In [3] it is shown that S is determined up to isomorphism by the pair of F-algebras (B,Q). (Actually, in [3], B is defined as $\left(\mathrm{End}_S \mathcal{I}^\vee\right)^{\mathrm{op}}$. Since sending a homomorphism $\alpha : \mathcal{I} \to \mathcal{I}$ to its transpose $\alpha^\vee : \mathcal{I}^\vee \to \mathcal{I}^\vee$ is an anti-isomorphism from $\mathrm{End}_S \mathcal{I}$ to $\mathrm{End}_S \mathcal{I}^\vee$, our B is the same as that in [3], and similarly for Q.) As discussed in [3], the algebras B and Q are Azumaya over their centers, which are respectively étale quadratic and cubic extensions of F. Moreover, these étale centers can be recovered from the action of $\mathrm{Gal}(\bar{F}/F)$ on the hexagon of (-1)-curves, as the action induces a 1-cocycle of $\mathrm{Gal}(\bar{F}/F)$ with values in $S_2 \times S_3$, inducing a pair of étale extensions of F, quadratic and cubic.

We end this section with two results about the endomorphism algebra of $\mathcal{T}$.

Lemma 2.1 *Let* $A := \mathrm{End}_S \mathcal{T}$. *Then*

$$A = \begin{pmatrix} B & \mathrm{Hom}_S(\mathcal{J},\mathcal{I}) & \mathrm{Hom}_S(\mathcal{O}_S,\mathcal{I}) \\ 0 & Q & \mathrm{Hom}_S(\mathcal{O}_S,\mathcal{J}) \\ 0 & 0 & F \end{pmatrix}.$$

Proof: It suffices to show $\mathrm{Hom}_{\bar{S}}(\overline{\mathcal{I}},\overline{\mathcal{J}}) = \mathrm{Hom}_{\bar{S}}(\overline{\mathcal{I}},\mathcal{O}_{\bar{S}}) = \mathrm{Hom}_{\bar{S}}(\overline{\mathcal{J}},\mathcal{O}_{\bar{S}}) = 0$. However, each of these three Hom-spaces is isomorphic to a direct sum of terms of the form $H^0(\bar{S},\mathcal{O}_{\bar{S}}(D))$ for a divisor D with $\deg D < 0$. But if D has a section then $D \sim D'$ for some effective D' so $\deg D = -D'.K \geq 0$. These Hom spaces are therefore zero. □

The projective dimension of a left T-module is denoted by $\mathrm{pdim}_T M$. The global homological dimension of T is defined and denoted by

$$\mathrm{gldim}\, T := \sup\{\mathrm{pdim}_T M \mid M \in \mathsf{Mod}\, T\}.$$

Proposition 2.2 $\mathrm{gldim}\, A \leq 2$.

Proof: Let R and S be rings and X an R-S-bimodule. If S is a semisimple ring, then

$$\mathrm{gldim}\begin{pmatrix} R & X \\ 0 & S \end{pmatrix} = \max\{\mathrm{pdim}_R X + 1, \mathrm{gldim}\, R\}.$$

(See [1, Prop. III.2.7].) Applying this result twice, first to

$$A' := \begin{pmatrix} B & \mathrm{Hom}(\mathcal{J},\mathcal{I}) \\ 0 & Q \end{pmatrix} \tag{2-4}$$

then to A with $R = A'$ and $S = F$, gives the desired result. □

3. Cohomology vanishing lemmas

We will prove several results about vanishing of cohomology and Ext-groups for sheaves on S. These results will be used in Section 4 to show that $\mathcal{T}$ is a tilting bundle and therefore induces an equivalence of derived categories.

A key step in proving that $\mathcal{T}$ is tilting is showing that $\mathrm{Ext}^i_S(\mathcal{T},\mathcal{T}) = 0$ for $i > 0$. This reduces, by flat base change, to proving that $\mathrm{Ext}^i_{\bar{S}}(\overline{\mathcal{T}},\overline{\mathcal{T}}) = 0$. Given the explicit description of $\overline{\mathcal{T}}$ as a direct sum of invertible sheaves, it suffices to prove that $h^1(D - D') = h^2(D - D') = 0$ for all D and D' belonging to the list

$$H, \quad H', \quad L_1 + M_2, \quad L_2 + M_3, \quad L_3 + M_1, \quad 0. \tag{3-1}$$

We will make repeated use of the relation $L_i + M_j \sim L_j + M_i$.

Proposition 3.1 *Let D and D' be divisors on $\bar{S}$ appearing in the list (3-1). Then*

$$-3 \le \deg(D - D') \le 3.$$

Furthermore,

1. *if* $\deg(D - D') = 1$*, then $D - D'$ is linearly equivalent to an exceptional line.*
2. *if* $\deg(D - D') = 2$*, then $D - D' \sim L_i + M_j$ for some $i \neq j \in \{1,2,3\}$.*
3. *if* $\deg(D - D') = 3$*, then $D - D'$ is linearly equivalent to either H or H'.*
4. *if* $\deg(D - D') = 0$*, then $D - D'$ is linearly equivalent to either* 0*, $L_i - L_j$, $L_i - M_i$, or $M_i - L_i$ for some $i, j \in \{1,2,3\}$.*
5. *if* $\deg(D - D') < 0$*, then $D - D'$ is linearly equivalent to either $-L_i$, or $-M_j$, or $-L_i - M_j$, or $-H$, or $-H'$.*

Proof: Exceptional lines have degree 1 so $\deg H = \deg H' = 3$ and $\deg(L_i + M_j) = 2$. It follows that the degree of $D - D'$ is between 3 and -3.

(1) If $\deg(D - D') = 1$, then D is linearly equivalent to H or H' and $D' = L_i + M_j$ for some i, j. It follows from (2-2) and (2-3) that $D - D'$ is linearly equivalent to an exceptional line, and every exceptional line can occur as $D - D'$.

(2) and (3) are obvious.

(4) In this case D and D' have the same degree.

If $\deg D = \deg D' = 2$, then $D = L_i + M_j$ and $D' = L_k + M_\ell$. By considering all possible i, j, k, ℓ, we see that $D - D'$ is linearly equivalent to a divisor of the form $L_i - L_j$.

If $\deg D = \deg D' = 3$, then, for example, $D \sim H$ and $D' \sim H'$, and $D - D' \sim L_i - M_i$. Switching the roles of H and H', we see $D - D' \sim M_i - L_i$. Finally, we may have $D - D' \sim 0$.

(5) This is the mirror of the cases (1)-(3). □

Corollary 3.2 *Suppose D is the difference of two divisors appearing in the list (3-1). If* $\deg D \geq -2$, *then there is an exceptional line E on $\bar{S}$ such that $D - E$ is also a difference of two divisors appearing in the list (3-1) and $D.E \geq -1$.*

Proof: This is established through case-by-case analysis using Proposition 3.1 to look at all the possibilities for D. □

A divisor D on $\bar{S}$ is good if $h^1(D) = h^2(D) = 0$.

Lemma 3.3 *The divisors $-H$ and $-H'$ on $\bar{S}$ are good.*

Proof: The existence of the morphisms $f, f' : \bar{S} \to \mathbb{P}^2_{\bar{F}}$ allows us to use the Leray spectral sequence. The arguments for $-H$ and $-H'$ are the same so we only prove the result for $-H$.

Because $\bar{S}$ is a blowup of $\mathbb{P}^2_{\bar{F}}$, $f_*\mathcal{O}_{\bar{S}} = \mathcal{O}_{\mathbb{P}^2_{\bar{F}}}$ and $R^j f_*\mathcal{O}_{\bar{S}} = 0$ if $j \geq 1$.

Since $\mathcal{O}_{\bar{S}}(-H) \cong f^*\mathcal{O}_{\mathbb{P}^2_{\bar{F}}}(-\ell)$, the projection formula gives

$$\begin{aligned} R^j f_*\mathcal{O}_{\bar{S}}(-H) &= R^j f_*\big(\mathcal{O}_{\bar{S}} \otimes f^*\mathcal{O}_{\mathbb{P}^2_{\bar{F}}}(-\ell)\big) \\ &\cong R^j f_*\mathcal{O}_{\bar{S}} \otimes \mathcal{O}_{\mathbb{P}^2_{\bar{F}}}(-\ell) \\ &\cong \begin{cases} \mathcal{O}_{\mathbb{P}^2_{\bar{F}}}(-\ell) & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases} \end{aligned}$$

The Leray spectral sequence

$$H^i(\mathbb{P}^2_{\bar{F}}, R^j f_*\mathcal{O}_{\bar{S}}(-H)) \Rightarrow H^{i+j}(\bar{S}, \mathcal{O}_{\bar{S}}(-H))$$

therefore degenerates to give

$$H^i(\bar{S}, \mathcal{O}_{\bar{S}}(-H)) \cong H^i(\mathbb{P}^2_{\bar{F}}, \mathcal{O}_{\mathbb{P}^2}(-\ell))$$

for all i. The result follows because $H^i(\mathbb{P}^2_{\bar{F}}, \mathcal{O}_{\mathbb{P}^2}(-\ell)) = 0$ for all i. □

Lemma 3.4 *Let C be any divisor on $\bar{S}$, and let E be one of the (-1)-curves. If $C - E$ is good and $C.E \geq -1$, then C is good.*

Proof: The long exact sequence in cohomology associated to

$$0 \to \mathcal{O}_{\bar{S}}(C - E) \to \mathcal{O}_{\bar{S}}(C) \to \mathcal{O}_E(C) \to 0$$

reads in part

$$\longrightarrow H^1(\bar{S}, \mathcal{O}_{\bar{S}}(C - E)) \longrightarrow H^1(\bar{S}, \mathcal{O}_{\bar{S}}(C)) \longrightarrow H^1(\bar{S}, \mathcal{O}_E(C)) \longrightarrow$$

$$\longrightarrow H^2(\bar{S}, \mathcal{O}_{\bar{S}}(C - E)) \longrightarrow H^2(\bar{S}, \mathcal{O}_{\bar{S}}(C)) \longrightarrow H^2(\bar{S}, \mathcal{O}_E(C)).$$

By hypothesis, the left-most term in each row is zero. The right-most term in each row is also zero because $H^i(\bar{S}, \mathcal{O}_E(C)) \cong H^i(\mathbb{P}^1_{\bar{F}}, \mathcal{O}_{\mathbb{P}^1}(C.E))$. Hence C is good. □

4. The tilting bundle $\mathcal{T}$

In this section, we show that $\mathcal{T}$ is a tilting bundle and prove our main result.

Proposition 4.1 *Let $i \geq 1$. Then* $\operatorname{Ext}^i_S(\mathcal{T}, \mathcal{T}) = 0$.

Proof: By flat base change it suffices to prove this when F is separably closed so we assume that $F = \bar{F}$. In that case $\operatorname{Ext}^i_S(\mathcal{T}, \mathcal{T})$ is isomorphic to a direct sum of terms of the form $H^i(S, \mathcal{O}_S(D - D'))$ where D and D' are divisors in the list (3-1).

It therefore suffices to show that $D - D'$ is good whenever D and D' are divisors in the list (3-1).

We argue by induction on $\deg(D - D')$. By Proposition 3.1, $-3 \leq \deg(D - D') \leq 3$. If $\deg(D - D') = -3$, then $D - D'$ is good by Lemma 3.3. Now suppose that $-2 \leq \deg(D - D') \leq 3$. By Corollary 3.2, there is an exceptional line E such that $D - D' - E$ is a difference of divisors in (3-1) and $(D - D').E \geq -1$. By the induction hypothesis, $D - D' - E$ is good, and it then follows from Lemma 3.4 that $D - D'$ is good. □

Since $\bar{S}$ is a del Pezzo surface of degree ≥ 6 it is a toric variety so we can, and will, make use of Cox's homogeneous coordinate ring for it [5].

Lemma 4.2 *Every $\mathcal{F} \in \mathsf{coh}\bar{S}$ has a finite resolution in which all terms are direct sums of invertible sheaves $\mathcal{O}_{\bar{S}}(D)$ for various divisors D on $\bar{S}$.*

Proof: Let A be Cox's homogeneous coordinate ring for $\bar{S}$ [5]. Then A is a polynomial ring with a grading by $\mathrm{Pic}(\bar{S})$. Let M be a finitely generated graded A-module. Then M has a finite projective resolution in the category of graded A-modules. By [9, Lemma 2.2], every finitely generated projective graded A-module is a direct sum of twists of A. The exact functor $\mathsf{Gr}(A,\mathrm{Pic}(\bar{S})) \to \mathsf{Qcoh}\bar{S}$, $M \rightsquigarrow \widetilde{M}$, described in [5, Thm. 3.11] sends the resolution of M to an exact sequence in $\mathsf{Qcoh}\bar{S}$ in which the right-most term is $\widetilde{M}$ and all other terms are direct sums of various $\mathcal{O}_{\bar{S}}(D)$, $D \in \mathrm{Div}(\bar{S})$. Given $\mathcal{F} \in \mathsf{coh}\bar{S}$, there is a finitely generated graded A-module M such that $\mathcal{F} \cong \widetilde{M}$. □

For the rest of this paper, we will work in the derived category. If D is a triangulated category, we denote the shift of an object $\mathcal{M}$ by $\mathcal{M}[1]$. Recall that a subcategory of D is *thick* (épaisse) if it is closed under isomorphisms, shifts, taking cones of morphisms, and taking direct summands of objects.

Let D be a triangulated category and $\mathcal{E}$ a set of objects in D. Then

- D^c denotes the full subcategory of D consisting of the compact objects, i.e., those objects C such that $\mathrm{Hom}_{\mathsf{D}}(C,-)$ commutes with direct sums;
- $\langle \mathcal{E} \rangle$ denotes the smallest thick full triangulated subcategory of D containing $\mathcal{E}$;
- $\mathcal{E}^{\perp}$ denotes the full subcategory of D consisting of objects $\mathcal{M}$ such that $\mathrm{Hom}_{\mathsf{D}}(E[i],\mathcal{M}) = 0$ for all $E \in \mathcal{E}$ and all $i \in \mathbb{Z}$.

We say that

- $\mathcal{E}$ generates D if $\mathcal{E}^{\perp} = 0$ and that
- D is compactly generated if $(\mathsf{D}^c)^{\perp} = 0$.

Clearly, if D is compactly generated and $\langle \mathcal{E} \rangle = \mathsf{D}^c$, then $\mathcal{E}$ generates D.

Theorem 4.3 (Ravenel and Neeman [8]. Also see Thm. 2.1.2 in [4]) *Let D be a compactly generated triangulated category. Then a set of objects $\mathcal{E} \subset \mathsf{D}^c$ generates D if and only if $\langle \mathcal{E} \rangle = \mathsf{D}^c$.* □

The unbounded derived categories $\mathsf{D}(\mathsf{Qcoh}S)$ and $\mathsf{D}(\mathsf{Qcoh}\bar{S})$ are compactly generated. Moreover, $\mathsf{D}(\mathsf{Qcoh}S)^c = \mathsf{D}^b(\mathsf{coh}S)$ and $\mathsf{D}(\mathsf{Qcoh}\bar{S})^c = \mathsf{D}^b(\mathsf{coh}\bar{S})$.

Tilting bundles. Let X be a projective scheme over a field k. A locally free sheaf $\mathcal{T} \in \mathsf{coh}X$ is a tilting bundle if it generates $\mathsf{D}(\mathsf{Qcoh}X)$ and $\mathrm{Ext}^i_X(\mathcal{T},\mathcal{T}) = 0$ for all $i > 0$.

Theorem 4.4 *$\overline{\mathcal{T}}$ generates $\mathsf{D}(\mathsf{Qcoh}\bar{S})$ and $\langle \overline{\mathcal{T}} \rangle = \mathsf{D}^b(\mathsf{coh}\bar{S})$.*

Proof: By Theorem 4.3, it suffices to show that $\langle\overline{\mathcal{T}}\rangle = \mathsf{D}^b(\mathsf{coh}\bar{S})$. Since $\langle\mathsf{coh}\bar{S}\rangle = \mathsf{D}^b(\mathsf{coh}\bar{S})$ it suffices to show that every coherent $\mathcal{O}_{\bar{S}}$-module belongs to $\langle\overline{\mathcal{T}}\rangle$.

If D is an effective divisor on $\bar{S}$ we write $\mathcal{I}_D$ for the ideal vanishing on D as a scheme. Thus $\mathcal{I}_D \cong \mathcal{O}_{\bar{S}}(-D)$. Whenever we write an arrow $\mathcal{O}_{\bar{S}}(-D) \to \mathcal{O}_{\bar{S}}$ it will be with the tacit understanding that this is the composition of an isomorphism $\mathcal{O}_{\bar{S}}(-D) \to \mathcal{I}_D$ followed by the inclusion $\mathcal{I}_D \to \mathcal{O}_{\bar{S}}$.

Since $M_3 \cdot (L_1 + M_2 + M_3) = 0$, $\mathcal{O}_{M_3} \cong \mathcal{O}_{M_3}(L_1 + M_2 + M_3)$. It follows from the exact sequences

$$0 \to \mathcal{O}_{\bar{S}}(L_1 + M_2) \to \mathcal{O}_{\bar{S}}(L_1 + M_2 + M_3) \to \mathcal{O}_{M_3}(L_1 + M_2 + M_3) \to 0$$

and

$$0 \to \mathcal{O}_{\bar{S}}(-M_3) \to \mathcal{O}_{\bar{S}} \to \mathcal{O}_{M_3} \to 0$$

that $\mathcal{O}_{M_3}$ and $\mathcal{O}_{\bar{S}}(-M_3)$ belong to $\langle\overline{\mathcal{T}}\rangle$. Hence $\mathcal{O}_E$ and $\mathcal{O}_{\bar{S}}(-E)$ belong to $\langle\overline{\mathcal{T}}\rangle$ for all exceptional lines E.

Since $L_i.L_k = 0$ if $i \neq k$, there is an exact sequence

$$0 \to \mathcal{O}_{\bar{S}}(-L_i - L_k) \to \mathcal{O}_{\bar{S}}(-L_i) \oplus \mathcal{O}_{\bar{S}}(-L_k) \to \mathcal{O}_{\bar{S}} \to 0.$$

Twisting by $L_i + M_j + L_k$, we obtain

$$0 \to \mathcal{O}_{\bar{S}}(M_j) \to \mathcal{O}_{\bar{S}}(M_j + L_k) \oplus \mathcal{O}_{\bar{S}}(L_i + M_j) \to \mathcal{O}_{\bar{S}}(L_i + M_j + L_k) \to 0.$$

Therefore, $\mathcal{O}_{\bar{S}}(M_j) \in \langle\overline{\mathcal{T}}\rangle$. From the exact sequence

$$0 \to \mathcal{O}_{\bar{S}} \to \mathcal{O}_{\bar{S}}(M_j) \to \mathcal{O}_{M_j}(M_j) \to 0,$$

we deduce that $\mathcal{O}_{M_j}(M_j) \in \langle\overline{\mathcal{T}}\rangle$.

It follows that $\mathcal{O}_E(E) \in \langle\overline{\mathcal{T}}\rangle$ for every exceptional curve E. But $\mathcal{O}_E$ is also in $\langle\overline{\mathcal{T}}\rangle$ so, because $\mathsf{D}^b(\mathsf{coh}\mathbb{P}^1_{\bar{F}})$ is generated by $\mathcal{O}_{\mathbb{P}^1_{\bar{F}}}$ and $\mathcal{O}_{\mathbb{P}^1_{\bar{F}}}(-1)$, it follows that $\mathsf{D}^b(\mathsf{coh}E) \subset \langle\overline{\mathcal{T}}\rangle$. Hence $\mathcal{O}_E(D) \in \langle\overline{\mathcal{T}}\rangle$ for all divisors D on $\bar{S}$.

Suppose $\mathcal{O}_{\bar{S}}(D) \in \langle\overline{\mathcal{T}}\rangle$. Then $\mathcal{O}_{\bar{S}}(D - E) \in \langle\overline{\mathcal{T}}\rangle$ because there is an exact sequence

$$0 \to \mathcal{O}_{\bar{S}}(D - E) \to \mathcal{O}_{\bar{S}}(D) \to \mathcal{O}_E(D) \to 0.$$

Likewise, $\mathcal{O}_{\bar{S}}(D + E) \in \langle\overline{\mathcal{T}}\rangle$ because there is an exact sequence

$$0 \to \mathcal{O}_{\bar{S}}(D) \to \mathcal{O}_{\bar{S}}(D + E) \to \mathcal{O}_E(D + E) \to 0.$$

It follows that $\langle\overline{\mathcal{T}}\rangle$ contains $\mathcal{O}_{\bar{S}}(D)$ for all $D \in \operatorname{Div}\bar{S}$ and therefore, by Lemma 4.2, contains $\mathcal{F}$ for every $\mathcal{F} \in \mathsf{coh}\bar{S}$. $\square$

When F is not separably closed $\mathcal{T}$ need not split as a direct sum of line bundles so the arguments in Theorem 4.4 can not be used to prove directly that $\langle\mathcal{T}\rangle = \mathsf{D}^b(\mathsf{coh} S)$. Instead we will show that $\mathcal{T}$ generates $\mathsf{D}(\mathsf{Qcoh} S)$ and then apply Theorem 4.3.

Theorem 4.5 *Let F be an arbitrary field. Then*

$$\mathrm{RHom}_S(\mathcal{T},-) : \mathsf{D}^b(\mathsf{coh} S) \to \mathsf{D}^b(\mathsf{mod} A)$$

is an equivalence of categories.

Proof: We will show that $\mathcal{T}$ generates $\mathsf{D}(\mathsf{Qcoh} S)$. It will then follow from Theorem 4.3 that

$$\langle\mathcal{T}\rangle = \mathsf{D}(\mathsf{Qcoh} S)^c = \mathsf{D}^b(\mathsf{coh} S).$$

By Proposition 4.1, $\mathrm{Ext}^i_S(\mathcal{T},\mathcal{T}) = 0$ for $i > 0$. By Proposition 2.2, $A = \mathrm{End}_S(\mathcal{T})$ has finite global dimension. Thus we have shown that $\mathcal{T}$ is a tilting bundle and our theorem will then follow directly from [2, Thm. 3.1.2] (or [6, Thm. 7.6]).

Let $\mathcal{M} \in \mathsf{D}(\mathsf{Qcoh} S)$ and suppose $\mathrm{RHom}_S(\mathcal{T},\mathcal{M}) = 0$. We must show that $\mathcal{M} = 0$.

Since $\mathcal{T}$ is locally free, $\mathcal{H}om_S(\mathcal{T},-)$ and $\mathcal{T}^\vee \otimes_S -$ are exact functors on $\mathsf{Qcoh} S$. Likewise, $\mathcal{H}om_{\bar{S}}(\overline{\mathcal{T}},-)$ and $\overline{\mathcal{T}}^\vee \otimes_{\bar{S}} -$ are exact functors on $\mathsf{Qcoh}\bar{S}$. Thus, for example, $\mathrm{R}\mathcal{H}om_S(\mathcal{T},\mathcal{M})$ can be computed on $\mathsf{D}(\mathsf{Qcoh} S)$ by applying $\mathcal{H}om_S(\mathcal{T},-)$ to each individual term in $\mathcal{M}$.

Consider the cartesian square

$$\begin{tikzcd} \bar{S} \arrow[r, "v"] \arrow[d, "q"'] & S \arrow[d, "p"] \\ \mathrm{Spec}(\bar{F}) \arrow[r, "u"'] & \mathrm{Spec}(F). \end{tikzcd}$$

Since u (and therefore v) is flat, the natural transformation

$$u^* Rp_* \to Rq_* v^*$$

is an isomorphism of functors from $\mathsf{D}(\mathsf{Qcoh} S)$ to $\mathsf{D}(\bar{F})$ [7, (3.18)]. We now have

$$\begin{aligned} 0 \;=\; u^*\mathrm{RHom}_S(\mathcal{T},\mathcal{M}) &\cong u^* Rp_* \mathrm{R}\mathcal{H}om_S(\mathcal{T},\mathcal{M}) \quad \text{by [7, p.85]} \\ &\cong Rq_* v^* \mathrm{R}\mathcal{H}om_S(\mathcal{T},\mathcal{M}) \quad \text{by [7, (3.18)]} \\ &\cong Rq_* v^* (\mathcal{T}^\vee \otimes^L_S \mathcal{M}) \\ &\cong Rq_* (\overline{\mathcal{T}}^\vee \otimes^L_{\bar{S}} Lv^*\mathcal{M}) \\ &\cong Rq_* \mathrm{R}\mathcal{H}om_{\bar{S}}(\overline{\mathcal{T}}, Lv^*\mathcal{M}) \\ &\cong \mathrm{RHom}_{\bar{S}}(\overline{\mathcal{T}}, Lv^*\mathcal{M}). \end{aligned}$$

But $\overline{\mathcal{T}}$ generates $\mathsf{D}(\mathsf{Qcoh}\bar{S})$ so $v^*\mathcal{M} = 0$. Since v^* is faithful, $\mathcal{M} = 0$, and we are done. □

Corollary 4.6 (cf. [3], Corollary 5.2) *The functor* $\mathrm{Hom}_S(\mathcal{T}, -) : \mathsf{coh}(S) \to \mathsf{mod}A$ *induces an isomorphism*

$$\mathrm{Hom}_S(\mathcal{T}, -) : K_*(S) \to K_*(F \times B \times Q).$$

Proof: It follows from Theorem 1.98 of [10] that the equivalence of derived categories found in Theorem 4.5 induces an isomorphism in K-theory

$$\mathrm{Hom}_S(\mathcal{T}, -) : K_*(\mathsf{coh}S) \to K_*(\mathsf{mod}A).$$

Moreover, A has a nilpotent ideal I so that A/I is isomorphic to its semi-simple quotient $F \times B \times Q$. Thus, it follows that the K-theory of A is isomorphic to that of $F \times B \times Q$, and we recover the isomorphism found in [3]. □

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