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The Four-Dimensional Sklyanin Algebras

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Abstract. The four-dimensional Sklyanin algebras are certain noncommutative graded algebras having the same Hilbert series as the polynomial ring on four indeterminates. Their structure and representation theory is intimately connected with the geometry of an elliptic curve (and a fixed translation) embedded in $\mathbb{P}^3$. This is an account of the work done on these algebras over the past four years.

Key words. Sklyanin algebras, graded algebras, noncommutative geometry, elliptic curves, representation theory.

1. Introduction

This is an expanded version of the talk I gave at the meeting in honour of M. Artin, held in Antwerp during May 1992.

Our purpose is to introduce the nonexpert to the structure and representation theory of the noncommutative algebras defined by Sklyanin in 1982. Of more importance than the algebras themselves are the methods and techniques used to study them. These methods suggest that there is a subject of 'noncommutative algebraic geometry': the Sklyanin algebras are a particularly interesting and suggestive example of what such a subject is about.

It is appropriate to discuss such matters at this conference in honour of M. Artin. As the historical account in the last section of this article makes clear, he has been at the forefront of the development of these new methods.

2. Definitions

Fix an elliptic curve $E = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\eta$ over $\mathbb{C}$, and a point $\tau \in E$ which is not in the four-torsion subgroup E_4. For each such pair (E, τ) we will define a graded algebra $A(E, \tau)$. There are several possible definitions, but the following has the advantage of being short, and easy to work with.

DEFINITION 2.1. [16] Fix a degree 4 line bundle $\mathscr{L}$ on E, and set $V = H^0(E, \mathscr{L})$. Identify $V \otimes V$ with $H^0(E \times E, \mathscr{L} \boxtimes \mathscr{L})$. Define the shifted diagonal

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$\Delta_\tau := \{(x, x+\tau) \mid x \in E\}$. Denote by M the set of fixed points for the involution $(x, y) \to (y+\tau, x-\tau)$ on $E \times E$. We say that a divisor D on $E \times E$ is allowable if D is stable under this involution, and M occurs in D with even multiplicity. The four-dimensional Sklyanin algebra associated to (E, τ) is defined to be the quotient of the tensor algebra,

$$A(E, \tau) := T(V)/\langle R \rangle,$$

where

$$R := \{f \in V \otimes V \mid f = 0, \text{ or } (f) = \Delta_\tau + D \text{ and } D \text{ is allowable}\}.$$

This definition was given by Odesskii and Feigin [16] in 1989. However, as the name suggests, these algebras were first defined by Sklyanin in 1982 [18]. Sklyanin's definition was nothing like the above. He begins with Baxter's 2×2 solution to the Yang–Baxter equation, and defines an algebra in terms of this. Some manipulations yield a description in terms of generators and relations:

$$A(E, \tau) = \mathbb{C}[x_0, x_1, x_2, x_3]$$

with relations

$$\begin{aligned} x_0x_1 - x_1x_0 &= \alpha_1(x_2x_3 + x_3x_2), & x_0x_1 + x_1x_0 &= x_2x_3 - x_3x_2,\\ x_0x_2 - x_2x_0 &= \alpha_2(x_3x_1 + x_1x_3), & x_0x_2 + x_2x_0 &= x_3x_1 - x_1x_3,\\ x_0x_3 - x_3x_0 &= \alpha_3(x_1x_2 + x_2x_1), & x_0x_3 + x_3x_0 &= x_1x_2 - x_2x_1, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ is determined by (E, τ) (see Section 12.4 for details). The parameter $\boldsymbol{\alpha}$ lies on the surface $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$, but not all points of this surface arise from some (E, τ).

Let A be an $\mathbb{N}$-graded k-algebra. If M is a $\mathbb{Z}$-graded A-module, we write $H_M(t) = \Sigma_n(\dim M_n)t^n$ for the *Hilbert series* of M. If $H_M(t) = g(t)(1-t)^{-k}$ for some $g(t) \in \mathbb{Z}[t, t^{-1}]$ satisfying $g(1) \neq 0$, then the *Gelfand–Kirillov dimension* of M is defined to be $d(M) := k$ and the *multiplicity* of M is defined to be $e(M) := g(1)$. Thus, $d(M) = 0$ if and only if M is finite-dimensional, and in that case $e(M)$ is its dimension. If A is a quotient of a polynomial ring, then $d(M)$ is the Krull dimension of M, and $e(M)$ is the usual multiplicity.

An A-module M is *d-critical* if $d(M) = d$ and $d(M/N) < d$ for all nonzero submodules $N \subset M$. If $d \in \mathbb{Z}$ then we define the *shift* of M by d to be the graded A-module $M[d]$ which equals M as an ungraded module, but is graded by $M[d]_i = M_{d+i}$.

3. Good Properties

It has emerged that the Sklyanin algebras are in many respects as well-behaved as commutative polynomial rings, except for the fact that they are noncommutative.

THEOREM 3.1 [21]. *$A(E, \tau)$ is a Noetherian domain, and has the same Hilbert series as the polynomial ring in four indeterminates. It has global homological dimension 4, and is a Koszul algebra, in the sense that $\mathrm{Ext}^*_A(\mathbb{C}, \mathbb{C}) \cong A^! := T(V^*)/\langle R^\perp \rangle$.*

For a noncommutative algebra, having finite global dimension is not as strong a condition as it is for a commutative ring. However, $A(E,\tau)$ does satisfy stronger technical conditions, which play a crucial role in the proof of some of the later results.

THEOREM 3.2 [12]. *Let M be a finitely generated $A(E,\tau)$-module.*

(*a*) $\mathrm{Ext}^i_A(N,A) = 0$ *for all submodules* $N \subset \mathrm{Ext}^j_A(M,A)$ *and all* $i < j$.
(*b*) *If* $M \neq 0$ *and* $j(M) := \min\{j \mid \mathrm{Ext}^j_A(M,A) \neq 0\}$ *then* $d(M) + j(M) = 4$.

Condition (a) is called the *Auslander condition*, and a ring which satisfies condition (b) is said to satisfy the *Cohen–Macaulay condition.*

4. A Problem

Describe all finite-dimensional irreducible representations of $A(E,\tau)$.

Sklyanin's original motivation for introducing $A(E,\tau)$ was that a natural question concerning the 'quantum inverse scattering method' could be rephrased as the problem of finding all finite-dimensional $A(E,\tau)$-modules. Of course, the first step is to find the simple modules.

Even without Sklyanin's question as motivation, the question of finding the finite-dimensional representations of an algebra is a central issue in noncommutative algebra. This question arises from the more fundamental question of finding solutions in matrices to a system of 'noncommutative polynomial equations', in the same way that the question of finding (or understanding) the solutions in some field to a system of polynomial equations gives rise to commutative algebra and algebraic geometry. In our context there is an obvious bijection

$$\{d\text{-dimensional } A\text{-modules}\} \leftrightarrow \{\text{solutions } x_0, x_1, x_2, x_3 \in M_d(\mathbb{C}) \text{ to } (2.1)\}.$$

In this talk we wish to emphasize the parallel between the methods of commutative algebra/algebraic geometry, and the methods which have been introduced in the past 5 or 6 years to study certain noncommutative graded algebras. These methods are particularly effective for solving the problem stated above.

It is helpful to think of our problem as analogous to that of finding the points of an affine variety. However, since $A(E,\tau)$ is a graded algebra, it is natural to approach this question through an analogy with projective algebraic geometry. Thus, the graded A-modules are of central interest. These modules will be studied as objects of the category $\mathrm{Proj}(A)$.

5. Proj(A)

DEFINITION 5.1. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \dots$ be a graded, connected, Noetherian algebra over a field k, and suppose that A is generated as a k-algebra by A_1. ('Connected' means that $A_0 = k$.) Define the following categories.

GrMod(A) is the category of all finitely generated $\mathbb{Z}$-graded A-modules with morphisms being the degree 0 A-module homomorphisms.

Tors(A) is the full subcategory consisting of the finite-dimensional graded A-modules.

Proj(A) is the quotient category GrMod(A)/Tors(A), which may be formed since Tors (A) is a Serre subcategory.

If A were a commutative algebra, and S the projective scheme determined by A, then Serre's Theorem [17] says that the category Proj(A) is equivalent to the category of coherent sheaves of $\mathcal{O}_S$-modules.

For those somewhat unfamiliar with the formation of a quotient category, we remind them that the objects are the same as the objects of the original category, but there are more morphisms. In particular, there are more isomorphisms. For example, in our case, if $f: M \to N$ is a degree 0 homorphism of graded A-modules, then f is an isomorphism Proj(A) if the kernel and cokernel of f are finite-dimensional.

The finite-dimensional simple A-modules, which are the objects of eventual interest, are not graded modules, but the next two results indicate how they are related to the irreducible objects in Proj(A). Recall that an *irreducible object* in an Abelian category is an object M whose only subobjects are (isomorphic to) M itself and 0. In particular, if F is an infinite-dimensional graded A-module, all of whose proper quotients are finite-dimensional, then F is irreducible in Proj(A). Such modules are familiar to ring theorists: a 1-critical graded A-module is irreducible in Proj(A).

PROPOSITION 5.2. [20]. *Let A be a graded, connected, Noetherian algebra over an algebraically closed field k, generated in degree 1. Every nontrivial finite-dimensional simple A-module is a quotient of a graded A-module which is irreducible in* Proj(A). *This graded module is unique up to isomorphism and shifting degree, as an object of* Proj(A).

The *trivial* A-module is $A/A_1 + A_2 + \dots$. It is simple since A is assumed to be connected. We will always ignore this simple module (cf. the irrelevant prime ideal); the words 'simple A-module' will always mean 'nontrivial simple A-module'.

The reader should be warned that it is possible for an irreducible object in Proj(A) to have no finite dimensional simple quotients other than the trivial module.

PROPOSITION 5.3. [20]. *Let A be a graded, connected, Noetherian algebra over an algebraically closed field k, generated in degree 1. If F is an irreducible object in* Proj(A) *having finite-dimensional simple quotients S_1 and S_2, then $S_1$and S_2 are equivalent simple modules in the sense that there is a commutative diagram:*

$$\begin{array}{ccc} A & \xrightarrow{\phi_\lambda} & A \\ \psi_1 \downarrow & & \downarrow \psi_2 \\ M_{d_1}(k) & \xrightarrow{\sim} & M_{d_2}(k) \end{array}$$

for some $\lambda \in k$, *where* $\phi_\lambda \in \mathrm{Aut}(A)$ *is given by* $\phi_\lambda(a) = \lambda^n a$ *for* $a \in A_n$, *and* $\psi_i : A \to \mathrm{End}_k(S_i) \cong \mathrm{M}_{d_i}(k)$ *is the map induced by the action of* A *on* S_i. *In particular,* $\dim(S_1) = \dim(S_2)$.

The equivalence in Proposition 5.3 is the noncommutative analogue of the equivalence relation used to construct projective space, viz. two points in affine space are represented by the same point in projective space if and only if they are scalar multiples of one another.

After (5.2) and (5.3) our original problem may be broken into the following subproblems:

(1) find/classify all the irreducible objects in Proj(A);
(2) decide which of these have a finite-dimensional simple quotient;
(3) find the dimension of the simple quotients.

6. Point Modules

Suppose that $R = \mathbb{C}[T_0, \dots, T_n]/J$ is a graded quotient ring of the commutative polynomial ring endowed with its usual graded structure. Let $\mathscr{V}(J) \subset \mathbb{P}^n$ be the projective variety cut out by J. To each point $p \in \mathscr{V}(J)$ we may associate the graded R-module $M(p) := R/I(p) \cong \mathbb{C}[T]$, where $I(p)$ is the ideal generated by the homogeneous polynomials vanishing at p. Since $\mathbb{C}[T]$ is a domain, every proper quotient of $M(p)$ is finite-dimensional, whence $M(p)$ is an irreducible object in Proj(R). It is an easy exercise to see that every irreducible object in Proj(R) is isomorphic to one of this form. This familiar example motivates the following definition.

DEFINITION 6.1 [4]. A point module is a graded A-module M such that

(1) M is cyclic, and
(2) $H_M(t) = (1 - t)^{-1}$.

Even if A is not commutative, a point module still determines a point of the projective space $\mathbb{P}(A_1^*)$ of lines in A_1^*, namely $(\mathrm{Ann}_{A_1}(M_0))^\perp$: since $\dim(M_0) = \dim(M_1) = 1$ and M is generated by M_0, the kernel of the map $A_1 \to \mathrm{Hom}(M_0, M_1)$ is of codimension 1.

The relevance of this definition to our problem is that every point module is an irreducible object in Proj(A). This is easy to see, since a point module is 1-critical: if a submodule contains M_i then it contains M_j for all $j \geqslant i$ since M is cyclic and A is generated in degree 1, whence the quotient is finite dimensional. Thus it is natural to determine the point modules, as a first approximation to understanding Proj(A).

THEOREM 6.2. [21]. *The point modules for* $A(E, \tau)$ *are in bijection with*

$$E \cup \{e_0, e_1, e_2, e_3\} \subset \mathbb{P}(A_1^*) = \mathbb{P}(H^0(E, \mathscr{L})^*) \cong \mathbb{P}^3,$$

where E is identified with its image in $\mathbb{P}^3$ *obtained via the ample line bundle* $\mathscr{L}$, *and the four points* e_i *are the singular points of the four singular quadrics which contain E.*

Notation. Write $M(p)$ for the point module corresponding to the point $p \in E \cup \{e_0, \ldots, e_3\}$.

From now on we will always identify E with its image in $\mathbb{P}(A_1^*)$. The reader should be aware of the following well-known facts. Firstly, E is defined by two quadratic equations, and consequently there is a pencil of quadric surfaces containing E. Exactly four of these quadrics are singular, and they may be labelled Q_i in such a way that $e_i = \mathrm{Sing}(Q_i)$.

The generators for the algebra given by (2.1) have the property that $x_j(e_i) = \delta_{ij}$. Thus, $M(e_i) \cong A/Ax_j + Ax_k + Ax_l$ where $\{i, j, k, l\} = \{0, 1, 2, 3\}$. These are the 'obvious' point modules, and have as quotients all the one-dimensional simple A-modules (i.e. the simple modules which are the easiest to find).

Theorem 6.2 shows that E, which was used in defining $A(E, \tau)$, may be recovered from knowledge of the point modules. So can τ. If $p \in E$ then the submodule of $M(p)$ consisting of the degree $\geqslant 1$ part is generated by M_1, and has Hilbert series $t(1 - t)^{-1}$. Hence, if we shift degree it becomes a point module. It is isomorphic to $M(p - \tau)$. In other words, there is a short exact sequence

$$0 \to M(p - \tau)[-1] \to M(p) \to \mathbb{C} \to 0.$$

Hence, one recovers τ from the point module data. For a commutative ring, a nonzero submodule of a point module is isomorphic to (a shift of) the original point module. Hence, the noncommutativity of A is due to τ and can be recognized from the point modules. There is also a short exact sequence

$$0 \to M(e_i)[-1] \to M(e_i) \to \mathbb{C} \to 0.$$

From the penultimate short exact sequence, one sees that if τ is of finite order, n say, then there is a nonzero degree n map $\psi\colon M(p) \to M(p)$. It is not difficult to show that for each $0 \neq \lambda \in \mathbb{C}$, the cokernel of $1 - \lambda\psi \in \mathrm{End}_A(M(p))$ is a simple A-module of dimension n, and that these are the only finite-dimensional simple quotients of $M(p)$. In particular, if τ is not of finite order, then $M(p)$ has no nontrivial simple quotients. This confirms the earlier comment that an irreducible object in $\mathrm{Proj}(A)$ may have no finite-dimensional simple quotients. The details of these arguments appear in [13, Section 5].

Define

$$\Gamma \subset \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*) \cong \mathbb{P}^3 \times \mathbb{P}^3$$

by

$$\Gamma := \{(x, y) \mid \exists \text{ a s.e.s. of point modules } 0 \to M(x)[-1] \to M(y) \to \mathbb{C} \to 0\}.$$

Such a definition may be made for any graded algebra A. If the image of $f \in A_1 \otimes A_1$ in A_2 under the multiplication map is zero (i.e. if f is a quadratic relation of A) then f must vanish on Γ. For the algebras $A(E, \tau)$, Theorem 6.2 says that $\Gamma \supset \Delta_\tau$, the shifted diagonal occurring in our definition of $A(E, \tau)$. This is not surprising when

$A(E, \tau)$ is defined as in Definition 2.1, but is a surprise when $A(E, \tau)$ is defined as in Sklyanin's original paper [18], or when $A(E, \tau)$ is defined by generators and relations. This is one reason why the Odesskii–Feigin definition is to be preferred, but see also Section 12.5.

7. Fat Points

We remarked in Section 6 that for the commutative ring $R = \mathbb{C}[T_0, \dots, T_n]/J$, the point modules are all the irreducible objects in Proj(R). However, for a noncommutative algebra there may be other irreducible objects in Proj. This is analogous to the fact that simple modules over a noncommutative algebra need not be 1-dimensional (even if the base field is algebraically closed). The other irreducible objects of Proj are called *fat points*.

PROPOSITION 7.1 [4]. *Let* $A = A(E, \tau)$. *Every irreducible object in* Proj (A) *has a representative* $F \in \mathrm{Gr\,Mod}(A)$ *such that*

(1) $F = F_0 \oplus F_1 \oplus \dots;$
(2) *F is generated by* F_0;
(3) $\dim F_j$ *is a constant,* $\dim F_j = e(F)$, *the multiplicity of* F;
(4) *F has no nonzero finite dimensional submodule.*

The proof of this result uses the homological properties of $A(E, \tau)$ given in Theorem 3.2.

DEFINITION 7.2. If F has the properties in Proposition 7.1 and $e(F) > 1$ we call F a fat point module, and its isomorphism class in Proj(A) a fat point.

Hence, our original problem of finding the finite-dimensional simples reduces, after (5.2), (5.3) and (7.1), to the problem of finding the fat points, and their simple quotients.

8. Line Modules

Our search for the fat points proceeds by first understanding another important class of modules, the line modules, and obtaining the fat points as quotients of these.

DEFINITION 8.1. A graded A-module M is a line module if

(1) M is cyclic and
(2) $H_M(t) = (1 - t)^{-2}$.

For the commutative algebra $R = \mathbb{C}[T_0, \dots, T_n]/J$, the line modules are in bijection with those lines in $\mathbb{P}^n$ which lie in $\mathcal{V}(J)$. Even in the non-commutative case, each line module determines a line in $\mathbb{P}(A_1^*)$, namely $(\mathrm{Ann}_{A_1}(M_0))^{\perp}$.

THEOREM 8.2. [13] *The line modules for* $A(E, \tau)$ *are in bijection with those lines in* $\mathbb{P}(A_1^*)$ *which are secant lines to* E.

Notation. If $p, q \in E$ then the secant line through p and q is denoted $\overline{pq}$ and the corresponding line module is denoted $M(p, q)$. We also write $M(l)$ for the line module corresponding to a line l.

DEFINITION 8.3. If F is an irreducible object in $\text{Proj}(A(E, \tau))$, and l is a secant line we say that F lies on l, or that l contains F, if there is a non-zero (degree 0) map $M(l) \to F$. (Since F is irreducible in $\text{Proj}(A)$, the image of the map is isomorphic to F in $\text{Proj}(A)$.)

PROPOSITION 8.4 [22]. *Every fat point lies on a secant line.*

Definition 8.3 is consistent with the geometry, since a point module lies on a secant line if and only if the secant line passes through the point in question. For example, if $p, q \in E$ there is a short exact sequence

$$0 \to M(p + \tau, q - \tau)[-1] \to M(p, q) \to M(p) \to 0.$$

Similarly, if $\overline{pq}$ passes through one of the singular points e_i there is a short exact sequence

$$0 \to M(p - \tau, q - \tau)[-1] \to M(p, q) \to M(e_i) \to 0.$$

There are no other ways in which a point module can lie on a secant line. These results are proved in [13].

After Proposition 8.4 our search for fat points proceeds by asking which lines contain fat points. The next result reduces this to the problem of understanding the maps between line modules.

PROPOSITION 8.5 [22]. *Let C be an irreducible object in* $\text{Proj}(A)$ *of multiplicity e. By Proposition* 8.4 *we may assume it is a quotient of a line module, say $M(p, q)$. Then there exists another line module $M(l)$ and a short exact sequence*

$$0 \to M(l)[-e] \to M(p, q) \to C \to 0.$$

9. Central Elements and Analogies

Sklyanin [18] found two linearly independent element of degree 2 in $A(E, \tau)$, which we shall label Ω_1, Ω_2. The linear combinations of these elements are analogous to the defining equations of the quadrics containing E.

Two results confirm that this is a good analogy. The first is that $B := A/\langle \Omega_1, \Omega_2 \rangle$ has as its point modules precisely the $M(p)$ for $p \in E$, and so in this respect resembles the homogeneous coordinate ring $\oplus_{n \geqslant 0} H^0(E, \mathscr{L}^{\otimes n})$. Actually, this is a small part of a result of Artin and van den Bergh [6] which states that $\text{Proj}(B)$ is equivalent to the category of coherent $\mathcal{O}_E$-modules.

The second piece of evidence for the reasonableness of the analogy comes from finding quadratic elements in $A(E, \tau)$ which annihilate line modules. The pencil of quadrics containing E may be labelled $Q(z)$ $(z \in E)$ in such a way that $Q(z) = Q(-z)$

and

$$Q(z) = \bigcup_{p+q=z} \overline{pq}.$$

(Of course, this depends on placing the identity at a particular place on E: the identity is placed so that four points of $E \subset \mathbb{P}^3$ are coplanar if and only if their sum is zero.) Thus, the next result is an analogue of the tautology that the defining equation of $Q(z)$ vanishes on the secant lines $\overline{pq}$ for which $p + q = \pm z$ and on no others.

THEOREM 9.1 [13]. *If $z \in E$ then there exists $0 \neq \Omega(z) \in \mathbb{C}\Omega_1 + \mathbb{C}\Omega_2$ (unique up to scalar multiple) such that*

$$\Omega(z) \cdot M(p, q) = 0 \Leftrightarrow p + q = z \quad \text{or} \quad p + q = -z - 2\tau.$$

The only equalities are $\Omega(z) = \Omega(-z - 2\tau)$.

The fact that $\Omega(z) = \Omega(-z - 2\tau)$ is another manifestation of the noncommutativity: the central element $\Omega(z)$ annihilates line modules corresponding to lines lying on two different quadrics, namely $Q(z)$ and $Q(-z - 2\tau)$ (one ruling from each quadric). Consideration of annihilators of line modules allows one to sharpen Proposition 8.5 as follows.

PROPOSITION 9.2 [22]. *Let C be an irreducible object in* Proj(A) *of multiplicity $e > 1$, and suppose C is a quotient of $M(p, q)$. Then*

(1) *there is a short exact sequence*

$$0 \to M(p - e\tau, q - e\tau)[-e] \to M(p, q) \to C \to 0$$

(2) *if $2e\tau \neq 0$ then $p + q = \omega + (e - 1)\tau$ for some $\omega \in E_2$.*

We will not discuss how one determines when there exist nonzero maps between line modules, but after that is done one may classify all the fat points. The next two sections give the results obtained.

10. When τ is of Infinite Order

The next result solves our original problem when τ is not of finite order.

THEOREM 10.1 [22]. *Suppose that τ is of infinite order. Let E_2 denote the 2-torsion subgroup of E.*

(1) *The fat points may be labelled as $\{F(\omega + k\tau) \mid \omega \in E_2, 0 \leqslant k \in \mathbb{Z}\}$.*
(2) *$F(\omega + k\tau)$ lies on the secant line $\overline{pq}$ if and only if $p + q = \omega + k\tau$.*
(3) *$e(F(\omega + k\tau)) = k + 1$.*
(4) *For each $\omega \in E_2$, $F(\omega) \cong M(e_i)$ where $e_i = \operatorname{Sing} Q(\omega)$.*
(5) *$F(\omega + k\tau)$ has a 1-parameter family of $(k + 1)$-dimensional simple quotients (cf. Proposition 5.3).*
(6) *These are all the finite dimensional simple $A(E, \tau)$-modules.*

Thus, for each $d \in \mathbb{N}$, there are four 1-parameter families of d-dimensional simple $A(E, \tau)$-modules, one for each element of E_2. All the simples within a particular 1-parameter family are equivalent in the sense of Proposition 5.3, and there are no other equivalences.

This result tells us that there is a similarity between the representation theory of $A(E, \tau)$ and that of both $\mathrm{U}(\mathfrak{gl}(2))$, the enveloping algebra of $\mathfrak{gl}(2)$, and $\mathrm{U}_q(\mathfrak{sl}(2))$, the quantized enveloping algebra of $\mathfrak{sl}(2)$. Recall that for each $d \in \mathbb{N}$, there is a 1-parameter family of simple d-dimensional $\mathfrak{gl}(2)$-modules, and if q is not a root of unity, then for each d there are precisely 4 simple $\mathrm{U}_q(\mathfrak{sl}(2))$-modules of dimension d. It is not inappropriate to think of the line modules $M(p, q)$ as analogous to Verma modules, and if we write $V(\omega + k\tau)$ for a suitable simple quotient of $F(\omega + k\tau)$ then $\omega + k\tau$ is like a 'highest weight'. This analogy is made clearer in [14].

As we will now explain, the similarity to $\mathrm{U}(\mathfrak{gl}(2))$ and $\mathrm{U}_q(\mathfrak{sl}(2))$ is not too surprising. Recall that when $A(E, \tau)$ was defined in Section 2, we required $4\tau \neq 0$. However, let's look at what happens when τ is 2-torsion. If $\tau = \frac{1}{2}(1 + \eta)$ then $\alpha_i = 0$ for all i; to see this one needs the description of the α_i in Section 12.4. From Equation (2.1) one sees that x_0 is now central, and that $A/\langle x_0 - 1\rangle \cong \mathrm{U}(\mathfrak{so}(3))$, and since our base field is $\mathbb{C}$, this is isomorphic to $\mathrm{U}(\mathfrak{sl}(2))$. The representation theory of $A(E, \frac{1}{2}(1 + \eta))$ is governed by that of $\mathrm{U}(\mathfrak{sl}(2))$ – for precise details see [14]. Hence, we may think of $A(E, \frac{1}{2}(1 + \eta))$ as the homogenization of $\mathrm{U}(\mathfrak{sl}(2))$, or as a graded version of $\mathrm{U}(\mathfrak{gl}(2))$ which is also of Gelfand–Kirillov dimension 4.

Now suppose that $\tau \in E_2$ but $\tau \neq \frac{1}{2}(1 + \eta)$. In this case exactly one $\alpha_i = 0$, and it is not difficult to see that there are elements $K_+, K_- \in A_1$ such that K_+K_- is central. Furthermore $A/\langle K_+K_- - 1\rangle \cong \mathrm{U}_q(\mathfrak{sl}(2))$ for a suitable q. The details of these statements are in [21, Section 1]. As one varies E so does q change, and every $\mathrm{U}_q(\mathfrak{sl}(2))$ occurs as a quotient of one of the algebras defined by relations (2.1).

Thus it is not inappropriate to think of $A(E, \tau)$ for a general τ as a 'deformation' of $\mathrm{U}(\mathfrak{gl}(2))$ or of $\mathrm{U}_q(\mathfrak{sl}(2))$.

As the next section will show there is a distinct difference between the cases when τ is of finite and infinite order. This is due to the fact that the center is 'small' when τ is of infinite order, and 'large' when τ is of finite order. In particular, compare the next two results.

THEOREM 10.2. [13]. *When τ is of infinite order, the center of $A(E, \tau)$ is $\mathbb{C}[\Omega_1, \Omega_2]$.*

11. When τ is of Finite Order

Write n for the order of τ and s for the order of 2τ.

THEOREM 11.1 [20] [23]. *If τ is of finite order, then $A(E, \tau)$ is a finite module over its center.*

THEOREM 11.2 [20]. *Suppose that τ is of finite order n, and let s be the order of 2τ.*

(1) *$A(E, \tau)$ satisfies a polynomial identity of degree 2n.*
(2) *If $g = \Pi_{\omega,k}\Omega(\omega + k\tau)$ where the product is taken over all $\omega \in E_2$ and all $0 \leqslant k < s - 1$ then $A[g^{-1}]_0$ is an Azumaya algebra of rank s^2 over its center.*
(3) *$A[g^{-1}]$ is Azumaya of rank n^2 over its center.*

THEOREM 11.3 [20]. *Suppose that τ is of finite order, and let s be the order of 2τ.*

(1) *All fat points are of multiplicity $\leqslant s$.*
(2) *The fat points of multiplicity s are parametrized by a rational 3-fold.*
(3) *The only fat points of multiplicity $<s$ are the fat points $\{F(\omega + k\tau) \mid \omega \in E_2, 0 < k < s - 1\}$ described in Theorem 10.1.*

THEOREM 11.4 [20]. *Suppose that τ is of finite order n, and let s be the order of 2τ.*

(1) *If F is a fat point of multiplicity s, then F has a 1-parameter family of simple quotients of dimension n (an equivalence class in the sense of 5.3).*
(2) *Each $M(e_i)$ has an equivalence class of 1-dimensional simple quotients.*
(3) *If $p \in E$ then $M(p)$ has an equivalence class of n-dimensional simple quotients, and no others.*
(4) *If $F = F(\omega + k\tau)$ for some $\omega \in E_2$ and $0 < k < s - 1$ as in Theorem 10.1, then F has an equivalence class of $(k + 1)$-dimensional simple quotients.*

Theorem 11.2.2 is best understood in the following context. Let S denote the projective scheme determined by $Z(A)$ the center of A. Then we may construct a sheaf $\mathscr{A}$ of $\mathcal{O}_S$-algebras as follows. For each homogeneous $0 \neq f \in Z(A)$ the sections of $\mathscr{A}$ over $S_{(f)} = \mathrm{Spec}(Z(A)[f^{-1}]_0)$ is $A[f^{-1}]_0$. Because A is a finite $Z(A)$-module, $\mathscr{A}$ is a coherent $\mathcal{O}_S$-module. Hence, (11.2.2) describes a dense open set over which $\mathscr{A}$ is Azumaya.

By Theorem 11.2.1, or by Theorem 11.3.2 and Theorem 11.4.1, there is a four-dimensional family of n-dimensional simple A-modules, and no simple modules of dimension $>n$. To give a more precise classification of these we must classify the fat points of multiplicity s. This is done as follows.

First, since a fat point lies on some line it is annihilated by some $\Omega(z)$, so we may fix z and parametrize the fat points killed by each $\Omega(z)$. The classification depends on whether or not $\Omega(z)$ divides the element g defined in 11.2.2. For this discussion, let's suppose that $\Omega(z)$ does not divide g.

The fat points will be classified in terms of which lines they lie on. Let $L(z)$ denote the set of secant lines $\{\overline{pq} \mid p + q = z\}$. We say that two secant lines are *equivalent* [1] if they contain infinitely many fat points in common. In turns out that (for this choice of z) the equivalence class of $\overline{pq} \in L(z)$ is $[\overline{pq}] = \{\overline{p + 2i\tau, q - 2i\tau} \mid i \in \mathbb{Z}\}$. If F is a fat point killed by $\Omega(z)$ then F lies on a line in $l_1 \in L(z)$ and on a line $l_2 \in L(-z - 2\tau)$, and l_1 and l_2 are unique up to equivalence. Conversely given lines $l_1 \in L(z)$ and $l_2 \in L(-z-2\tau)$ there is a fat point lying on both these. Hence, the fat points killed by $\Omega(z)$ are in bijection with pairs $([l_1], [l_2])$.

If $\sigma: E \to E$ is defined by $\sigma(p) = p + \tau$, then $(\sigma^2)^*\mathscr{L}_s \cong \mathscr{L}_s$ where $\mathscr{L}_s = \mathscr{L} \otimes \sigma^*\mathscr{L} \otimes \ldots \otimes (\sigma^*)^{s-1}\mathscr{L}$. We define $\mathscr{L}'$ to be the descent of $\mathscr{L}_s$ to $E' = E/\langle 2\tau \rangle$ (see [23]), and identify E' with its image in $\mathbb{P}' = \mathbb{P}(H^0(E', \mathscr{L}')^*) \cong \mathbb{P}^3$ under the obvious embedding. Again there is a pencil of quadrics containing E' and if z' denotes the image of z in E', then the hypothesis on $\Omega(z)$ ensures that

$$Q(z') = \bigcup \{\overline{p'q'} \mid p' + q' = \pm z'\}$$

is a smooth quadric containing E'. Let $L(z')$ and $L(-z')$ denote the two rulings on $Q(z')$. Hence, the equivalence classes of lines in $L(z)$ are in bijection with the lines in $L(z')$. Similarly, the equivalence classes of lines in $L(-z-2\tau)$ are in bijection with the lines in $L(-z')$.

It is now possible to establish a bijection between the fat points killed by $\Omega(z)$ and the points of $Q(z')\backslash E'$ as follows. First observe that there is a bijection between points of $Q(z')$ and points of $L(z') \times L(-z')$: a point corresponds to the line pair which intersects at that point. Hence, if F is a fat point killed by $\Omega(z)$, lying on $l_1 \in L(z)$ and on $l_2 \in L(-z-2\tau)$, then F corresponds to the point $[l_1] \cap [l_2] \in Q(z')$.

12. Further Remarks

These final remarks add some detail to issues which were treated rather cursorily in the main text.

(1) An A-module M is called a *Cohen–Macaulay* module if $\operatorname{Ext}^i_A(M, A) = 0$ whenever $i \neq j(M)$ (see Theorem 3.2 for the definition of $j(M)$). By [13], the point modules and line modules are precisely the Cohen–Macaulay modules of GK dimension 1 and 2 respectively which are of multiplicity 1. Thus these are precisely the modules with the nicest homological properties. We remind the reader that over the commutative polynomial ring $R = k[T_0, \ldots, T_n]$ the Cohen–Macaulay modules of multiplicity 1 are precisely the modules $R/\langle y_1, \ldots, y_m \rangle$ where the y_j are linear forms, i.e. such modules are in bijection with the linear subspaces of $\mathbb{P}^n$.

(2) Generalizing Definitions 7.2 and 8.1, one may also define *plane modules* for $A(E, \tau)$. By [13] these are exactly the Cohen–Macaulay modules of GK dimension 3 and multiplicity 1, and they are in bijection with the planes in $\mathbb{P}^3$. We call any cyclic graded module having the same Hilbert series as a polynomial ring, a *linear module*. It can be shown that if L is a linear $A(E, \tau)$-module, then $L \cong A/AW$ where $W \subset A_1$ is the space of linear forms vanishing on L. This is easy to prove for plane modules once one knows the Hilbert series of A, and that A is a domain. It is also fairly easy to prove for point modules. However, the only proof I know for line modules uses the fact that A has the Cohen–Macaulay property and satisfies the Auslander condition. This is (to me) somewhat unsatisfying.

(3) The two proofs of Theorem 11.1 in [20] and [23], showing that A is finite over its center, are completely different. The proof in [23] (which is modelled on that in [3]) is direct, first determining the center of $B := A/\langle \Omega_1, \Omega_2 \rangle$ (which mainly involves

geometric arguments), and then pulling the result up to A using the fact that Ω_1, Ω_2 is a regular sequence. The proof in [20] is indirect. It first shows there are enough finite dimensional simple A-modules of bounded dimension to separate elements of A, whence A satisfies a polynomial identity. Thus $A[z^{-1}]$ is finite over its center for some central $z \in A$. However, since A has good homological properties, one may apply a result of Stafford [24] to conclude that A is a maximal order, and hence finite over its center.

(4) The coefficients α_i in the defining equations (2.1) are

$$\alpha_1 = \left(\frac{\theta_{11}(\tau)\theta_{00}(\tau)}{\theta_{01}(\tau)\theta_{10}(\tau)}\right)^2,$$

$$\alpha_2 = -\left(\frac{\theta_{11}(\tau)\theta_{01}(\tau)}{\theta_{00}(\tau)\theta_{10}(\tau)}\right)^2,$$

$$\alpha_3 = -\left(\frac{\theta_{11}(\tau)\theta_{10}(\tau)}{\theta_{00}(\tau)\theta_{01}(\tau)}\right)^2$$

where the θ_{ab} are Jacobi's four theta functions with respect to $\mathbb{Z} \oplus \mathbb{Z}\eta$ having zeroes at

$$\left(\frac{1+a}{2}\right)\eta + \left(\frac{1+b}{2}\right).$$

(5) In [13] there is another geometric definition of the defining relations of $A(E, \tau)$: the space of relations for $A(E, \tau)$ is precisely the subspace of $V \otimes V$ consisting of those f which vanish on Δ_τ and on all (e_i, e_i). Conversely, the subvariety of $\mathbb{P}^3 \times \mathbb{P}^3$ cut out by the defining relations is $\Delta_\tau \cup \{(e_i, e_i)\}$. Since this is the graph of an automorphism of $E \cup \{e_i\}$, it follows from [4] that the point modules are in bijection with the points of $E \cup \{e_i\}$. We also remark that $\Delta_\tau \cup \{(e_i, e_i)\}$ is the variety Γ defined at the end of Section 6.

13. A Brief History

Around 1985–1986 M. Artin and W. Schelter [2] classified all three-dimensional regular graded algebras (the definition of 'regular' is not the usual commutative one – the only commutative algebra which belongs to their class is the polynomial ring). The classification includes a classification of all graded algebras $A = k[x, y, z]$, generated by 3 degree 1 elements, such that

(a) A has the same Hilbert series as the commutative polynomial ring,
(b) every A-module has a projective resolution of length 3,
(c) $Ext^i_A(k, A) = 0$, if $i < 3$ and, k if $i = 3$.

They prove that the algebras satisfying (a)–(c) fall into 6 classes, the most interesting of which is the one they labelled 'Type A'. The algebras in this class are now called three-dimensional Sklyanin algebras since they resemble the four-dimensional Sklyanin algebras. We will briefly discuss the Type A algebras.

Although Artin and Schelter were able to show that the generic Type A algebra is regular (in their sense), they were unable to determine precisely which Type A algebras were regular. This problem was solved by Artin, Tate and van den Bergh [4]. Among the new methods which they introduced was the notion of a point module. The point modules for a Type A algebra are parametrized by an elliptic curve, E say, lying in $\mathbb{P}^2 = \mathbb{P}(A_1^*)$. Shifting the grading on a point module corresponds to translation by a point $\tau \in E$ (cf. Section 6). The algebra A can be defined in terms of the data (E, τ) as follows. Let R_A denote the subspace of $A_1 \otimes A_1$ consisting of those bilinear forms which vanish at all points $(x, x + \tau), x \in E$. Then $A \cong T(A_1)/\langle R_A \rangle$. In [5] the notion of a line module is also introduced and it is shown that the line modules are in bijection with the lines in $\mathbb{P}^2$. Furthermore, a point module is a quotient of a given line module if and only if the corresponding point lies in the intersection of E with the corresponding line.

One of the main steps in understanding A is the passage to the ring $B := A/\langle g \rangle$, where g is a homogeneous cubic central element, first found by Schelter's program 'Affine'. The algebra B is a 'twisted homogeneous coordinate ring' of E; such algebras are studied by Artin and van den Bergh [6], and it follows from their work that the category Proj(B) is equivalent to the category of coherent $\mathcal{O}_E$-modules. Thus, B is well understood.

The structure of A depends delicately on whether or not τ is of finite order. If τ is of infinite order, then the center of A is $k[g]$, whereas A is a finite module over its center when τ is of finite order. The latter is the more complicated case, since then A must have a three-dimensional family of finite-dimensional simple modules, and it is a natural problem to understand these. One approach to doing this is to construct from A a sheaf $\mathscr{A}$ of $\mathcal{O}_S$-algebras where S is the projective scheme determined by the center of A: if $0 \neq u$ is a central element of A then

$$\mathcal{O}_S(S_{(u)}) = Z(A)[u^{-1}]_0 \quad \text{and} \quad \mathscr{A}(S_{(u)}) = A[u^{-1}]_0.$$

The center $\mathscr{Z}$ of A is defined by $\mathscr{Z}(S_{(u)}) = Z(S_{(u)}))$ and the scheme Spec $\mathscr{Z}$ is defined in the usual way. In [1] Artin shows that Spec $\mathscr{Z} \cong \mathbb{P}^2$ if the order of τ is not divisible by 3; it is not always the case that $S \cong \mathbb{P}^2$. For more complete information on the center see [23]. In studying $\mathscr{A}$, Artin introduced the notion of a 'fat point', and it plays a central role in his proof that Spec $\mathscr{Z} \cong \mathbb{P}^2$.

The other strand of our story begins in 1982–1983 with two papers of E. K. Sklyanin [18, 19]. In these papers, Sklyanin defined and began a study of the algebras $A(E, \tau)$. He defined $A(E, \tau)$ in terms of Baxter's solution to the Yang–Baxter equation [7]. Baxter's solution is given in terms of certain theta functions on an elliptic curve E, and the coefficients in the defining relations of Sklyanin's algebras are also given in terms of theta functions evaluated at a certain point τ of E. Sklyanin was interested in finding all the finite-dimensional irreducible representations of

$A(E, \tau)$, and although he found many finite-dimensional representations he was unable to show they were simple, and did not know whether he had found them all. Despite Sklyanin's work, little was known about the structure of $A(E, \tau)$. For example, Sklyanin conjectured that the Hilbert series of $A(E, \tau)$ is the same as that of the polynomial ring in 4 variables. In 1989, J. T. Stafford and the present author were able to verify this conjecture [21]. The methods used to verify the conjecture were those introduced by Artin, Tate, and Van den Bergh. In particular, the proof proceeds by finding the point modules for $A(E, \tau)$, and passing to the ring $B := A/\langle\Omega_1, \Omega_2\rangle$ where Ω_1, Ω_2 are central elements found by Sklyanin. As for the Type A algebras, B is a twisted homogeneous coordinate ring of E, so good properties of B can be pulled back to A.

In a separate development, around 1989, Odesskii and Feigin [15, 16], gave a more geometric definition of Sklyanin's algebra which did not refer to the Yang–Baxter equation. In doing so they were able to construct a much larger class of algebras (on n generators with $\binom{n}{2}$ relations), depending on an elliptic curve E and a point $\tau \in E$, which included both the Artin–Schelter Type A algebras and the algebras of Sklyanin. They also showed that these algebras have point modules parametrized by E (and perhaps some other points) and shifting degree corresponds to translation on E. They also introduce the notion of line modules, plane modules etc., and show how these are related to the geometry of E embedded as a degree n curve in $\mathbb{P}^{n-1}$. These two papers are full of fascinating observations and working out the details of the structure of these higher dimensional Sklyanin algebras will be an interesting project.

In a different direction, I. Cherednik [10, 11] used Belavin's [8, 9] generalization of Baxter's solution to the Yang–Baxter equation, to construct generalizations of Sklyanin's algebras. Cherednik constructs a family of graded algebras on n^2 generators (again depending on an elliptic curve E and a point τ). Just as Sklyanin's algebras ($n = 2$) can be seen as deformations of $U(\mathfrak{gl}(2))$ (see Section 10), so are Cherednik's algebras deformations of $U(\mathfrak{gl}(n))$ in an analogous way. Our understanding of Cherednik's algebras is rather poor.

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