THE SIMPLE $\mathcal{D}$-MODULE ASSOCIATED TO THE INTERSECTION HOMOLOGY COMPLEX FOR A CLASS OF PLANE CURVES

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Communicated by C.A. Weibel
Received 1 March 1986

Let $X$ be an irreducible plane algebraic curve over an algebraically closed field $k$ of characteristic zero. Suppose that $X$ is analytically irreducible at all points. Let $\mathcal{D}(\mathbb{A}^2)$ be the ring of differential operators on $\mathbb{A}^2$. This paper gives a direct algebraic proof that $\mathcal{O}(\mathbb{A}^2 \setminus X)/\mathcal{O}(\mathbb{A}^2)$ is a simple $\mathcal{D}(\mathbb{A}^2)$-module. This may also be proved via the Riemann–Hilbert correspondence.

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero, $Y$ a non-singular affine algebraic variety and $X \subseteq Y$ a hypersurface defined by an irreducible $f \in \mathcal{O}(Y)$. Write $\mathcal{D} = \mathcal{D}(Y)$ for the ring of differential operators on $Y$, and consider $\mathcal{O}(Y \setminus X) = \mathcal{O}(Y)$ as a left $\mathcal{D}$-module. By Bernstein [1] and Kashiwara [6] it is of finite length as a $\mathcal{D}$-module. It is not difficult to show that $\mathcal{O}(Y \setminus X)/\mathcal{O}(Y)$ contains a unique simple $\mathcal{D}$-submodule – this is denoted $\mathcal{L}(X, Y)$. Its existence is proved in greater generality in [2]. The problem is to determine $\mathcal{L}(X, Y)$.

We prove the following:

Theorem. Let $X \subseteq \mathbb{A}^2$ be an irreducible curve. Let $\tilde{X}$ denote the normalisation of $X$, and suppose that $\pi: \tilde{X} \to X$, the natural projection, is injective. Then $\mathcal{L}(X, \mathbb{A}^2) = \mathcal{O}(\mathbb{A}^2 \setminus X)/\mathcal{O}(\mathbb{A}^2)$.

This theorem is known to the $\mathcal{D}$-modules experts. It was pointed out to the author, by J.-L. Brylinski and the referee, that it is a consequence of the Riemann–Hilbert correspondence. However the present paper gives the first purely algebraic proof of this result.

One motivation for wanting to describe $\mathcal{L}(X, Y)$ is that the de Rham complex satisfies $\text{DR}(\mathcal{L}(X, Y)) = \text{IC}_{Y}[-1]$, the Goresky–MacPherson intersection homology complex for the middle perversity [4].

There are a number of results describing a generator for $\mathcal{L}(X, Y)$; see for

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example [2] and [9]. In the case of an arbitrary irreducible curve \(X \subset \mathbb{A}^2\) defined by \(f\), it is remarked in [2, Introduction] that \(\mathcal{L}(X, Y)\) is generated by \(f^{-1}df/dx\) when \(\mathcal{O}(\mathbb{A}^2) = k[x, y]\). Of course, this does not make it possible to decide if a given element of \(\mathcal{O}(\mathbb{A}^2)/\mathcal{O}(\mathbb{A}^2)\) belongs to \(\mathcal{L}(X, \mathbb{A}^2)\).

The theorem above extends the well-known fact that if \(X \subset \mathbb{A}^2\) is non-singular, then \(\mathcal{O}(\mathbb{A}^2 \setminus X)/\mathcal{O}(\mathbb{A}^2)\) is a simple \(\mathcal{D}(\mathbb{A}^2)\)-module. Recall that the usual proof of this fact, if \(X\) is defined by \(f \in \mathcal{O}(\mathbb{A}^2)\), then the ideal of \(\mathcal{O}(\mathbb{A}^2)\) generated by \(df/dx\), \(df/dy\) and \(f\) equals \(\mathcal{O}(\mathbb{A}^2)\). Of course such a result is not available to us when \(X\) is singular.

The idea of the proof is to extend some of the work in [7]. Let \(X\) be any curve, \(\tilde{X}\) its normalisation, and let \(\pi: \tilde{X} \rightarrow X\) the natural projection. In [7, §3] it is shown that if \(\pi\) is injective, then \(\mathcal{O}(X)\) is a simple \(\mathcal{D}(X)\)-module and furthermore, that \(\mathcal{D}(X)\) is Morita equivalent to \(\mathcal{D}(\tilde{X})\). In Section 2 we extend this result as follows. Set \(R = k[x, y] = \mathcal{O}(\mathbb{A}^2)\), let \(f \in R\) define the curve \(X\), and suppose that \(\pi: \tilde{X} \rightarrow X\) is injective; then for all \(n \in \mathbb{N}\), \(R/f^nR\) is a simple \(\mathcal{D}(R/f^nR)\)-module (Corollary 2.9). Recall that as in [7, §1], \(\mathcal{D}(A)\) is defined for any commutative \(k\)-algebra \(A\). Having established this, the main theorem is a relatively straightforward consequence. The proof is given in Section 3; by reading Section 3 first the reader will understand the necessity of Section 2.

Let us recall some of the notation of [7]. Let \(A \subset C\) be commutative \(k\)-algebras. If \(D \in \mathcal{D}(C)\) and \(f \in C\), write \(D * f := D(f)\) for the evaluation of the differential operator \(D\) on \(f\). Define \(\mathcal{D}(C, A) := \{D \in \mathcal{D}(C) \mid D * f \in A\ \text{for all} \ f \in C\}\), and \(\mathcal{D}(C, A)^* \subset C = \{D * f \mid D \in \mathcal{D}(C, A)\ \text{and} \ f \in C\}\).

2. Differential operators on \(k[x, y]/(f^n)\)

Set \(R = k[x, y] = \mathcal{O}(\mathbb{A}^2)\), let \(f \in R\) be irreducible, and let \(X \subset \mathbb{A}^2\) denote the curve defined by \(f\). Set \(A = R/f^nR\).

The goal of this section is to show that there are inclusions of \(k\)-algebras

\[A \subset \mathcal{O}(X) \otimes k[z]/(z^n) \subset \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n) \subset \text{Fract} A\]

and that \(A\) is of finite codimension in \(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)\). When the projection \(\pi: \tilde{X} \rightarrow X\) from the normalisation is injective, the arguments of [7, Propositions 3.3, 3.4] may be adapted to show that \(\mathcal{D}(A)\) is Morita equivalent to \(\mathcal{D}(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n))\). But the latter is isomorphic to \(\mathcal{D}(\tilde{X}) \otimes_k M_\ast(k)\), where \(M_\ast(k)\) is the ring of \(n \times n\) matrices over \(k\). Hence \(\mathcal{D}(A)\) is Morita equivalent to \(\mathcal{D}(\tilde{X})\), and in particular is a simple ring. It follows that \(A\) is a simple \(\mathcal{D}(A)\)- module, and this is the result which is carried forward to Section 3, and used there to establish the main theorem of the paper.

Proposition 2.1. Let \(R = k[t_1, \ldots, t_d] = \mathcal{O}(\mathbb{A}^d)\), and suppose that \(f \in R\) is irreducible, defining a hypersurface \(X\). Let \(n \in \mathbb{N}\). Then there is a \(k\)-algebra homomorphism

\[\phi: R \rightarrow \mathcal{D}(\mathcal{O}(\mathbb{A}^d))/\mathcal{O}(\mathbb{A}^d)\].

In particular, if \(\pi: \tilde{X} \rightarrow X\) then \(\pi^\ast\phi: \mathcal{O}(X) \otimes k[z]/(z^n) \rightarrow \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)

is a morphism of \(\mathcal{D}(\mathcal{O}(\tilde{X}))\)-modules, which we denote by \(\phi\).
such that ker $\theta = f^n R$.

Proof. Since $k = \cap_{i=1}^d k[t_1, \ldots, \hat{t}_j, \ldots, t_d]$ (where $\hat{t}_j$ means 'omit $t_j$') we may assume, without loss of generality, that $f \not\in k[t_2, \ldots, t_d]$.

Let $\psi : R \to \mathcal{O}(X)$ be the natural map with kernel $f R$. Define $\theta$ by

$$\theta(t_1) = \psi(t_1) + z, \quad \theta(t_i) = \psi(t_i) \quad (2 \leq i \leq d).$$

and extend $\theta$ to a $k$-algebra homomorphism. By Taylor's theorem it follows, for $g \in R$, that

$$\theta(g) = g(\psi(t_1) + z, \psi(t_2), \ldots, \psi(t_d))$$

$$= \sum_{j=0}^{n-1} \frac{1}{j!} \psi\left( \frac{\partial^j g}{\partial t_1^j} \right) z^j.$$

Hence, ker $\theta = \{ g \in R \mid \partial^j g / \partial t_1^j \in f R \text{ for all } 0 \leq j < n \}$.

Sublemma. If $\partial^j g / \partial t_1^j \in f R$ for all $0 \leq j < n$, then $g \in f^n R$.

Proof. By induction on $n$. The sublemma is true if $n = 1$. Assuming that the sublemma holds for $n - 1$, we may suppose that $g \in f^{n-1} R$. Since $\partial^j / \partial t_1^j (\partial g / \partial t_1) \in f R$ for all $0 \leq j < n - 1$, the induction hypothesis ensures that $\partial g / \partial t_1 \in f^{n-1} R$.

Write $g = f^{n-1} h$. Then

$$\frac{\partial g}{\partial t_1} = (n-1)f^{n-2} h \frac{\partial f}{\partial t_1} + f^{n-1} \frac{\partial h}{\partial t_1} \in f^{n-1} R.$$

Hence $h \partial f / \partial t_1 \in f R$. By assumption, deg$_n(f) \geq 1$, and so (by a degree argument) it follows that $\partial f / \partial t_1 \not\in f R$. Hence, as $f R$ is a prime ideal, $h \in f R$. Thus $g \in f^n R$ as required. \qed

We now turn to the case we are interested in, namely $d = 2$. Write $A = R / f^n R$, and $A' = \mathcal{O}(X) \otimes_k k[z] / (z^n)$.

Lemma 2.2. Let $R = k[x, y] = \mathcal{O}(\mathbb{A}^2)$. Let $0 \neq f \in R$ be irreducible defining a curve $X \subseteq \mathbb{A}^2$. Let $n \in \mathbb{N}$, and let

$$\theta : R / f^n R \to \mathcal{O}(X) \otimes_k k[z] / (z^n)$$

be the injective algebra homomorphism obtained in Proposition 2.1. Then
(a) \( \theta(R/f^nR) \) is of finite codimension (as a \( k \)-vector space) in
\[ \mathcal{O}(X) \otimes_k k[z]/(z^n); \]

(b) We have inclusions
\[ R/f^nR \subseteq \mathcal{O}(X) \otimes_k k[z]/(z^n) \subseteq \text{Fract}(R/f^nR) \]

where \text{Fract}( ) denotes the ring of fractions.

**Proof.** (a) Write \( V = R/f^nR, U = \mathcal{O}(X) \otimes_k k[z]/(z^n) \). There are filtrations of these vector spaces
\[ V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0, \]
and
\[ U = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = 0 \]
where \( V_i = A f^i \) are \( U_i = A' z^i \). The injection \( \theta: V \to U \) satisfies \( \theta(V_i) \subseteq U_i \) and \( \theta(V) \cap U_i = \theta(V_i) \). Hence if we consider the short exact sequences below, and the injections induced by \( \theta \), we obtain an injection \( \gamma: V_i/V_{i+1} \to U_i/U_{i+1} \) making the following diagram commute:

\[
\begin{array}{ccc}
0 & \to & V_i/V_{i+1} \\
\downarrow & & \downarrow \\
0 & \to & U_i/U_{i+1}
\end{array}
\]

The aim is to show that \( \dim_k(U/\theta(V)) < \infty \), and the idea is to show by induction on \( i \) that \( \dim_k(U_i/\theta(V)) < \infty \). It is clear that if \( \dim_k(U_{i+1}/\theta(V_{i+1})) < \infty \) and \( \dim_k((U_i/U_{i+1})/\gamma(V_i/V_{i+1})) < \infty \), then \( \dim_k(U_i/\theta(V_i)) < \infty \). Hence one need only show that \( \gamma(V_i/V_{i+1}) \) is of finite codimension in \( U_i/U_{i+1} \); note that as \( V_n = 0 \) and \( U_n = 0 \) this will also start the induction process.

Note that \( U_i/U_{i+1} \) and \( V_i/V_{i+1} \) are both free \( \mathcal{O}(X) \)-modules of rank 1. However, \( \gamma \) is an \( \mathcal{O}(X) \)-module homomorphism because \( \theta(f' r) = \theta(f') \theta(r) \), and \( \theta(r) = \psi(r) \in A' z^i \); hence \( \theta(f' r) = \theta(f') \psi(r) \pmod{U_{i+1}} \). As \( \gamma \neq 0 \), we have that \( \gamma(V_i/V_{i+1}) \) is a nonzero \( \mathcal{O}(X) \)-submodule of the cyclic \( \mathcal{O}(X) \)-module \( U_i/U_{i+1} \). Hence, \( \gamma(V_i/V_{i+1}) \) is necessarily of finite codimension in \( U_i/U_{i+1} \). This establishes (a).

(b) We now consider \( A \subseteq A' \). The regular elements of \( A \) are precisely those elements not in \( fA \). There exists \( \xi \in A \) which is transcendental over \( k \), and \( k[\xi] \cap fA = 0 \). Hence, \( k(\xi) \subseteq \text{Fract} A \). Since \( A'/A \) is a finite-dimensional vector space, there exists \( 0 \neq p(\xi) \in k[\xi] \) such that \( A'p(\xi) \subseteq A \). As \( p(\xi) \) is a unit in \( \text{Fract} A \), we have \( A' \subseteq \text{Fract} A \). This completes the proof of (b). \( \square \)

**Corollary 2.3.** There are inclusions of \( k \)-algebras
\[ R/f^nR \subseteq \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n) \subseteq \text{Fract}(R/f^nR) \]
and \( R/f^nR \) is of finite codimension in \( \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n) \). The map
\[ \text{Spec}(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)) \to \text{Spec}(R/f^nR) \] is the natural projection \( \pi: \tilde{X} \to X \).
**Proof.** The inclusion of the $k$-algebras is obtained from the previous lemma together with the natural inclusion $\mathcal{O}(X) \subseteq \mathcal{O}(\tilde{X})$. The fact that $R/f^nR$ is of finite codimension, follows from the previous lemma, and the fact that $\mathcal{O}(X)$ is of finite codimension in $\mathcal{O}(\tilde{X})$. The fact that $\theta : R/f^nR \to \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)$ induces an isomorphism after factoring out both these algebras by their nilpotent radicals ensures that the map on the spectra is as claimed. □

**Proposition 2.4.** $\mathcal{D}(k[z]/(z^n)) \cong M_n(k)$, the ring of $n \times n$ matrices over $k$.

**Proof.** Write $T = k[z]/(z^n)$ and set $J_T = \ker(\mu_T : T \otimes_k T \to T)$, where $\mu_T$ is the multiplication map. Then $J_T$ is generated by $1 \otimes z - z \otimes 1$; hence $J_T^n = 0$. Now

$$\mathcal{D}(T) = \lim_{\rightarrow m} \text{Hom}_T(T \otimes_k T/J_T^n, T) = \text{Hom}_T(T \otimes_k T, T)$$

$$\cong M_n(k). \quad \square$$

**Proposition 2.5.**

$$\mathcal{D}(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)) \cong \mathcal{D}(\tilde{X}) \otimes_k M_n(k).$$

**Proof.** Write $C = \mathcal{O}(\tilde{X})$, $T = k[z]/(z^n)$, $B = C \otimes_k T$. Denote by $J_B$, $J_C$, $J_T$ the kernel of the multiplication maps $B \otimes_k B \to B$, $C \otimes_k C \to C$ and $T \otimes_k T \to T$ respectively. Note that, when we consider $C \otimes_k C$ and $T \otimes_k T$ as subalgebras of $B \otimes_k B$, we have $J_C, J_T \subseteq J_B$. Furthermore, since $J_B$ is generated by $\{1 \otimes b - b \otimes 1 : b \in B\}$ it follows that $J_B = C \otimes C \otimes J_T + J_C \otimes T \otimes T$.

Thus for all $m \in \mathbb{N}$

$$J_B^{2m} \subseteq C \otimes C \otimes J_T^m + J_C^m \otimes T \otimes T \subseteq J_B^m.$$ Since, $J_B^{2n} = 0$, we have for $m \geq 2n$,

$$J_B^{2m} \subseteq J_C^m \otimes T \otimes T \subseteq J_B^m.$$ Thus the sequence $J_C^m \otimes T \otimes T$ is cofinal with the sequence $J_B^m$. Hence

$$\mathcal{D}(B) = \lim_{\rightarrow m} \text{Hom}_B(B \otimes B/(J_B^m \otimes T \otimes T), B)$$

$$= \lim_{\rightarrow m} \text{Hom}_{C \otimes T}((C \otimes C/J_C^m) \otimes T \otimes T, C \otimes T)$$

$$= \lim_{\rightarrow m} \text{Hom}_C(C \otimes C/J_C^m, C) \otimes \text{Hom}_T(T \otimes T, T)$$

$$= \mathcal{D}(C) \otimes M_n(k). \quad \square$$
Corollary 2.6. The $k$-algebras $\mathcal{D}(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n))$ and $\mathcal{D}(\tilde{X})$ are Morita equivalent. \(\square\)

Remark. Thus $\mathcal{D}(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n))$ inherits the 'good' properties of $\mathcal{D}(\tilde{X})$; in particular it is a simple, noetherian ring of global homological dimension 1.

We now consider the inclusion $A = R/f^nR \subseteq \mathcal{O}(\tilde{X}) \otimes_k k[z]/(z^n)$.

Proposition 2.7. Write $C = \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)$. The following are equivalent:

(a) $\mathcal{D}(A)$ is a simple ring;
(b) $\mathcal{D}(C, A) * C = A$;
(c) $\mathcal{D}(A) = \text{End}_{\mathcal{D}(C)} \mathcal{D}(C, A)$;
(d) $\mathcal{D}(A)$ and $\mathcal{D}(C)$ are Morita equivalent.

Proof. Exactly the same proof as [7, Proposition 3.3] will work. \(\square\)

Proposition 2.8. When $\pi: \tilde{X} \to X$ is injective, then all the equivalent conditions of Proposition 2.7 are satisfied.

Proof. We shall prove that (b) is satisfied. The proof imitates that of [7, Theorem 3.4].

Consider $A = R/f^nR \subseteq C = \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)$. Let $M$ be a maximal ideal of $A$, and let $Q$ be the unique (after Corollary 2.3) maximal ideal of $C$ containing $M$. Write $S = A \setminus M$, and note that these are regular elements of $A$. Let $P$ be the maximal ideal of $\mathcal{O}(\tilde{X})$ given by $P = Q \cap \mathcal{O}(\tilde{X})$. Then $A_S \subseteq C_S = \mathcal{O}(\tilde{X})_P \otimes k[z]/(z^n)$. Set $m = P\mathcal{O}(\tilde{X})_P$. As $(\mathcal{O}(\tilde{X})_P, m)$ is a 1-dimensional regular local ring, we may choose $t \in P$ such that $m = t\mathcal{O}(\tilde{X})_P$. Let $\partial = \text{Der} \mathcal{O}(\tilde{X})_P$ satisfy $\partial(t) = 1$. By setting $\partial(z) = 0$ we may extend $\partial$ to a derivation on $C_S$.

After Proposition 2.4, $k[z]/(z^n)$ is a simple $\mathcal{D}(k[z]/(z^n))$-module, so there exists $D_1 \in \mathcal{D}(k[z]/(z^n))$ such that $D_1(1) = 1$ and $D_1(z_i) = 0$ for $0 < i < n$. (Explicitly $D_1$ is a scalar multiple of $\prod_{i=1}^{n-1} (z \partial_z - i)$ where $\partial_z$ is the derivation $d/dz$.) Extend $D_1$ to an $\mathcal{O}(\tilde{X})_P$-linear map on $C_S$, so that $D_1 \in \mathcal{D}(C_S)$.

After Corollary 2.3, $A_S$ is of finite codimension in $C_S$, so for some $r \in \mathbb{N}$, $t' C_S \subseteq A_S$. Write $C_S = t' C_S \oplus V$ where $V$ is the $k$-vector space with basis $B = \{t' z^j | 0 \leq j < r, 0 \leq i < n\}$. Consider $D_2 = \prod_{i=1}^{n-1} (t \partial_z - i) \in \mathcal{D}(C_S)$. Then $D = D_1D_2 \in \mathcal{D}(C_S)$ and satisfies $D(t' z^i) = 0$ for all $t' z^i \in B \setminus \{1\}$, and $0 \neq D(1) \in k$. Furthermore, $D(t' C_S) \subseteq t' C_S$. Thus $D \in \mathcal{D}(C_S, A_S)$ and $1 \in D * C_S$. There exists $s \in S$ such that $sD \in \mathcal{D}(C, A)$, and $s \in D * C$. Hence $D * C \not\subseteq M$. Since $M$ was arbitrary, it follows that $\mathcal{D}(C, A) * C = A$ as required. \(\square\)

Corollary 2.9. If $\pi: \tilde{X} \to X$ is injective, then $R/f^nR$ is a simple $\mathcal{D}(R/f^nR)$-module.
A simple $\mathcal{D}$-module

Proof. After Corollary 2.6 and Propositions 2.7 and 2.8 it has been established that $\mathcal{D}(R/f^nR)$ is Morita equivalent to $\mathcal{D}(\tilde{X})$. Hence $\mathcal{D}(R/f^nR)$ is a simple ring. Consequently $R/f^nR$ is a simple $\mathcal{D}(R/f^nR)$-module, because any proper factor module would have a nonzero annihilator. □

3. The $\mathcal{D}$-module $k[x, y]/k[x, y]$

Keep the notation of Section 2. Recall the definition of the idealiser in [7, §1.5]. Write $\mathcal{D} = \mathcal{D}(k[x, y]) = \mathcal{D}(A^2)$, and define

$$\mathfrak{l}(\mathcal{D}f^n) = \{D \in \mathcal{D} \mid f^nD \in \mathcal{D}f^n\}.$$

Notice that $\mathcal{D}f^n$ becomes a two-sided ideal of $\mathfrak{l}(\mathcal{D}f^n)$. If $k[x, y]/k[x, y]$ is viewed as an $\mathfrak{l}(\mathcal{D}f^n)$-module, then $f^{-n}k[x, y]/k[x, y]$ becomes an $\mathfrak{l}(\mathcal{D}f^n)$-submodule which is annihilated by $\mathcal{D}f^n$.

Lemma 3.1. Suppose that, for all $n \in \mathbb{N}$, $f^{-n}k[x, y]/k[x, y]$ is a simple $\mathfrak{l}(\mathcal{D}f^n)/\mathcal{D}f^n$-module. Then $k[x, y]/k[x, y]$ is a simple $\mathcal{D}$-module.

Proof. Suppose that the hypothesis holds and let $m, m'$ be elements of $k[x, y]/k[x, y]$ with $m \neq 0$. For $n$ sufficiently large, $m, m' \in f^{-n}k[x, y]/k[x, y]$. Hence $m' \in \mathfrak{l}(\mathcal{D}f^n) \cdot m \subseteq \mathcal{D}m$, and the result follows. □

Notice that there is a $k$-algebra isomorphism $\varphi : \mathfrak{l}(\mathcal{D}f^n) \rightarrow \mathfrak{l}(f^n\mathcal{D})$ given by $\varphi(D) = D'$, where, for $D \in \mathfrak{l}(\mathcal{D}f^n)$, $D' \in \mathcal{D}$ is the element such that $f^nD = D'f^n$. One sees that $\varphi$ induces an isomorphism

$$\psi : \mathfrak{l}(\mathcal{D}f^n)/\mathcal{D}f^n \rightarrow \mathfrak{l}(f^n\mathcal{D})/f^n\mathcal{D}.$$

However, recall the following:

Proposition 3.2 (Smith and Stafford [7, (1.6)]). $\mathfrak{l}(f^n\mathcal{D})/f^n\mathcal{D} \cong \mathcal{D}(k[x, y]/(f^n))$. □

Thus, in a natural way, $f^{-n}k[x, y]/k[x, y]$ is given the structure of a left $\mathcal{D}(k[x, y]/(f^n))$-module. The key observation is now:

Lemma 3.3. As a $\mathcal{D}(k[x, y]/(f^n))$-module, $f^{-n}k[x, y]/k[x, y]$ is isomorphic to $k[x, y]/(f^n)$.

Proof. This is routine. Just chase the isomorphisms above together with that in Proposition 3.2, as presented in [7, §1.6], and combine these with the natural $k[x, y]$-module isomorphism $k[x, y]/(f^n) \rightarrow f^{-n}k[x, y]/k[x, y]$. □
Hence we obtain the theorem of the introduction:

**Theorem.** If $\pi: \tilde{X} \to X$ is injective, then $k[x, y]/k[x, y]$ is a simple $\mathcal{D}$-module.

**Proof.** Combine Lemmas 3.1 and 3.3, Proposition 3.2 and Corollary 2.7. □

**Remark.** In retrospect, the key to the proof is the fact that $\mathcal{D}(k[x, y]/(f^n))$ is a simple ring. However, [7, §1.5, 1.6] $\mathcal{D}(k[x, y]/(f^n)) \cong \text{End}_{\mathcal{D}}(\mathcal{D}/f^n\mathcal{D})$. Thus if we had known to start that $\mathcal{D}/f^n\mathcal{D}$ were isomorphic to a direct sum of $n$ copies of $\mathcal{D}/f\mathcal{D}$, we would have had at once that $\mathcal{D}(k[x, y]/(f^n)) \cong \mathcal{D}(k[x, y]/(f)) \otimes_k M_n(k)$, and hence the ring is simple (when $\pi$ is injective).

Although our result does not imply the splitting of $\mathcal{D}/f^n\mathcal{D}$, we have been informed by Van den Essen and Van Doorn that they can prove that, if $\pi: \tilde{X} \to X$ is injective, then $\text{Ext}^1_{\mathcal{D}}(\mathcal{D}/f\mathcal{D}, \mathcal{D}/f\mathcal{D}) = 0$. This of course guarantees the splitting of $\mathcal{D}/f^n\mathcal{D}$, and so would give a quicker proof of our Corollary 2.7. Their work, to appear in [8], also shows that $k[x, y]/k[x, y]$ is a simple $\mathcal{D}$-module. The two approaches are quite different (although they also begin with the Morita equivalence of $\mathcal{D}(X)$ and $\mathcal{D}(\tilde{X})$ established in [7]).

**References**


