Journal of Pure and Applied Algebra 50 (1988) 287–294
North-Holland

THE SIMPLE $\mathscr{D}$-MODULE ASSOCIATED TO THE INTERSECTION HOMOLOGY COMPLEX FOR A CLASS OF PLANE CURVES

S.P. SMITH*

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

Communicated by C.A. Weibel
Received 1 March 1986

Let X be an irreducible plane algebraic curve over an algebraically closed field k of characteristic zero. Suppose that X is analytically irreducible at all points. Let $\mathscr{D}(\mathbb{A}^2)$ be the ring of differential operators on $\mathbb{A}^2$. This paper gives a direct algebraic proof that $\mathcal{O}(\mathbb{A}^2 \backslash X)/\mathcal{O}(\mathbb{A}^2)$ is a simple $\mathscr{D}(\mathbb{A}^2)$-module. This may also be proved via the Riemann–Hilbert correspondence.

1. Introduction

Let k be an algebraically closed field of characteristic zero, Y a non-singular affine algebraic variety and $X \subseteq Y$ a hypersurface defined by an irreducible $f \in \mathcal{O}(Y)$. Write $\mathscr{D} = \mathscr{D}(Y)$ for the ring of differential operators on Y, and consider $\mathcal{O}(Y \backslash X) = \mathcal{O}(Y)_f$ as a left $\mathscr{D}$-module. By Bernstein [1] and Kashiwara [6] it is of finite length as a $\mathscr{D}$-module. It is not difficult to show that $\mathcal{O}(Y \backslash X)/\mathcal{O}(Y)$ contains a unique simple $\mathscr{D}$-submodule – this is denoted $\mathscr{L}(X, Y)$. Its existence is proved in greater generality in [2]. The problem is to determine $\mathscr{L}(X, Y)$.

We prove the following:

Theorem. *Let $X \subseteq \mathbb{A}^2$ be an irreducible curve. Let $\tilde{X}$ denote the normalisation of X, and suppose that $\pi : \tilde{X} \to X$, the natural projection, is injective. Then $\mathscr{L}(X, \mathbb{A}^2) = \mathcal{O}(\mathbb{A}^2 \backslash X)/\mathcal{O}(\mathbb{A}^2)$.*

This theorem is known to the $\mathscr{D}$-modules experts. It was pointed out to the author, by J.-L. Brylinski and the referee, that it is a consequence of the Riemann–Hilbert correspondence. However the present paper gives the first purely algebraic proof of this result.

One motivation for wanting to describe $\mathscr{L}(X, Y)$ is that the de Rham complex satisfies $\mathrm{DR}(\mathscr{L}(X, Y)) = \mathrm{IC}_Y[-1]$, the Goresky–MacPherson intersection homology complex for the middle perversity [4].

There are a number of results describing a generator for $\mathscr{L}(X, Y)$; see for

* Current address: Department of Mathematics, University of Washington, Seattle, WA 98195, U.S.A.

example [2] and [9]. In the case of an arbitrary irreducible curve $X \subset \mathbb{A}^2$ defined by f, it is remarked in [2, Introduction] that $\mathcal{L}(X, Y)$ is generated by $f^{-1}\mathrm{d}f/\mathrm{d}x$ when $\mathcal{O}(\mathbb{A}^2) = k[x, y]$. Of course, this does not make it possible to decide if a given element of $\mathcal{O}(\mathbb{A}^2)_f/\mathcal{O}(\mathbb{A}^2)$ belongs to $\mathcal{L}(X, \mathbb{A}^2)$.

The theorem above extends the well-known fact that if $X \subset \mathbb{A}^2$ is non-singular, then $\mathcal{O}(\mathbb{A}^2 \backslash X)/\mathcal{O}(\mathbb{A}^2)$ is a simple $\mathcal{D}(\mathbb{A}^2)$-module. Recall that the usual proof of this uses the fact that, if X is defined by $f \in \mathcal{O}(\mathbb{A}^2)$, then the ideal of $\mathcal{O}(\mathbb{A}^2)$ generated by $\mathrm{d}f/\mathrm{d}x$, $\mathrm{d}f/\mathrm{d}y$ and f equals $\mathcal{O}(\mathbb{A}^2)$. Of course such a result is not available to us when X is singular.

The idea of the proof is to extend some of the work in [7]. Let X be any curve, $\tilde{X}$ its normalisation, and let $\pi : \tilde{X} \to X$ the natural projection. In [7, §3] it is shown that if π is injective, then $\mathcal{O}(X)$ is a simple $\mathcal{D}(X)$-module and furthermore, that $\mathcal{D}(X)$ is Morita equivalent to $\mathcal{D}(\tilde{X})$. In Section 2 we extend this result as follows. Set $R = k[x, y] = \mathcal{O}(\mathbb{A}^2)$, let $f \in R$ define the curve X, and suppose that $\pi : \tilde{X} \to X$ is injective; then for all $n \in \mathbb{N}$, R/f^nR is a simple $\mathcal{D}(R/f^nR)$-module (Corollary 2.9). Recall that as in [7, §1], $\mathcal{D}(A)$ is defined for any commutative k-algebra A. Having established this, the main theorem is a relatively straightforward consequence. The proof is given in Section 3; by reading Section 3 first the reader will understand the necessity of Section 2.

Let us recall some of the notation of [7]. Let $A \subset C$ be commutative k-algebras. If $D \in \mathcal{D}(C)$ and $f \in C$, write $D * f := D(f)$ for the evaluation of the differential operator D on f. Define $\mathcal{D}(C, A) := \{D \in \mathcal{D}(C) \mid D * f \in A \text{ for all } f \in C\}$, and $\mathcal{D}(C, A) * C = \{D * f \mid D \in \mathcal{D}(C, A) \text{ and } f \in C\}$.

2. Differential operators on $k[x, y]/(f^n)$

Set $R = k[x, y] = \mathcal{O}(\mathbb{A}^2)$, let $f \in R$ be irreducible, and let $X \subseteq \mathbb{A}^2$ denote the curve defined by f. Set $A = R/f^nR$.

The goal of this section is to show that there are inclusions of k-algebras

$$A \subseteq \mathcal{O}(X) \otimes k[z]/(z^n) \subseteq \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n) \subseteq \operatorname{Fract} A$$

and that A is of finite codimension in $\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)$. When the projection $\pi : \tilde{X} \to X$ from the normalisation is injective, the arguments of [7, Propositions 3.3, 3.4] may be adapted to show that $\mathcal{D}(A)$ is Morita equivalent to $\mathcal{D}(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n))$. But the latter is isomorphic to $\mathcal{D}(\tilde{X}) \otimes_k M_n(k)$, where $M_n(k)$ is the ring of $n \times n$ matrices over k. Hence $\mathcal{D}(A)$ is Morita equivalent to $\mathcal{D}(\tilde{X})$, and in particular is a simple ring. It follows that A is a simple $\mathcal{D}(A)$-module, and this is the result which is carried forward to Section 3, and used there to establish the main theorem of the paper.

Proposition 2.1. *Let $R = k[t_1, \ldots, t_d] = \mathcal{O}(\mathbb{A}^d)$, and suppose that $f \in R$ is irreducible, defining a hypersurface X. Let $n \in \mathbb{N}$. Then there is a k-algebra homomorphism*

$$\theta : R \to \mathcal{O}(X) \otimes_k k[z]/(z^n)$$

such that $\ker \theta = f^n R$.

Proof. Since $k = \bigcap_{j=1}^{d} k[t_1, \ldots, \hat{t}_j, \ldots, t_d]$ (where $\hat{t}_j$ means 'omit t_j') we may assume, without loss of generality, that $f \not\in k[t_2, \ldots, t_d]$.

Let $\psi : R \to \mathcal{O}(X)$ be the natural map with kernel fR. Define θ by

$$\theta(t_1) = \psi(t_1) + z , \qquad \theta(t_i) = \psi(t_i) \quad (2 \le i \le d) .$$

and extend θ to a k-algebra homomorphism. By Taylor's theorem it follows, for $g \in R$, that

$$\begin{aligned} \theta(g) &= g(\psi(t_1) + z, \psi(t_2), \ldots, \psi(t_d)) \\ &= \sum_{j=0}^{n-1} \frac{1}{j!} \psi\left(\frac{\partial^j g}{\partial t_1^j}\right) z^j . \end{aligned}$$

Hence, $\ker \theta = \{g \in R \mid \partial^j g / \partial t_1^j \in fR \text{ for all } 0 \le j < n\}$. It is clear that $f^n R \subseteq \ker \theta$, and the reverse inclusion follows from the

Sublemma. *If* $\partial^j g/\partial t_1^j \in fR$ *for all* $0 \le j < n$, *then* $g \in f^n R$.

Proof. By induction on n. The sublemma is true if $n = 1$. Assuming that the sublemma holds for $n - 1$, we may suppose that $g \in f^{n-1}R$. Since $\partial^j/\partial t_1^j(\partial g/\partial t_1) \in fR$ for all $0 \le j < n-1$, the induction hypothesis ensures that $\partial g/\partial t_1 \in f^{n-1}R$. Write $g = f^{n-1}h$. Then

$$\frac{\partial g}{\partial t_1} = (n-1) f^{n-2} h \frac{\partial f}{\partial t_1} + f^{n-1} \frac{\partial h}{\partial t_1} \in f^{n-1} R .$$

Hence $h\, \partial f/\partial t_1 \in fR$. By assumption, $\deg_{t_1}(f) \ge 1$, and so (by a degree argument) it follows that $\partial f/\partial t_1 \not\in fR$. Hence, as fR is a prime ideal, $h \in fR$. Thus $g \in f^n R$ as required. □□

We now turn to the case we are interested in, namely $d = 2$. Write $A = R/f^n R$, and $A' = \mathcal{O}(X) \otimes_k k[z]/(z^n)$.

Lemma 2.2. *Let* $R = k[x, y] = \mathcal{O}(\mathbb{A}^2)$. *Let* $0 \ne f \in R$ *be irreducible defining a curve* $X \subseteq \mathbb{A}^2$. *Let* $n \in \mathbb{N}$, *and let*

$$\theta : R/f^n R \to \mathcal{O}(X) \otimes_k k[z]/(z^n)$$

be the injective algebra homomorphism obtained in Proposition 2.1. *Then*

(a) $\theta(R/f^nR)$ *is of finite codimension (as a k-vector space) in*

$$\mathcal{O}(X) \otimes_k k[z]/(z^n) ;$$

(b) *We have inclusions*

$$R/f^nR \subseteq \mathcal{O}(X) \otimes_k k[z]/(z^n) \subseteq \mathrm{Fract}(R/f^nR)$$

where Fract( ) *denotes the ring of fractions.*

Proof. (a) Write $V = R/f^nR$, $U = \mathcal{O}(X) \otimes_k k[z]/(z^n)$. There are filtrations of these vector spaces $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$, and $U = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = 0$ where $V_i = Af^i$ are $U_i = A'z^i$. The injection $\theta : V \to U$ satisfies $\theta(V_i) \subseteq U_i$ and $\theta(V) \cap U_i = \theta(V_i)$. Hence if we consider the short exact sequences below, and the injections induced by θ, we obtain an injection $\gamma : V_i/V_{i+1} \to U_i/U_{i+1}$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 \to & V_{i+1} & \to & V_i & \to & V_i/V_{i+1} & \to 0 \\
 & \downarrow & & \downarrow & & & \\
0 \to & U_{i+1} & \to & U_i & \to & U_i/U_{i+1} & \to 0
\end{array}$$

The aim is to show that $\dim_k(U/\theta(V)) < \infty$, and the idea is to show by induction on i that $\dim_k(U_i/\theta(V_i)) < \infty$. It is clear that if $\dim_k(U_{i+1}/\theta(V_{i+1})) < \infty$ and $\dim_k((U_i/U_{i+1})/\gamma(V_i/V_{i+1})) < \infty$, then $\dim_k(U_i/\theta(V_i)) < \infty$. Hence one need only show that $\gamma(V_i/V_{i+1})$ is of finite codimension in U_i/U_{i+1}; note that as $V_n = 0$ and $U_n = 0$ this will also start the induction process.

Note that U_i/U_{i+1} and V_i/V_{i+1} are both free $\mathcal{O}(X)$-modules of rank 1. However, γ is an $\mathcal{O}(X)$-module homomorphism because $\theta(f^ir) = \theta(f^i)\theta(r)$, and $\theta(r) - \psi(r) \in A'z$; hence $\theta(f^ir) = \theta(f^i)\psi(r) \pmod{U_{i+1}}$. As $\gamma \neq 0$, we have that $\gamma(V_i/V_{i+1})$ is a nonzero $\mathcal{O}(X)$-submodule of the cyclic $\mathcal{O}(X)$-module U_i/U_{i+1}. Hence, $\gamma(V_i/V_{i+1})$ is necessarily of finite codimension in U_i/U_{i+1}. This establishes (a).

(b) We now consider $A \subseteq A'$. The regular elements of A are precisely those elements not in fA. There exists $\xi \in A$ which is transcendental over k, and $k[\xi] \cap fA = 0$. Hence, $k(\xi) \subseteq \mathrm{Fract}\, A$. Since A'/A is a finite-dimensional vector space, there exists $0 \neq p(\xi) \in k[\xi]$ such that $A'p(\xi) \subseteq A$. As $p(\xi)$ is a unit in Fract A, we have $A' \subseteq \mathrm{Fract}\, A$. This completes the proof of (b). □

Corollary 2.3. *There are inclusions of k-algebras*

$$R/f^nR \subseteq \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n) \subseteq \mathrm{Fract}(R/f^nR)$$

and R/f^nR *is of finite codimension in* $\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)$. *The map* $\mathrm{Spec}(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)) \to \mathrm{Spec}(R/f^nR)$ *is the natural projection* $\pi : \tilde{X} \to X$.

Proof. The inclusion of the k-algebras is obtained from the previous lemma together with the natural inclusion $\mathcal{O}(X) \subseteq \mathcal{O}(\tilde{X})$. The fact that R/f^nR is of finite codimension, follows from the previous lemma, and the fact that $\mathcal{O}(X)$ is of finite codimension in $\mathcal{O}(\tilde{X})$. The fact that $\theta : R/f^nR \to \mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)$ induces an isomorphism after factoring out both these algebras by their nilpotent radicals ensures that the map on the spectra is as claimed. □

Proposition 2.4. *$\mathscr{D}(k[z]/(z^n)) \cong M_n(k)$, the ring of $n \times n$ matrices over k.*

Proof. Write $T = k[z]/(z^n)$ and set $J_T = \ker(\mu_T : T \otimes_k T \to T)$, where μ_T is the multiplication map. Then J_T is generated by $1 \otimes z - z \otimes 1$; hence $J_T^{2n} = 0$. Now

$$\begin{aligned}\mathscr{D}(T) &= \varinjlim_m \operatorname{Hom}_T(T \otimes_k T/J_T^m, T) = \operatorname{Hom}_T(T \otimes_k T, T)\\ &\cong M_n(k)\,. \qquad \square\end{aligned}$$

Proposition 2.5.

$$\mathscr{D}(\mathcal{O}(\tilde{X}) \otimes k[z]/(z^n)) \cong \mathscr{D}(\tilde{X}) \otimes_k M_n(k)\,.$$

Proof. Write $C = \mathcal{O}(\tilde{X})$, $T = k[z]/(z^n)$, $B = C \otimes_k T$. Denote by J_B, J_C, J_T the kernel of the multiplication maps $B \otimes_k B \to B$, $C \otimes_k C \to C$ and $T \otimes_k T \to T$ respectively. Note that, when we consider $C \otimes_k C$ and $T \otimes_k T$ as subalgebras of $B \otimes_k B$, we have $J_C, J_T \subseteq J_B$. Furthermore, since J_B is generated by $\{1 \otimes b - b \otimes 1 \mid b \in B\}$ it follows that

$$J_B = C \otimes C \otimes J_T + J_C \otimes T \otimes T\,.$$

Thus for all $m \in \mathbb{N}$

$$J_B^{2m} \subseteq C \otimes C \otimes J_T^m + J_C^m \otimes T \otimes T \subseteq J_B^m\,.$$

Since, $J_T^{2n} = 0$, we have for $m \geq 2n$,

$$J_B^{2m} \subseteq J_C^m \otimes T \otimes T \subseteq J_B^m\,.$$

Thus the sequence $J_C^m \otimes T \otimes T$ is cofinal with the sequence J_B^m. Hence

$$\begin{aligned}\mathscr{D}(B) &= \varinjlim_m \operatorname{Hom}_B(B \otimes B/(J_C^m \otimes T \otimes T), B)\\ &= \varinjlim_m \operatorname{Hom}_{C\otimes T}((C \otimes C/J_C^m) \otimes T \otimes T, C \otimes T)\\ &\simeq \varinjlim_m \operatorname{Hom}_C(C \otimes C/J_C^m, C) \otimes \operatorname{Hom}_T(T \otimes T, T)\\ &= \mathscr{D}(C) \otimes M_n(k)\,. \qquad \square\end{aligned}$$

Corollary 2.6. *The k-algebras* $\mathscr{D}(\mathscr{O}(\tilde{X}) \otimes k[z]/(z^n))$ *and* $\mathscr{D}(\tilde{X})$ *are Morita equivalent.* □

Remark. Thus $\mathscr{D}(\mathscr{O}(\tilde{X}) \otimes k[z]/(z^n))$ inherits the 'good' properties of $\mathscr{D}(\tilde{X})$; in particular it is a simple, noetherian ring of global homological dimension 1.

We now consider the inclusion $A = R/f^nR \subseteq \mathscr{O}(\tilde{X}) \otimes_k k[z]/(z^n)$.

Proposition 2.7. *Write* $C = \mathscr{O}(\tilde{X}) \otimes k[z]/(z^n)$. *The following are equivalent*:
(a) $\mathscr{D}(A)$ *is a simple ring*;
(b) $\mathscr{D}(C, A) * C = A$;
(c) $\mathscr{D}(A) = \mathrm{End}_{\mathscr{D}(C)} \mathscr{D}(C, A)$;
(d) $\mathscr{D}(A)$ *and* $\mathscr{D}(C)$ *are Morita equivalent.*

Proof. Exactly the same proof as [7, Proposition 3.3] will work. □

Proposition 2.8. *When* $\pi : \tilde{X} \to X$ *is injective, then all the equivalent conditions of Proposition* 2.7 *are satisfied.*

Proof. We shall prove that (b) is satisfied. The proof imitates that of [7, Theorem 3.4].

Consider $A = R/f^nR \subseteq C = \mathscr{O}(\tilde{X}) \otimes k[z]/(z^n)$. Let M be a maximal ideal of A, and let Q be the unique (after Corollary 2.3) maximal ideal of C containing M. Write $S = A \backslash M$, and note that these are regular elements of A. Let P be the maximal ideal of $\mathscr{O}(\tilde{X})$ given by $P = Q \cap \mathscr{O}(\tilde{X})$. Then $A_S \subseteq C_S = \mathscr{O}(\tilde{X})_P \otimes k[z]/(z^n)$. Set $\mathfrak{m} = P\mathscr{O}(\tilde{X})_P$. As $(\mathscr{O}(\tilde{X})_P, \mathfrak{m})$ is a 1-dimensional regular local ring, we may choose $t \in P$ such that $\mathfrak{m} = t\mathscr{O}(\tilde{X})_P$. Let $\partial \in \mathrm{Der}\, \mathscr{O}(\tilde{X})_P$ satisfy $\partial(t) = 1$. By setting $\partial(z) = 0$ we may extend ∂ to a derivation on C_S.

After Proposition 2.4, $k[z]/(z^n)$ is a simple $\mathscr{D}(k[z]/(z^n))$-module, so there exists $D_1 \in \mathscr{D}(k[z]/(z^n))$ such that $D_1(1) = 1$ and $D_1(z^i) = 0$ for $0 < i < n$. (Explicitly D_1 is a scalar multiple of $\prod_{j=1}^{n-1} (z\partial_z - i)$ where ∂_z is the derivation d/dz). Extend D_1 to an $\mathscr{O}(\tilde{X})_P$-linear map on C_S, so that $D_1 \in \mathscr{D}(C_S)$.

After Corollary 2.3, A_S is of finite codimension in C_S, so for some $r \in \mathbb{N}$, $t^rC_S \subseteq A_S$. Write $C_S = t^rC_S \oplus V$ where V is the k-vector space with basis $B = \{t^jz^i \mid 0 \le j < r, 0 \le i < n\}$. Consider $D_2 = \prod_{j=1}^{r-1} (t\partial - j) \in \mathscr{D}(C_S)$. Then $D = D_1D_2 \in \mathscr{D}(C_S)$ and satisfies $D(t^jz^i) = 0$ for all $t^jz^i \in B \backslash \{1\}$, and $0 \neq D(1) \in k$. Furthermore, $D(t^rC_S) \subseteq t^rC_S$. Thus $D \in \mathscr{D}(C_S, A_S)$ and $1 \in D * C_S$. There exists $s \in S$ such that $sD \in \mathscr{D}(C, A)$, and $s \in D * C$. Hence $D * C \not\subseteq M$. Since M was arbitrary, it follows that $\mathscr{D}(C, A) * C = A$ as required. □

Corollary 2.9. *If* $\pi : \tilde{X} \to X$ *is injective, then* R/f^nR *is a simple* $\mathscr{D}(R/f^nR)$*-module.*

Proof. After Corollary 2.6 and Propositions 2.7 and 2.8 it has been established that $\mathscr{D}(R/f^nR)$ is Morita equivalent to $\mathscr{D}(\tilde{X})$. Hence $\mathscr{D}(R/f^nR)$ is a simple ring. Consequently R/f^nR is a simple $\mathscr{D}(R/f^nR)$-module, because any proper factor module would have a nonzero annihilator. □

3. The $\mathscr{D}$-module $k[x, y]_f/k[x, y]$

Keep the notation of Section 2. Recall the definition of the idealiser in [7, §1.5]. Write $\mathscr{D} = \mathscr{D}(k[x, y]) = \mathscr{D}(\mathbb{A}^2)$, and define

$$\mathbb{I}(\mathscr{D}f^n) = \{D \in \mathscr{D} \mid f^nD \in \mathscr{D}f^n\} .$$

Notice that $\mathscr{D}f^n$ becomes a two-sided ideal of $\mathbb{I}(\mathscr{D}f^n)$. If $k[x, y]_f/k[x, y]$ is viewed as an $\mathbb{I}(\mathscr{D}f^n)$-module, then $f^{-n}k[x, y]/k[x, y]$ becomes an $\mathbb{I}(\mathscr{D}f^n)$-submodule which is annihilated by $\mathscr{D}f^n$.

Lemma 3.1. *Suppose that, for all $n \in \mathbb{N}$, $f^{-n}k[x, y]/k[x, y]$ is a simple $\mathbb{I}(\mathscr{D}f^n)/\mathscr{D}f^n$-module. Then $k[x, y]_f/k[x, y]$ is a simple $\mathscr{D}$-module.*

Proof. Suppose that the hypothesis holds and let m, m' be elements of $k[x, y]_f/k[x, y]$ with $m \neq 0$. For n sufficiently large, $m, m' \in f^{-n}k[x, y]/k[x, y]$. Hence $m' \in \mathbb{I}(\mathscr{D}f^n) \cdot m \subseteq \mathscr{D}m$, and the result follows. □

Notice that there is a k-algebra isomorphism $\varphi : \mathbb{I}(\mathscr{D}f^n) \to \mathbb{I}(f^n\mathscr{D})$ given by $\varphi(D) = D'$, where, for $D \in \mathbb{I}(\mathscr{D}f^n)$, $D' \in \mathscr{D}$ is the element such that $f^nD = D'f^n$. One sees that φ induces an isomorphism

$$\psi : \mathbb{I}(\mathscr{D}f^n)/\mathscr{D}f^n \to \mathbb{I}(f^n\mathscr{D})/f^n\mathscr{D} .$$

However, recall the following:

Proposition 3.2 (Smith and Stafford [7, (1.6)]). $\mathbb{I}(f^n\mathscr{D})/f^n\mathscr{D} \cong \mathscr{D}(k[x, y]/(f^n))$. □

Thus, in a natural way, $f^{-n}k[x, y]/k[x, y]$ is given the structure of a left $\mathscr{D}(k[x, y]/(f^n))$-module. The key observation is now:

Lemma 3.3. *As a $\mathscr{D}(k[x, y]/(f^n))$-module, $f^{-n}k[x, y]/k[x, y]$ is isomorphic to $k[x, y]/(f^n)$.*

Proof. This is routine. Just chase the isomorphisms above together with that in Proposition 3.2, as presented in [7, §1.6], and combine these with the natural $k[x, y]$-module isomorphism $k[x, y]/(f^n) \to f^{-n}k[x, y]/k[x, y]$. □

Hence we obtain the theorem of the introduction:

Theorem. *If $\pi : \tilde{X} \to X$ is injective, then $k[x, y]_f / k[x, y]$ is a simple $\mathscr{D}$-module.*

Proof. Combine Lemmas 3.1 and 3.3, Proposition 3.2 and Corollary 2.7. $\square$

Remark. In retrospect, the key to the proof is the fact that $\mathscr{D}(k[x, y]/(f^n))$ is a simple ring. However, [7, §1.5, 1.6] $\mathscr{D}(k[x, y]/(f^n)) \cong \operatorname{End}_{\mathscr{D}}(\mathscr{D}/f^n\mathscr{D})$. Thus if we had known to start that $\mathscr{D}/f^n\mathscr{D}$ were isomorphic to a direct sum of n copies of $\mathscr{D}/f\mathscr{D}$, we would have had at once that $\mathscr{D}(k[x, y]/(f^n)) \cong \mathscr{D}(k[x, y]/(f)) \otimes_k M_n(k)$, and hence the ring is simple (when π is injective). Although our result does not imply the splitting of $\mathscr{D}/f^n\mathscr{D}$, we have been informed by Van den Essen and Van Doorn that they can prove that, if $\pi : \tilde{X} \to X$ is injective, then $\operatorname{Ext}^1_{\mathscr{D}}(\mathscr{D}/f\mathscr{D}, \mathscr{D}/f\mathscr{D}) = 0$. This of course guarantees the splitting of $\mathscr{D}/f^n\mathscr{D}$, and so would give a quicker proof of our Corollary 2.7. Their work, to appear in [8], also shows that $k[x, y]_f / k[x, y]$ is a simple $\mathscr{D}$-module. The two approaches are quite different (although they also begin with the Morita equivalence of $\mathscr{D}(X)$ and $\mathscr{D}(\tilde{X})$ established in [7]).

References

[1] J.N. Bernstein, Analytic continuation of generalized functions with respect to a parameter, Functional Anal. Appl. 6 (1972) 26–40.

[2] J.-L. Brylinski, Le classe fondamentale d'une variété algébrique engendre le $\mathscr{D}$-module qui calcule sa cohomologie d'intersection, in: A. Galligo, M. Granger, Ph. Maisonobe, eds., Systèmes Differentiels et singularités Asterique 130 (1985) 260–271.

[3] J-L. Brylinski and M. Kashiwara, Kazhdan–Lusztig conjecture and holonomic systems, Invent. Math. 64 (1981) 387–410.

[4] M. Goresky and R. Macpherson, Intersection Homology II, Invent. Math. 72 (1983) 77–130.

[5] A. Grothendieck, Elements de Géométrie Algébrique, Publ. Math. IHES 24 (1965).

[6] M. Kashiwara, b-functions on holonomic systems, Invent. Math. 38 (1976) 33–54.

[7] S.P. Smith and J.T. Stafford, Differential operators on an affine curve, Proc. London Math. Soc., to appear.

[8] A. Van Den Essen and R. Van Doorn, $\mathscr{D}_n$-modules with support on a curve, Preprint, University of Nijmegen, 1986.

[9] K. Vilonen, Intersection homology $\mathscr{D}$-module on local complete intersections with isolated singularities, Invent. Math. 81 (1985) 107–114.