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SHEAVES OF NONCOMMUTATIVE ALGEBRAS AND THE BEILINSON-BERNSTEIN EQUIVALENCE OF CATEGORIES

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ABSTRACT. Let X be an irreducible algebraic variety defined over a field k, let $\mathscr{R}$ be a sheaf of (noncommutative) noetherian k-algebras on X containing the sheaf of regular functions $\mathcal{O}$ and let R be the ring of global sections. We show that under quite reasonable abstract hypotheses (concerning the existence of a faithfully flat overring of R obtained from the local sections of $\mathscr{R}$) there is an equivalence between the category of R-modules and the category of sheaves of $\mathscr{R}$-modules which are quasicoherent as $\mathcal{O}$-modules. This shows that the equivalence of categories established by Beilinson and Bernstein as the first step in their proof of the Kazhdan-Lusztig conjectures (where R is a primitive factor ring of the enveloping algebra of a complex semisimple Lie algebra, and $\mathscr{R}$ is a sheaf of twisted differential operators on a generalised flag variety) is valid for more fundamental reasons than is apparent from their work.

1. Introduction.

1.1. This paper is motivated by the Beilinson-Bernstein Theorem [**BB**]. Very briefly, this says that for certain primitive factor rings D_λ of the enveloping algebra of a complex semisimple Lie algebra there is an associated sheaf $\mathscr{D}_\lambda$ of noncommutative algebras over a complex projective algebraic variety X such that $D_\lambda = \Gamma(X, \mathscr{D}_\lambda)$ and there is an equivalence between the category of left D_λ-modules and the category of sheaves of left $\mathscr{D}_\lambda$-modules which are quasicoherent as $\mathcal{O}$-modules ($\mathcal{O}$ is the structure sheaf of X, and is a subsheaf of $\mathscr{D}_\lambda$). In this paper we abstract some of the essential features of their construction in an attempt to understand what ring theoretic properties of D_λ ensure this equivalence of categories. We show that the equivalence of categories follows from the existence of a faithfully flat overring of D_λ.

Let us give the details.

1.2. Let k be any field and X an irreducible algebraic variety over k. Let $\mathscr{R}$ be a sheaf of noncommutative noetherian k-algebras over X. Set $R = \Gamma(X, \mathscr{R})$ and suppose that R has a classical ring of quotients Q. If U is an open affine subset of X write $R_U = \Gamma(U, \mathscr{R})$. We make the following assumptions concerning $\mathscr{R}$ from which the Theorem below will be deduced.

(i) The structure sheaf $\mathcal{O}$ of X is a subsheaf of $\mathscr{R}$ and $\mathscr{R}$ is a quasicoherent sheaf of left $\mathcal{O}$-modules.

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(ii) If $U \subset X$ is an open affine subset of X, then R_U is a subalgebra of Q containing R and is generated as a right or left R-module by $\Gamma(U, \mathcal{O})$.

(iii) There is a finite open affine cover (U_α) for X such that the diagonal embedding $R \to \oplus R_\alpha$ (where $R_\alpha = \Gamma(U_\alpha, \mathcal{R})$) obtained from the restriction maps makes $\oplus R_\alpha$ a faithfully flat right R-module.

Write R-Mod for the category of left R-modules and $\mathcal{R}$-$\mathcal{M}od$ for the category of sheaves of left $\mathcal{R}$-modules which are quasicoherent as $\mathcal{O}$-modules.

THEOREM. *There is an equivalence between* R-Mod *and* $\mathcal{R}$-$\mathcal{M}od$ *given by the mutually inverse functors* $M \to \mathcal{R} \otimes_R M$ *and* $\mathcal{M} \to \Gamma(X, \mathcal{M})$.

1.3. It is straightforward to check that $\mathcal{D}_\lambda$ satisfies (i) and (ii) above. Although (iii) follows from the equivalence of categories established in **[BB]** we have not been able to establish its truth on a priori grounds except in some special cases. However, there are reasons for expecting this may be possible (see §3.1) and then the Theorem would imply the result of Beilinson and Bernstein. Joseph and Stafford **[JS]** have given a direct proof of the flatness condition in (iii), so one only needs to find a direct proof of the "faithfulness" condition.

1.4. This paper is primarily written for noncommutative ring theorists. The paper can be read without a knowledge of the work of Beilinson and Bernstein, but we expect that an understanding of their construction would make the work here more meaningful.

2. Proof of the Theorem.

2.1. The Theorem will be proved using the language of torsion theory. We recall the standard terminology and results which we require. The reader is refered to **[St]** for a thorough treatment and proofs.

Let R be a ring and A a ring containing R (with the same identity). If A is flat as a right R-module and $A \otimes_R A \simeq A$ (under the map $a \otimes b \to ab$) as an A-A bimodule we call A a *perfect* (*left*) *localisation* of R. Denote by $\mathcal{T}$ the class of all left R-modules M satisfying $A \otimes_R M = 0$. We call $\mathcal{T}$ the *torsion class* associated to A. Because A_R is flat, the class $\mathcal{T}$ is closed under submodules, factor modules and extensions. If M is an R-module there is a largest submodule of M contained in $\mathcal{T}$; we denote this submodule by $\tau(M)$ and call it the *torsion submodule* of M. If $\tau(M) = M$ we call M a *torsion module*, and if $\tau(M) = 0$ we say that M is *torsion-free*. Denote by $\mathcal{F}$ the class of left ideals I of R such that R/I is a torsion module. We call $\mathcal{F}$ the *Gabriel filter* associated to A.

If $a \in A$, there exists an I in $\mathcal{F}$ such that $Ia \subset R$. If M is an R-module, then $\tau(M) = \{m \in M | Im = 0 \text{ for some } I \text{ in } \mathcal{F}\}$. The kernel of the natural map ϕ: $M \to A \otimes_R M$ is $\tau(M)$ and $A \otimes M/\phi(M)$ is a torsion module. If I and J are in $\mathcal{F}$ so are $I \cap J$ and IJ. The filter $\mathcal{F}$ may be characterized as those left ideals I of R such that $AI = A$. These facts will be used without comment in what follows.

2.2. Consider the situation described in §1.2. For simplicity we shall write $U_{\alpha_1 \cdots \alpha_n} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$, and $R_{\alpha_1 \cdots \alpha_n} = \Gamma(U_{\alpha_1 \cdots \alpha_n}, \mathcal{R})$. If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules we shall write $M = \Gamma(X, \mathcal{M})$ and $M_{\alpha_1 \cdots \alpha_n} = \Gamma(U_{\alpha_1 \cdots \alpha_n}, \mathcal{M})$. Abusing notation, we shall write $\rho_{\alpha_1 \cdots \alpha_n}$ for the restriction map $M_{\beta_1 \cdots \beta_m} \to M_{\alpha_1 \cdots \alpha_n}$ for any subset

$\{\beta_1,\ldots,\beta_m\} \subset \{\alpha_1,\ldots,\alpha_n\}$. No confusion should arise as the domain of $\rho_{\alpha_1\cdots\alpha_n}$ will always be clear from the context.

The stalk of $\mathscr{R}$ at $x \in X$ will be denoted by R_x. If $V \subset X$ is an open subset, denote $\Gamma(V, \mathcal{O})$ by A_V, and write A_x for the stalk of $\mathcal{O}$ at x.

2.3. Let us point out some immediate consequences of the assumptions made in §1.2.

The quasicoherence of $\mathscr{R}$ ensures that if $U \subset V$ are open affine subsets of X, then $R_U = A_U \otimes_{A_V} R_V$. If $V \subset X$ is open affine, then there is an equivalence between the categories R_V-Mod (of left R_V-modules) and $\mathscr{R}|_V\text{-}\mathscr{M}od$ (of sheaves of left $\mathscr{R}|_V$-modules which are quasicoherent as $\mathcal{O}|_V$-modules). If M is an R_V-module, then view M as an A_V-module and form the sheaf $\tilde{M}$ (the notation is that of [H]). If $U \subset V$ is an open affine subset, then

$$\Gamma(U, \tilde{M}) = A_U \otimes_{A_V} M = A_U \otimes_{A_V} (R_V \otimes_{R_V} M) = R_U \otimes_{R_V} M,$$

so $\tilde{M}$ is a sheaf of $\mathscr{R}|_V$-modules with global sections M. Also, if $\mathscr{M} \in \mathscr{R}|_V\text{-}\mathscr{M}od$, then $\mathscr{M} \simeq \Gamma(V, \mathscr{M})^{\sim}$ and $\Gamma(V, \mathscr{M})$ is an R_V-module. Hence the equivalence between R_V-Mod and $\mathscr{R}|_V\text{-}\mathscr{M}od$.

2.4. It is not strictly necessary to assume that R has a classical ring of fractions Q. One could replace Q in the hypotheses by the maximal flat epimorphic extension of R. The existence of Q is only used in applying the following result.

PROPOSITION [St, CHAPTER XI, PROPOSITION 2.4]. *If A is a ring with $R \subset A \subset Q$ and A_R is flat, then A is a perfect left localisation of R.*

COROLLARY. *If $U \subset U_\alpha$ and U is open affine, then R_U is a perfect left localisation of R.*

PROOF. By assumption, R_α is flat as a right R-module, so is a perfect left localisation by the Proposition.

Put $A_\alpha = \Gamma(U_\alpha, \mathcal{O})$. As $\mathscr{R}$ is quasicoherent, $R_U = A_U \otimes_{A_\alpha} R_\alpha$. Now, as A_U is flat as an A_α-module, R_U is flat as a right R_α-module. Hence R_U is flat as a right R-module. Apply the Proposition to R_U.

Thus the hypotheses in §1.2 ensure that each $R_{\alpha_1\cdots\alpha_n}$ is a perfect left localisation of R. We shall denote the torsion class associated to the perfect left localisation R_α of R by $\mathscr{T}_\alpha$. The associated Gabriel filter will be denoted by $\mathscr{F}_\alpha$, and the associated torsion functor will be denoted by τ_α.

2.5. LEMMA. *Let R be a noetherian ring with ring of fractions Q. Let A_1 and A_2 be perfect left localisations of R. Suppose that each A_i is generated as either a right or left R-module by a subset S_i and that the elements of S_1 commute with the elements of S_2. Let B denote the subring of Q generated by A_1 and A_2. Then there is an isomorphism of A_1–A_2 bimodules $A_1 \otimes_R A_2 \to B$ given by $a_1 \otimes a_2 \to a_1a_2$, and hence there is an isomorphism of R-R bimodules $A_1 \otimes_R A_2 \simeq A_2 \otimes_R A_1$.*

PROOF. If A is a perfect left localisation of R, the map $A \otimes_R A \to A$ given by $a \otimes a' \to aa'$ is an A-A bimodule isomorphism.

The natural maps

$$A_1 \otimes_R B \to A_1 \otimes_R (A_1 \otimes_{A_1} B) \to (A_1 \otimes_R A_1) \otimes_{A_1} B \to A_1 \otimes_{A_1} B \to B$$

are all isomorphisms of A_1-B bimodules, so the composition gives $A_1 \otimes_R B \simeq B$ by $a \otimes b \to ab$.

Tensor the exact sequence $0 \to A_2 \to B \to B/A_2 \to 0$ with the flat R-module A_1. To prove the first part of the lemma it is enough to prove that $A_1 \otimes_R (B/A_2) = 0$, because one may then compose the isomorphisms $A_1 \otimes_R A_2 \to A_1 \otimes_R B \to B$ to get the result.

A typical element of B is a finite sum of terms of the form $a_1 a_2$ with $a_1 \in A_1$ and $a_2 \in A_2$ (just use the fact that S_1 and S_2 commute, and that $A_1 = S_1 R = RS_1$). Let $\mathscr{F}$ denote the Gabriel filter associated to A_1. Given $a_1 \in A_1$, $a_2 \in A_2$ pick I in $\mathscr{F}$ such that $Ia_1 \subset R$; whence $Ia_1 a_2 \subset A_2$. So the image of $a_1 a_2$ in B/A_2 is in the torsion submodule (associated to A_1). As every element of B/A_2 is a finite sum of torsion elements, B/A_2 is itself torsion. In other words $A_1 \otimes_R (B/A_2) = 0$.

We have shown $A_1 \otimes_R A_2 \simeq B$ as A_1-A_2 bimodules, hence as R-R bimodules. Reversing the roles of A_2 and A_1, we also have $A_2 \otimes_R A_1 \simeq B$ as R-R bimodules. The lemma follows.

In the commutative case the isomorphism $A_1 \otimes_R A_2 \to A_2 \otimes_R A_1$ would just be $a_1 \otimes a_2 \to a_2 \otimes a_1$. In the noncommutative case the map is not so clear; first $a_1 \otimes a_2$ is mapped to $a_1 a_2 \in B$, then $a_1 a_2$ is expressed as a sum of terms of the form $a_2' a_1'$ with $a_2' \in A_2$, $a_1' \in A_1$ and each $a_2' a_1'$ is mapped to $a_2' \otimes a_1' \in A_2 \otimes A_1$.

2.6. LEMMA. *$R_{\alpha_1} \otimes_R \cdots \otimes_R R_{\alpha_n} \simeq R_{\alpha_1 \cdots \alpha_n}$ as R_{α_1}-R_{α_n} bimodules.*

PROOF. For each $m \leqslant n$, $R_{\alpha_1 \cdots \alpha_m}$ is a perfect left localization of R (as remarked in §2.4). Now apply Lemma 2.5 inductively. Notice that to apply 2.5 one must use the fact that $\Gamma(U_\alpha \cap U_\beta, \mathscr{O})$ is generated by the images (under the restriction maps) of $\Gamma(U_\alpha, \mathscr{O})$ and $\Gamma(U_\beta, \mathscr{O})$, and this is guaranteed by the assumption that X is a variety.

2.7. LEMMA. *Let N be an $R_{\alpha_1 \cdots \alpha_n}$-module such that $R_{\alpha_1 \cdots \alpha_{n+1}} \otimes N = 0$, where the tensor product is over $R_{\alpha_1 \cdots \alpha_n}$. Then $R_{\alpha_{n+1}} \otimes_R N = 0$.*

PROOF.

$$R_{\alpha_{n+1}} \otimes_R N = R_{\alpha_{n+1}} \otimes_R \left(R_{\alpha_1 \cdots \alpha_n} \otimes N \right) = \left(R_{\alpha_{n+1}} \otimes_R R_{\alpha_1 \cdots \alpha_n} \right) \otimes N,$$

where the second tensor product is over $R_{\alpha_1 \cdots \alpha_n}$. Now apply the previous lemma.

2.8. LEMMA. *Let $\mathscr{M} \in \mathscr{R}$-$\mathscr{M}od$ and put $M = \Gamma(X, \mathscr{M})$. If ρ_α: $M \to M_\alpha$ is the restriction map, then* $\ker \rho_\alpha = \tau_\alpha(M)$.

PROOF. If $m \in \tau_\alpha(M)$, then $Im = 0$ for some $I \in \mathscr{F}_\alpha$. Hence

$$\rho_\alpha(m) \in R_\alpha \rho_\alpha(m) = R_\alpha I \rho_\alpha(m) = R_\alpha \rho_\alpha(Im) = 0.$$

So $\tau_\alpha(M) \subset \ker \rho_\alpha$.

Put $N = \ker \rho_\alpha$, and $N_\beta = \rho_\beta(N)$. As $\mathscr{R}$ is quasicoherent, $\rho_{\alpha\beta}(N_\beta) = R_{\alpha\beta} \otimes_{R_\beta} N_\beta$. But $\rho_{\alpha\beta}(N_\beta) = \rho_{\alpha\beta}\rho_\beta(N) = \rho_{\alpha\beta}\rho_\alpha(N) = 0$. Hence, by the previous lemma, $R_\alpha \otimes_R N_\beta = 0$, and so each $N_\beta \in \mathscr{T}_\alpha$.

Pick $n \in N$, and put $n_\beta = n|_{U_\beta}$. By the previous paragraph, for each β there exists $I_\beta \in \mathscr{F}_\alpha$ with $I_\beta n_\beta = 0$. Put $I = \bigcap_\beta I_\beta$. Thus $I \in \mathscr{F}_\alpha$ and $In_\beta = 0$ for all β. But $In_\beta = In|_{U_\beta}$; as In is locally zero and the U_β cover X, In is globally zero. That is $In = 0$, and hence $n \in \tau_\alpha(M)$.

2.9. LEMMA. *Let* $\mathscr{M} \in \mathscr{R}\text{-}\mathscr{M}od$, *and put* $M = \Gamma(X, \mathscr{M})$. *Let* $\rho_\alpha\colon M \to M_\alpha = \Gamma(U_\alpha, \mathscr{M})$ *be the restriction map. Then* $M_\alpha/\rho_\alpha(M)$ *is in* $\mathscr{T}_\alpha$ *(that is,* $R_\alpha \otimes_R (M_\alpha/\rho_\alpha(M)) = 0$*).*

PROOF. We need to show that for each $m \in M_\alpha$, there exists $I \in \mathscr{F}_\alpha$ with $Im \subset \rho_\alpha(M)$.

Let $m \in M_\alpha$. Put $m_\beta = \rho_{\alpha\beta}(m) \in M_{\alpha\beta}$. As $\mathscr{R}$ is quasicoherent, $R_{\alpha\beta} \otimes_{R_\beta} (M_{\alpha\beta}/\rho_{\alpha\beta}(M_\beta)) = 0$, so by Lemma 2.7, $R_\alpha \otimes_R (M_{\alpha\beta}/\rho_{\alpha\beta}(M_\beta)) = 0$. So for each β, there exists $I_\beta \in \mathscr{F}_\alpha$ with $I_\beta m_\beta \subset \rho_{\alpha\beta}(M_\beta)$. Put $J = \bigcap_\beta I_\beta \in \mathscr{F}_\alpha$. If $y \in J$ we can pick for each β an element $m'_\beta \in M_\beta$ with $ym_\beta = \rho_{\alpha\beta}(m'_\beta) = m'_\beta|_{U_{\alpha\beta}}$. Notice that $m'_\beta|_{U_{\alpha\beta\gamma}} = ym_\beta|_{U_{\alpha\beta\gamma}} = ym|_{U_{\alpha\beta\gamma}}$ so the elements $m'_\beta|_{U_{\beta\gamma}}$ and $m'_\gamma|_{U_{\beta\gamma}}$ have the same restriction to $U_{\alpha\beta\gamma}$. In other words $(m'_\beta|_{U_{\beta\gamma}} - m'_\gamma|_{U_{\beta\gamma}}) \in M_{\beta\gamma}$ restricts to zero on $U_{\alpha\beta\gamma}$, and hence this element is in the kernel of $\rho_{\alpha\beta\gamma}\colon M_{\beta\gamma} \to M_{\alpha\beta\gamma}$. But this kernel is $\tau_\alpha(M_{\beta\gamma})$ so for each pair (β, γ) there exists $I_{\beta\gamma} \in \mathscr{F}_\alpha$ with $I_{\beta\gamma}(m'_\beta|_{U_{\beta\gamma}} - m'_\gamma|_{U_{\beta\gamma}}) = 0$. Put $J' = \bigcap_{\beta,\gamma} I_{\beta\gamma} \in \mathscr{F}_\alpha$. If $z \in J'$, then $zm'_\beta|_{U_{\beta\gamma}} = zm'_\gamma|_{U_{\beta\gamma}}$, so the elements zm'_β glue to give a global section; that is, there exists $\overline{m} \in M$ with $\overline{m}|_{U_\beta} = zm'_\beta$ for all β. By definition, $m_\alpha = m$, so $m'_\alpha = ym_\alpha = ym$. Hence $\overline{m}|_{U_\alpha} = zym$ and $zym \in \rho_\alpha(M)$. As $z \in J'$ was arbitrary, $J'ym \in \rho_\alpha(M)$.

The choice of J' depended on y. Pick elements $y_1, \ldots, y_s$ which generate J and pick corresponding left ideals $J'_1, \ldots, J'_s$. Put $J'' = J'_1 \cap \cdots \cap J'_s$. We still have $J'' \in \mathscr{F}_\alpha$ and $J''_i m \in \rho_\alpha(M)$ for $i = 1, \ldots, s$. Hence if $I = J''y_1 + \cdots + J''y_s$, then $Im \in \rho_\alpha(M)$.

It remains to show that $I \in \mathscr{F}_\alpha$. As $R/J \in \mathscr{T}_\alpha$, it is enough to show that $J/I \in \mathscr{T}_\alpha$. As J is generated by the y_i, it is enough to show that each $Ry_i + I/I \in \mathscr{T}_\alpha$. But this module is a homomorphic image of R/J'' which is itself in $\mathscr{T}_\alpha$.

2.10. PROPOSITION (NOTATION AS IN LEMMA 2.8). *The restriction map* $\rho_\alpha\colon M \to M_\alpha$ *extends to an isomorphism* $\phi\colon R_\alpha \otimes_R M \to M_\alpha$.

PROOF. Let ϕ denote the canonical extension of ρ_α; that is $\phi(r \otimes m) = r\rho_\alpha(m)$ for $r \in R_\alpha$, $m \in M$. The diagram

$$\begin{array}{ccc} M & \xrightarrow{j} & R_\alpha \otimes_R M \\ & \rho_\alpha \searrow & \downarrow \phi \\ & & M_\alpha \end{array}$$

commutes, where $j(m) = 1 \otimes m$.

As $j(M)$ is an essential submodule of $R_\alpha \otimes_R M$, if $\ker\phi \neq 0$, then $\ker\phi \cap j(M) \neq 0$. Pick $m \in M$ with $j(m) \neq 0$ but $\phi j(m) = 0$. Then $\rho_\alpha(m) = 0$ but $j(m) \neq 0$. This is a contradiction as $\ker\rho_\alpha = \tau_\alpha(M) = \ker j$ by Lemma 2.8. Hence ϕ is injective.

Let $m \in M_\alpha$. By Lemma 2.9, there exists $I \in \mathscr{F}_\alpha$ with $Im \subset \rho_\alpha(M)$. But $m \in R_\alpha m = R_\alpha Im \subset R_\alpha\rho_\alpha(M) = im\phi$. Hence ϕ is surjective.

2.11. COROLLARY. *Let* $\mathscr{M} \in \mathscr{R}\text{-}\mathscr{M}od$ *and put* $M = \Gamma(X, \mathscr{M})$. *Then* $\mathscr{M} \simeq \mathscr{R} \otimes_R M$.

PROOF. If $U \subset X$ is any open affine subset, the restriction map $M \to M_U = \Gamma(U, \mathscr{M})$ extends to a map $R_U \otimes_R M \to M_U$. Hence there is a morphism of presheaves $\mathscr{R} \otimes_R M \to \mathscr{M}$, and so a morphism of sheaves $\mathscr{R} \otimes_R M \to \mathscr{M}$.

Because $\mathscr{R}$ is quasicoherent, $\mathscr{R} \otimes_R M|_{U_\alpha} \simeq \mathscr{R}|_{U_\alpha} \otimes_R M$, and so for any open affine $V \subset U_\alpha$, $\Gamma(V, \mathscr{R} \otimes_R M) = R_V \otimes_R M$. A consequence of the previous proposition is that if $V \subset U_\alpha$ is open affine then the natural map $R_V \otimes_R M \to M_V = \Gamma(V, \mathscr{M})$ is an isomorphism. Hence the morphism $\mathscr{R} \otimes_R M \to \mathscr{M}$ induces isomorphisms on the stalks, and so is an isomorphism itself.

In other words, each $\mathscr{M} \in \mathscr{R}\text{-}\mathscr{M}od$ is generated by its global sections.

2.12. PROPOSITION. *Let N be an R-module and put $\mathscr{M} = \mathscr{R} \otimes_R N$. Then $\Gamma(X, \mathscr{M}) = N$.*

PROOF. Put $M = \Gamma(X, \mathscr{M})$. First we show there exists an injective map from N into M.

Define a map $N \to M$ by associating with each $n \in N$ the global section s of $\mathscr{M}$ such that $s(x) = 1 \otimes n \in R_x \otimes_R N$ for each $x \in X$, where R_x is the stalk of $\mathscr{R}$ at x. Pick U_α containing x. As $\mathscr{R}$ is quasicoherent, $R_x = A_x \otimes_{A_\alpha} R_\alpha$ and $R_x \otimes_R N = A_x \otimes_{A_\alpha} R_\alpha \otimes_R N$. If $s = 0$ then $1 \otimes 1 \otimes n \in A_x \otimes_{A_\alpha} R_\alpha \otimes_R N$ is zero for all $x \in U_\alpha$. But as $\prod_{x \in U_\alpha} A_x$ is faithfully flat over A_α, $1 \otimes n$ considered as an element of $R_\alpha \otimes_R N$ is zero. Hence $1 \otimes n \in R_\alpha \otimes_R N$ is zero for all α. But $\oplus R_\alpha$ is faithfully flat over R so $n = 0$.

As R_α is flat over R, the map above extends to an injective map $R_\alpha \otimes_R N \to R_\alpha \otimes_R M$. As $\mathscr{R}$ is quasicoherent, the restriction of the presheaf $\mathscr{R} \otimes_R N$ to U_α is a sheaf, and hence $\Gamma(U_\alpha, \mathscr{M}) = R_\alpha \otimes_R N$. So the map $R_\alpha \otimes_R N \to R_\alpha \otimes_R M$ is a bijection by Proposition 2.10. In particular, as R_α is flat, $R_\alpha \otimes_R (M/N) = 0$. But this is true for all α, so the faithful flatness of $\oplus R_\alpha$ implies that $M/N = 0$. Hence $M = N$ as required.

It should be remarked that the quasicoherence of $\mathscr{R}$ implies that $\mathscr{R} \otimes_R N$ is also quasicoherent.

2.13. PROOF OF THE THEOREM. The Theorem is an immediate consequence of Corollary 2.11 and Proposition 2.12. We have shown that the functors $M \to \mathscr{R} \otimes_R M$ and $\mathscr{M} \to \Gamma(X, \mathscr{M})$ are mutually inverse to one another when considered as maps from the objects of one category to the objects of the the other category. It is immediate that an R-module homomorphism $M \to M'$ extends to a morphism of presheaves $\mathscr{R} \otimes_R M \to \mathscr{R} \otimes_R M'$, and hence gives a morphism of sheaves $\mathscr{M} \to \mathscr{M}'$. Likewise a morphism $\mathscr{M} \to \mathscr{M}'$ gives a morphism $\Gamma(X, \mathscr{M}) \to \Gamma(X, \mathscr{M}')$. It follows that the two functors are inverse to one another when considered as maps on the morphisms.

2.14. In **[BB]** the equivalence of categories is proved by showing that every $\mathscr{M} \in \mathscr{D}_\lambda\text{-}\mathscr{M}od$ is generated by its global sections and that $H^i(X, \mathscr{M}) = 0$ for $i > 0$. We have been unable to show under the hypotheses of §1.2 that $H^i(X, \mathscr{M}) = 0$ for $i > 0$. However, if we assume that $\mathscr{R}$ is flat over X (that is, R_x is a flat A_x-module for each $x \in X$) then we have the following result. We remark first that it is easy to show each $\mathscr{D}_\lambda$ satisfies this condition (the stalk of $\mathscr{D}_\lambda$ at x is just the ring of differential operators on the regular local ring A_x, and the ring of differential operators on A_x is free as an A_x-module).

PROPOSITION. *If $\mathscr{R}$ is flat over X, then each $\mathscr{M} \in \mathscr{R}\text{-}\mathscr{M}od$ satisfies $H^i(X, \mathscr{M}) = 0$ for $i > 0$.*

PROOF. Put $M = \Gamma(X, \mathscr{M})$ and take an injective resolution $0 \to M \to E_1 \to E_2 \to \cdots$ of the R-module M. By the equivalence of categories $0 \to \mathscr{M} \to \mathscr{R} \otimes_R E_1 \to \mathscr{R} \otimes_R E_2 \to \cdots$ is an injective resolution in the category $\mathscr{R}\text{-}\mathscr{M}od$.

CLAIM. If E is an injective R-module, then $\mathscr{R} \otimes_R E$ is injective in the category of sheaves of quasicoherent $\mathscr{O}$-modules.

Let $0 \to \mathfrak{G}' \to \mathfrak{G}$ be an exact sequence of quasicoherent $\mathscr{O}$-modules and suppose f: $\mathfrak{G}' \to \mathscr{R} \otimes_R E$ is a morphism of $\mathscr{O}$-modules. To prove the claim it suffices to show that there exists a morphism g: $\mathfrak{G} \to \mathscr{R} \otimes_R E$ whose restriction to $\mathfrak{G}'$ is f.

The natural map $\mathscr{R} \otimes_{\mathscr{O}} \mathfrak{G}' \to \mathscr{R} \otimes_{\mathscr{O}} \mathfrak{G}$ is injective on stalks as each R_x is a flat A_x-module, so is an injective morphism of $\mathscr{O}$-modules. A tensor product of quasicoherent $\mathscr{O}$-modules is again quasicoherent, so both these modules are in $\mathscr{R}\text{-}\mathscr{M}od$.

The morphism f induces a morphism $\bar{f}$: $\mathscr{R} \otimes_{\mathscr{O}} \mathfrak{G}' \to \mathscr{R} \otimes_R E$. Now as $\mathscr{R} \otimes_R E$ is injective in $\mathscr{R}\text{-}\mathscr{M}od$, $\bar{f}$ extends to a map $\bar{g}$: $\mathscr{R} \otimes_{\mathscr{O}} \mathfrak{G} \to \mathscr{R} \otimes_R E$. The map g is then the composition $\mathfrak{G} \to \mathscr{R} \otimes \mathfrak{G} \to \mathscr{R} \otimes E$. This proves the claim.

Now, by [**H**, III, Example 3.6], $\mathscr{R} \otimes_R E$ is flasque so we may calculate cohomology from the sequence $0 \to \mathscr{M} \to \mathscr{R} \otimes E_1 \to \mathscr{R} \otimes E_2 \to \cdots$ after applying $\Gamma(X, -)$. But, by the equivalence of categories, this gives the exact sequence $0 \to M \to E_1 \to E_2 \to \cdots$. Hence $H^i(X, \mathscr{M}) = 0$ for $i > 0$.

2.15. As remarked in §1.3, condition (iii) of the assumptions in §1.2 follows from the equivalence of categories. A precise statement of this is the following

PROPOSITION. *Suppose that $\mathscr{R}$ satisfies conditions* (i) *and* (ii) *of* §1.2, *and that there is an equivalence between the categories $\mathscr{R}\text{-}\mathscr{M}od$ and R-*Mod *given by the mutually inverse functors $M \to \mathscr{R} \otimes_R M$, $\mathscr{M} \to \Gamma(X, \mathscr{M})$. Then condition* (iii) *of* §1.2 *also holds.*

PROOF. First we show that if $U \subset X$ is open affine, then R_U is flat as a right R-module.

Let $0 \to M_1 \to M_2$ be an exact sequence of R-modules. Put $\mathscr{M}_i = \mathscr{R} \otimes_R M_i$. The assumption implies that $0 \to \mathscr{M}_1 \to \mathscr{M}_2$ is exact in $\mathscr{R}\text{-}\mathscr{M}od$. Hence $0 \to \Gamma(U, \mathscr{M}_1) \to \Gamma(U, \mathscr{M}_2)$ is an exact sequence of R_U-modules. But, as $\mathscr{R}$ is quasicoherent and U is open affine, $\Gamma(U, \mathscr{M}_i) = R_U \otimes_R M_i$ (this is because the presheaf $\mathscr{R} \otimes M|_U$ is already a sheaf over U). Hence $0 \to R_U \otimes_R M_1 \to R_U \otimes_R M_2$ is exact, and R_U is flat.

Suppose (U_α) is any open affine cover for X. Put $R_\alpha = \Gamma(U_\alpha, \mathscr{R})$ and suppose M is an R-module with $R_\alpha \otimes_R M = 0$ for all α. As just remarked, $R_\alpha \otimes_R M = \Gamma(U_\alpha, \mathscr{R} \otimes M)$. So $\mathscr{R} \otimes M$ is locally zero, hence globally zero. The equivalence of categories implies that $M = \Gamma(X, \mathscr{R} \otimes_R M)$ is also zero. Hence $\oplus R_\alpha$ is faithfully flat.

In the proof of the proposition the choice of the U_α was arbitrary, so we have

COROLLARY. *If $\mathscr{R}$ satisfies the conditions of* §1.2, *then condition* (iii) *will hold for any open affine cover (U_α) of X.*

3. Comments and remarks.

3.1. We adopt here the notation of [**BB**]. The underlying topological space of the sheaf $\mathcal{D}_\lambda$ is $X = G/B$, the flag variety of the connected complex semisimple Lie group corresponding to the complex semisimple Lie algebra $\mathbf{g}$. The cover (U_α) in this case could be obtained by taking the translates of the large Bruhat cell under the action of the Weyl group; that is each $U_\alpha = wBw_0B$ for some $w \in W$, the Weyl group, where w_0 is the long element of W. Each R_α is isomorphic to the Weyl algebra A_n, where $n = \dim X$. In [**HS**] it was shown that the restriction map $D_\lambda = \Gamma(X, \mathcal{D}_\lambda) \to \Gamma(Bw_0B, \mathcal{D}_\lambda) = A$ is (up to an automorphism of $\mathbf{g}$ and an automorphism of A) the Conze embedding $D_{w_0\lambda} \to A$ obtained through the action of $\mathbf{g}$ on the Verma module $M(w_0\lambda)$. In order to establish (iii) directly one would need to show (among other things) that the embedding $D_{w_0\lambda} \to A$ makes A flat as a right $D_{w_0\lambda}$-module.

It is shown in [**JS**] for λ dominant regular that A is the union of the right $D_{w_0\lambda}$-modules $L(w_0\lambda, w_0\mu)$ (of $\mathbf{g}$-finite maps from the Verma module $M(w_0\lambda)$ to $M(w_0\mu)$), where the union is taken over all dominant regular μ such that $\mu - \lambda$ is dominant integral. It already follows from [**BG**] that the $L(w_0\lambda, w_0\mu)$ are finitely generated projective right $D_{w_0\lambda}$-modules, and hence A is flat as a right $D_{w_0\lambda}$-module.

3.2. The hypothesis that X was irreducible was made only to avoid the possibility that some of the $U_{\alpha_1\cdots\alpha_n}$ might be empty and therefore that some of the $R_{\alpha_1\cdots\alpha_n}$ might be zero. If one assumes that X has a finite number of irreducible components X_i, and the hypotheses in §1.2 are adjusted so that $\mathcal{R}|_{X_i}$ satisfies (i)–(iii), then one can still show that R-Mod and $\mathcal{R}$-$\mathcal{M}od$ are equivalent.

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