SHEAVES OF NONCOMMUTATIVE ALGEBRAS AND THE BEILINSON-BERNSTEIN EQUIVALENCE OF CATEGORIES

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Abstract. Let $X$ be an irreducible algebraic variety defined over a field $k$, let $\mathcal{A}$ be a sheaf of (noncommutative) noetherian $k$-algebras on $X$ containing the sheaf of regular functions $\mathcal{O}$ and let $R$ be the ring of global sections. We show that under quite reasonable abstract hypotheses (concerning the existence of a faithfully flat overring of $R$ obtained from the local sections of $\mathcal{A}$) there is an equivalence between the category of $R$-modules and the category of sheaves of $\mathcal{A}$-modules which are quasicoherent as $\mathcal{O}$-modules. This shows that the equivalence of categories established by Beilinson and Bernstein as the first step in their proof of the Kazhdan-Lusztig conjectures (where $R$ is a primitive factor ring of the enveloping algebra of a complex semisimple Lie algebra, and $\mathcal{A}$ is a sheaf of twisted differential operators on a generalised flag variety) is valid for more fundamental reasons than is apparent from their work.

1. Introduction.

1.1. This paper is motivated by the Beilinson-Bernstein Theorem [BB]. Very briefly, this says that for certain primitive factor rings $D_\lambda$ of the enveloping algebra of a complex semisimple Lie algebra there is an associated sheaf $\mathcal{D}_\lambda$ of noncommutative algebras over a complex projective algebraic variety $X$ such that $D_\lambda = \Gamma(X, \mathcal{D}_\lambda)$ and there is an equivalence between the category of left $D_\lambda$-modules and the category of sheaves of left $\mathcal{D}_\lambda$-modules which are quasicoherent as $\mathcal{O}$-modules ($\mathcal{O}$ is the structure sheaf of $X$, and is a subsheaf of $\mathcal{D}_\lambda$). In this paper we abstract some of the essential features of their construction in an attempt to understand what ring theoretic properties of $D_\lambda$ ensure this equivalence of categories. We show that the equivalence of categories follows from the existence of a faithfully flat overring of $D_\lambda$.

Let us give the details.

1.2. Let $k$ be any field and $X$ an irreducible algebraic variety over $k$. Let $\mathcal{A}$ be a sheaf of noncommutative noetherian $k$-algebras over $X$. Set $R = \Gamma(X, \mathcal{A})$ and suppose that $R$ has a classical ring of quotients $Q$. If $U$ is an open affine subset of $X$ write $R_U = \Gamma(U, \mathcal{A})$. We make the following assumptions concerning $\mathcal{A}$ from which the Theorem below will be deduced.

(i) The structure sheaf $\mathcal{O}$ of $X$ is a subsheaf of $\mathcal{A}$ and $\mathcal{A}$ is a quasicoherent sheaf of left $\mathcal{O}$-modules.
(ii) If $U \subset X$ is an open affine subset of $X$, then $R_U$ is a subalgebra of $Q$ containing $R$ and is generated as a right or left $R$-module by $\Gamma(U, \mathcal{O})$.

(iii) There is a finite open affine cover $(U_a)$ for $X$ such that the diagonal embedding $R \to \bigoplus R_a$ (where $R_a = \Gamma(U_a, \mathcal{O})$) obtained from the restriction maps makes $\bigoplus R_a$ a faithfully flat right $R$-module.

Write $R$-$\text{Mod}$ for the category of left $R$-modules and $\mathcal{R} M \text{od}$ for the category of sheaves of left $\mathcal{R}$-modules which are quasicoherent as $\mathcal{O}$-modules.

**Theorem.** There is an equivalence between $R$-$\text{Mod}$ and $\mathcal{R} M \text{od}$ given by the mutually inverse functors $M \to \mathcal{R} \otimes_R M$ and $\mathcal{M} \to \Gamma(X, \mathcal{M})$.

1.3. It is straightforward to check that $\mathcal{R}_a$ satisfies (i) and (ii) above. Although (iii) follows from the equivalence of categories established in [BB] we have not been able to establish its truth on a priori grounds except in some special cases. However, there are reasons for expecting this may be possible (see §3.1) and then the Theorem would imply the result of Beilinson and Bernstein. Joseph and Stafford [JS] have given a direct proof of the flatness condition in (iii), so one only needs to find a direct proof of the “faithfulness” condition.

1.4. This paper is primarily written for noncommutative ring theorists. The paper can be read without a knowledge of the work of Beilinson and Bernstein, but we expect that an understanding of their construction would make the work here more meaningful.

2. **Proof of the Theorem.**

2.1. The Theorem will be proved using the language of torsion theory. We recall the standard terminology and results which we require. The reader is referred to [St] for a thorough treatment and proofs.

Let $R$ be a ring and $A$ a ring containing $R$ (with the same identity). If $A$ is flat as a right $R$-module and $A \otimes_R A = A$ (under the map $a \otimes b \to ab$) as an $A$-$A$ bimodule we call $A$ a perfect (left) localisation of $R$. Denote by $\mathcal{F}$ the class of all left $R$-modules $M$ satisfying $A \otimes_R M = 0$. We call $\mathcal{F}$ the torsion class associated to $A$. Because $A_R$ is flat, the class $\mathcal{F}$ is closed under submodules, factor modules and extensions. If $M$ is an $R$-module there is a largest submodule of $M$ contained in $\mathcal{T}$; we denote this submodule by $\tau(M)$ and call it the torsion submodule of $M$. If $M$ is a torsion module, and if $\tau(M) = 0$ we say that $M$ is torsion-free. Denote by $\mathcal{G}$ the Gabriel filter associated to $A$.

If $a \in A$, there exists an $I$ in $\mathcal{F}$ such that $Ia \subset R$. If $M$ is an $R$-module, then $\tau(M) = \{ m \in M \mid Im = 0 \text{ for some } I \text{ in } \mathcal{F} \}$. The kernel of the natural map $\phi: M \to A \otimes_R M$ is $\tau(M)$ and $A \otimes M/\phi(M)$ is a torsion module. If $I$ and $J$ are in $\mathcal{F}$ so are $I \cap J$ and $IJ$. The filter $\mathcal{F}$ may be characterized as those left ideals $I$ of $R$ such that $R/I$ is a torsion module. We call $\mathcal{F}$ the Gabriel filter associated to $A$.

2.2. Consider the situation described in §1.2. For simplicity we shall write $U_{a_1 \ldots a_n} = U_{a_1} \cap \cdots \cap U_{a_n}$ and $R_{a_1 \ldots a_n} = \Gamma(U_{a_1 \ldots a_n}, \mathcal{O})$. If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules we shall write $M = \Gamma(X, \mathcal{M})$ and $M_{a_1 \ldots a_n} = \Gamma(U_{a_1 \ldots a_n}, \mathcal{M})$. Abusing notation, we shall write $\rho_{a_1 \ldots a_n}$ for the restriction map $M_{\beta_1 \ldots \beta_m} \to M_{a_1 \ldots a_n}$ for any subset
\{ \beta_1, \ldots, \beta_m \} \subset \{ \alpha_1, \ldots, \alpha_n \}. No confusion should arise as the domain of \( \rho_{\alpha_1 \cdots \alpha_n} \) will always be clear from the context.

The stalk of \( \mathcal{O} \) at \( x \in X \) will be denoted by \( R_x \). If \( V \subset X \) is an open subset, denote \( \Gamma(V, \mathcal{O}) \) by \( A_V \), and write \( A_x \) for the stalk of \( \mathcal{O} \) at \( x \).

2.3. Let us point out some immediate consequences of the assumptions made in §1.2.

The quasicoherence of \( \mathcal{O} \) ensures that if \( U \subset V \) are open affine subsets of \( X \), then \( R_U = A_U \otimes_{A_V} R_V \). If \( V \subset X \) is open affine, then there is an equivalence between the categories \( \mathcal{O}_V \text{-Mod} \) (of left \( \mathcal{O}_V \text{-modules} \)) and \( \mathcal{O}_V \text{-Mod} \) (of sheaves of left \( \mathcal{O}_V \text{-modules} \) which are quasicoherent as \( \mathcal{O}_V \text{-modules} \)). If \( M \) is an \( \mathcal{O}_V \text{-module} \), then view \( M \) as an \( \mathcal{O}_V \text{-module} \) and form the sheaf \( \tilde{M} \) (the notation is that of [H]). If \( U \subset V \) is an open affine subset, then

\[ \Gamma(U, \tilde{M}) = A_U \otimes_{A_V} M = A_U \otimes_{A_V} (R_V \otimes_{R_U} M) = R_U \otimes_{R_V} M, \]

so \( \tilde{M} \) is a sheaf of \( \mathcal{O}_V \text{-modules} \) with global sections \( M \). Also, if \( M \in \mathcal{O}_V \text{-Mod} \), then \( M = \Gamma(V, M) \) and \( \Gamma(V, M) \) is an \( \mathcal{O}_V \text{-module} \). Hence the equivalence between \( \mathcal{O}_V \text{-Mod} \) and \( \mathcal{O}_V \text{-Mod} \).

2.4. It is not strictly necessary to assume that \( \mathcal{O} \) has a classical ring of fractions \( Q \). One could replace \( Q \) in the hypotheses by the maximal flat epimorphic extension of \( R \). The existence of \( Q \) is only used in applying the following result.

**Proposition [St, Chapter XI, Proposition 2.4].** If \( A \) is a ring with \( R \subset A \subset Q \) and \( A_R \) is flat, then \( A \) is a perfect left localisation of \( R \).

**Corollary.** If \( U \subset U_a \) and \( U \) is open affine, then \( R_U \) is a perfect left localisation of \( R \).

**Proof.** By assumption, \( R_a \) is flat as a right \( R \)-module, so is a perfect left localisation by the Proposition.

Put \( A_x = \Gamma(U_a, \mathcal{O}) \). As \( \mathcal{O} \) is quasicoherent, \( R_U = A_U \otimes_{A_x} R_a \). Now, as \( A_U \) is flat as an \( A_x \text{-module} \), \( R_U \) is flat as a right \( A_x \text{-module} \). Hence \( R_U \) is flat as a right \( R \)-module. Apply the Proposition to \( R_U \).

Thus the hypotheses in §1.2 ensure that each \( R_{\alpha_1 \cdots \alpha_n} \) is a perfect left localisation of \( R \). We shall denote the torsion class associated to the perfect left localisation \( R_a \) of \( R \) by \( \mathcal{T}_a \). The associated Gabriel filter will be denoted by \( \mathcal{T}_a \), and the associated torsion functor will be denoted by \( _\mathcal{T}_a \).

2.5. **Lemma.** Let \( R \) be a noetherian ring with ring of fractions \( Q \). Let \( A_1 \) and \( A_2 \) be perfect left localisations of \( R \). Suppose that each \( A_i \) is generated as either a right or left \( R \)-module by a subset \( S_i \) and that the elements of \( S_1 \) commute with the elements of \( S_2 \). Let \( B \) denote the subring of \( Q \) generated by \( A_1 \) and \( A_2 \). Then there is an isomorphism of \( A_1 - A_2 \) bimodules \( A_1 \otimes_R A_2 \to B \) given by \( a_1 \otimes a_2 \to a_1 a_2 \), and hence there is an isomorphism of \( R-R \) bimodules \( A_1 \otimes_R A_2 = A_2 \otimes_R A_1 \).

**Proof.** If \( A \) is a perfect left localisation of \( R \), the map \( A \otimes_R A \to A \) given by \( a \otimes a' \to aa' \) is an \( A-A \) bimodule isomorphism.

The natural maps

\[ A_1 \otimes_R B \to A_1 \otimes (A_1 \otimes_A B) \to (A_1 \otimes_R A_1) \otimes_A B \to A_1 \otimes A_1 B \to B \]
are all isomorphisms of $A_1$-$B$ bimodules, so the composition gives $A_1 \otimes_R B \simeq B$ by $a \otimes b \to ab$.

Tensor the exact sequence $0 \to A_2 \to B \to B/A_2 \to 0$ with the flat $R$-module $A_1$. To prove the first part of the lemma it is enough to prove that $A_1 \otimes_R (B/A_2) = 0$, because one may then compose the isomorphisms $A_1 \otimes_R A_2 \to A_1 \otimes_R B \to B$ to get the result.

A typical element of $B$ is a finite sum of terms of the form $a_1a_2$ with $a_1 \in A_1$ and $a_2 \in A_2$ (just use the fact that $S_1$ and $S_2$ commute, and that $A_1 = S_1 R = R S_1$). Let $F$ denote the Gabriel filter associated to $A_1$. Given $a_1 \in A_1$, $a_2 \in A_2$ pick $I$ in $F$ such that $I a_1 \subset R$; whence $I a_1 a_2 \subset A_2$. So the image of $a_1a_2$ in $B/A_2$ is in the torsion submodule (associated to $A_1$). As every element of $B/A_2$ is a finite sum of torsion elements, $B/A_2$ is itself torsion. In other words $A_1 \otimes_R (B/A_2) = 0$.

We have shown $A_1 \otimes_R A_2 \simeq B$ as $A_1$-$A_2$ bimodules, hence as $R$-$R$ bimodules. Reversing the roles of $A_2$ and $A_1$, we also have $A_2 \otimes_R A_1 \simeq B$ as $R$-$R$ bimodules. The lemma follows.

In the commutative case the isomorphism $A_1 \otimes_R A_2 \to A_2 \otimes_R A_1$ would just be $a_1 \otimes a_2 \to a_2 \otimes a_1$. In the noncommutative case the map is not so clear; first $a_1 \otimes a_2$ is mapped to $a_1a_2 \in B$, then $a_1a_2$ is expressed as a sum of terms of the form $a'_2 a'_1$ with $a'_2 \in A_2$, $a'_1 \in A_1$ and each $a'_2 a'_1$ is mapped to $a'_2 \otimes a'_1 \in A_2 \otimes A_1$.

2.6. Lemma. $R_{a_1} \otimes_R \cdots \otimes_R R_{a_n} \simeq R_{a_1 \cdots a_n}$ as $R_{a_1 \cdots a_n}$ bimodules.

Proof. For each $m \leq n$, $R_{a_1 \cdots a_m}$ is a perfect left localization of $R$ (as remarked in §2.4). Now apply Lemma 2.5 inductively. Notice that to apply 2.5 one must use the fact that $\Gamma(U_a \cap U_{a'}, \emptyset)$ is generated by the images (under the restriction maps) of $\Gamma(U_a, \emptyset)$ and $\Gamma(U_{a'}, \emptyset)$, and this is guaranteed by the assumption that $X$ is a variety.

2.7. Lemma. Let $N$ be an $R_{a_1 \cdots a_n}$-module such that $R_{a_1 \cdots a_{n+1}} \otimes N = 0$, where the tensor product is over $R_{a_1 \cdots a_{n+1}}$. Then $R_{a_{n+1}} \otimes_R N = 0$.

Proof. $R_{a_{n+1}} \otimes_R N = R_{a_{n+1}} \otimes_R \left( R_{a_1 \cdots a_n} \otimes N \right) = \left( R_{a_{n+1}} \otimes_R R_{a_1 \cdots a_n} \right) \otimes N,$ where the second tensor product is over $R_{a_1 \cdots a_n}$. Now apply the previous lemma.

2.8. Lemma. Let $M \in \mathcal{R}, M \text{mod}$ and put $M = \Gamma(X, \mathcal{M})$. If $\rho_a: M \to M_a$ is the restriction map, then $\ker \rho_a = \tau_a(M)$.

Proof. If $m \in \tau_a(M)$, then $I m = 0$ for some $I \in \mathcal{I}_a$. Hence $\rho_a(m) \in R_a \rho_a(m) = R_a I \rho_a(m) = R_a \rho_a(I m) = 0$. So $\tau_a(M) \subset \ker \rho_a$.

Put $N = \ker \rho_a$, and $N_B = \rho_B(N)$. As $\mathcal{I}$ is quasicoherent, $\rho_{a\beta}(N_B) = R_{a\beta} \otimes_R N_B$. But $\rho_{a\beta}(N_B) = \rho_{a\beta \rho_a}(N) = \rho_{a\beta \rho_a}(N) = 0$. Hence, by the previous lemma, $R_a \otimes_R N_B = 0$, and so each $N_B \in \mathcal{F}_a$.

Pick $n \in N$, and put $n_B = n|_{U_B}$. By the previous paragraph, for each $\beta$ there exists $I_\beta \subset \mathcal{I}_a$ with $I_\beta | U_\beta = 0$. Put $I \cap I_\beta$. Thus $I \in \mathcal{I}_a$ and $I \cap \beta = 0$ for all $\beta$. But $I \cap \beta = I n|_{U_\beta}$ as $I n$ is locally zero and the $U_\beta$ cover $X$, $I n$ is globally zero. That is $In = 0$, and hence $n \in \tau_a(M)$.
2.9. **Lemma.** Let $\mathcal{M} \in \mathcal{R}$-mod, and put $M = \Gamma(X, \mathcal{M})$. Let $\rho_a: M \to M_a = \Gamma(U_a, \mathcal{M})$ be the restriction map. Then $M_a/\rho_a(M)$ is in $\mathcal{F}_a$ (that is, $R_a \otimes_R (M_a/\rho_a(M)) = 0$).

**Proof.** We need to show that for each $m \in M_a$, there exists $I \in \mathcal{F}_a$ with $Im \subseteq \rho_a(M)$.

Let $m \in M_a$. Put $m_\beta = \rho_{a\beta}(m) \in M_{a\beta}$. As $\mathcal{R}$ is quasicoherent, $R_{a\beta} \otimes_R (M_{a\beta}/\rho_{a\beta}(M_{a\beta})) = 0$. So for each $\beta$, there exists $I_\beta \in \mathcal{F}_a$ with $I_\beta m_\beta \subseteq \rho_{a\beta}(M_{a\beta})$. Put $J = \bigcap I_\beta \in \mathcal{F}_a$. If $y \in J$ we can pick for each $\beta$ an element $m'_\beta \in M_\beta$ with $ym_\beta = \rho_{a\beta}(m'_\beta) = m'_\beta|_{U_\alpha}$.

Notice that $m'_\beta|_{U_{a\beta}} = ym_\beta|_{U_{a\beta}} = ym|_{U_\alpha}$, so the elements $m'_\beta|_{U_\alpha}$ and $m'_\gamma|_{U_\alpha}$ have the same restriction to $U_{a\alpha}$. In other words, $m'_\beta|_{U_{a\beta}} = m'_\gamma|_{U_{a\beta}}$, restrics to zero on $U_{a\beta}$, and hence this element is in the kernel of $\rho_{a\beta}(M_\beta)/M_{a\beta}$. But this kernel is $\tau_a(M_\beta)$ so for each pair $(\beta, \gamma)$ there exists $m_{\beta\gamma} \in \mathcal{F}_a$ with $m_{\beta\gamma}|_{U_{a\beta}} - m_{\gamma\alpha}|_{U_{a\gamma}} = 0$. Put $J' = \bigcap I_{\beta\gamma} \in \mathcal{F}_a$. If $z \in J'$, then $zm_{\beta\gamma}|_{U_{a\beta}} = zm_{\gamma\alpha}|_{U_{a\gamma}}$, so the elements $zm_{\beta\gamma}$ glue to give a global section; that is, there exists $m \in M$ with $m|_{U_\alpha} = zm$ for all $\beta$. By definition, $m_\alpha = m$, so $m'_\alpha = ym_\alpha = zm$. Hence $m|_{U_\alpha} = zm$ and $zm \in \rho_a(M)$. As $z \in J'$ was arbitrary, $J'y \in \rho_a(M)$.

The choice of $J'$ depended on $y$. Pick elements $y_1, \ldots, y_s$ which generate $J$ and pick corresponding left ideals $J'_1, \ldots, J'_s$. Put $J'' = J'_1 \cap \cdots \cap J'_s$. We still have $J'' \in \mathcal{F}_a$ and $J''m \in \rho_a(M)$ for $i = 1, \ldots, s$. Hence if $I = J''y_1 + \cdots + J''y_s$, then $Im \in \rho_a(M)$.

It remains to show that $I \in \mathcal{F}_a$. As $R/J \in \mathcal{F}_a$, it is enough to show that $J/I \in \mathcal{F}_a$. As $J$ is generated by the $y_i$, it is enough to show that each $Ry_i + I/I \in \mathcal{F}_a$. But this module is a homomorphic image of $R/J''$ which is itself in $\mathcal{F}_a$.

2.10. **Proposition (Notation as in Lemma 2.8).** The restriction map $\rho_a: M \to M_a$ extends to an isomorphism $\phi: R_a \otimes_R M \to M_a$.

**Proof.** Let $\phi$ denote the canonical extension of $\rho_a$, that is $\phi(r \otimes m) = r\rho_a(m)$ for $r \in R_a, m \in M$. The diagram

$$
\begin{array}{ccc}
M & \xrightarrow{j} & R_a \otimes_R M \\
\rho_a \downarrow & & \downarrow \phi \\
M_a & & 
\end{array}
$$

commutes, where $j(m) = 1 \otimes m$.

As $j(M)$ is an essential submodule of $R_a \otimes_R M$, if ker $\phi \neq 0$, then ker $\phi \cap j(M)$ $\neq 0$. Pick $m \in M$ with $j(m) \neq 0$ but $\phi(j(m)) = 0$. Then $\rho_a(m) = 0$ but $j(m) \neq 0$. This is a contradiction as ker $\rho_a = \tau_a(M) = \ker j$ by Lemma 2.8. Hence $\phi$ is injective.

Let $m \in M_a$. By Lemma 2.9, there exists $I \in \mathcal{F}_a$ with $Im \subseteq \rho_a(M)$. But $m \in R_a m = R_a Im \subseteq R_a \rho_a(M) = \text{im} \phi$. Hence $\phi$ is surjective.

2.11. **Corollary.** Let $\mathcal{M} \in \mathcal{R}$-mod and put $M = \Gamma(X, \mathcal{M})$. Then $\mathcal{M} \cong \mathcal{R} \otimes_R M$.

**Proof.** If $U \subseteq X$ is any open affine subset, the restriction map $M \to M_U = \Gamma(U, \mathcal{M})$ extends to a map $R_U \otimes_R M \to M_U$. Hence there is a morphism of presheaves $\mathcal{R} \otimes_R M \to \mathcal{M}$, and so a morphism of sheaves $\mathcal{R} \otimes_R M \to \mathcal{M}$. 

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Because $\mathcal{R}$ is quasicoherent, $\mathcal{R} \otimes_R M|_{U_a} \cong \mathcal{R}|_{U_a} \otimes_R M$, and so for any open affine $V \subset U_a$, $\Gamma(V, \mathcal{R} \otimes_R M) = R_V \otimes_R M$. A consequence of the previous proposition is that if $V \subset U_a$ is open affine then the natural map $R_V \otimes_R M \to M_V = \Gamma(V, \mathcal{M})$ is an isomorphism. Hence the morphism $\mathcal{R} \otimes_R M \to \mathcal{M}$ induces isomorphisms on the stalks, and so is an isomorphism itself.

In other words, each $\mathcal{M} \in \mathcal{D}_\lambda \text{-Mod}$ is generated by its global sections.

2.12. Proposition. Let $N$ be an $R$-module and put $\mathcal{M} = \mathcal{R} \otimes_R N$. Then $\Gamma(X, \mathcal{M}) = N$.

Proof. Put $M = \Gamma(X, \mathcal{M})$. First we show there exists an injective map from $N$ into $M$.

Define a map $N \to M$ by associating with each $n \in N$ the global section $s$ of $\mathcal{M}$ such that $s(x) = 1 \otimes n \in R_x \otimes_R N$ for each $x \in X$, where $R_x$ is the stalk of $\mathcal{R}$ at $x$. Pick $U_a$ containing $x$. As $\mathcal{R}$ is quasicoherent, $R_x = A_x \otimes_{A_a} R_a$ and $R_x \otimes_R N = A_x \otimes_{A_a} R_a \otimes_R N$. If $s = 0$ then $1 \otimes 1 \otimes n \in A_x \otimes_{A_a} R_a \otimes_R N$ is zero for all $x \in U_a$. But as $\mathcal{R}$ is quasicoherent, the restriction of the presheaf $\mathcal{R} \otimes_R N$ to $U_a$ is a sheaf, and hence $\Gamma(U_a, \mathcal{M}) = R_a \otimes_R N$. So the map $R_a \otimes_R N \to R_a \otimes_R M$ is a bijection by Proposition 2.10. In particular, as $\mathcal{R}$ is flat, $R_a \otimes_R (M/N) = 0$. But this is true for all $a$, so the faithful flatness of $\bigoplus R_a$ implies that $M/N = 0$. Hence $M = N$ as required.

It should be remarked that the quasicoherence of $\mathcal{R}$ implies that $\mathcal{R} \otimes_R N$ is also quasicoherent.

2.13. Proof of the Theorem. The Theorem is an immediate consequence of Corollary 2.11 and Proposition 2.12. We have shown that the functors $M \to \mathcal{R} \otimes_R M$ and $\mathcal{M} \to \Gamma(X, \mathcal{M})$ are mutually inverse to one another when considered as maps from the objects of one category to the objects of the other category. It is immediate that an $R$-module homomorphism $M \to M'$ extends to a morphism of presheaves $\mathcal{R} \otimes_R M \to \mathcal{R} \otimes_R M'$, and hence gives a morphism of sheaves $\mathcal{M} \to \mathcal{M}'$. Likewise a morphism $\mathcal{M} \to \mathcal{M}'$ gives a morphism $\Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{M}')$. It follows that the two functors are inverse to one another when considered as maps on the morphisms.

2.14. In [BB] the equivalence of categories is proved by showing that every $\mathcal{M} \in \mathcal{D}_\lambda \text{-Mod}$ is generated by its global sections and that $H^i(X, \mathcal{M}) = 0$ for $i > 0$. We have been unable to show under the hypotheses of §1.2 that $H^i(X, \mathcal{M}) = 0$ for $i > 0$. However, if we assume that $\mathcal{R}$ is flat over $X$ (that is, $R_x$ is a flat $A_x$-module for each $x \in X$) then we have the following result. We remark first that it is easy to show each $\mathcal{D}_\lambda$ satisfies this condition (the stalk of $\mathcal{D}_\lambda$ at $x$ is just the ring of differential operators on the regular local ring $A_x$, and the ring of differential operators on $A_x$ is free as an $A_x$-module).
Proposition. If $\mathcal{R}$ is flat over $X$, then each $\mathcal{M} \in \mathcal{R} Mod$ satisfies $H^i(X, \mathcal{M}) = 0$ for $i > 0$.

Proof. Put $M = \Gamma(X, \mathcal{M})$ and take an injective resolution $0 \to M \to E_1 \to E_2 \to \cdots$ of the $R$-module $M$. By the equivalence of categories $\mathcal{M} \to R \otimes_R E_1 \to R \otimes_R E_2 \to \cdots$ is an injective resolution in the category $\mathcal{R} Mod$.

Claim. If $E$ is an injective $R$-module, then $R \otimes_R E$ is injective in the category of sheaves of quasicoherent $\mathcal{O}$-modules.

Let $0 \to \mathcal{O}' \to \mathcal{O}$ be an exact sequence of quasicoherent $\mathcal{O}$-modules and suppose $f: \mathcal{O}' \to \mathcal{O} \otimes_R E$ is a morphism of $\mathcal{O}$-modules. To prove the claim it suffices to show that there exists a morphism $g: \mathcal{O} \to \mathcal{O} \otimes_R E$ whose restriction to $\mathcal{O}'$ is $f$.

The natural map $R \otimes \mathcal{O}' \to R \otimes \mathcal{O}$ is injective on stalks as each $R_x$ is a flat $\mathcal{O}_x$-module, so is an injective morphism of $\mathcal{O}$-modules. A tensor product of quasicoherent $\mathcal{O}$-modules is again quasicoherent, so both these modules are in $\mathcal{R} Mod$.

The morphism $f$ induces a morphism $\mathcal{O} \otimes_R \mathcal{O}' \to \mathcal{O} \otimes_R E$. Now as $\mathcal{O} \otimes_R E$ is injective in $\mathcal{R} Mod$, $f$ extends to a map $\mathcal{O} \to \mathcal{O} \otimes_R E$. The map $g$ is then the composition $\mathcal{O} \to \mathcal{O} \otimes \mathcal{O} \to \mathcal{O} \otimes E$. This proves the claim.

Now, by [H, III, Example 3.6], $R \otimes_R E$ is flasque so we may calculate cohomology from the sequence $0 \to \mathcal{O} \to \mathcal{O} \otimes_R E_1 \to \mathcal{O} \otimes_R E_2 \to \cdots$ after applying $\Gamma(X, -)$. But, by the equivalence of categories, this gives the exact sequence $0 \to M \to E_1 \to E_2 \to \cdots$. Hence $H^i(X, \mathcal{M}) = 0$ for $i > 0$.

2.15. As remarked in §1.3, condition (iii) of the assumptions in §1.2 follows from the equivalence of categories. A precise statement of this is the following

Proposition. Suppose that $\mathcal{R}$ satisfies conditions (i) and (ii) of §1.2, and that there is an equivalence between the categories $\mathcal{R} Mod$ and $R$-Mod given by the mutually inverse functors $M \to \mathcal{R} \otimes_R M$, $\mathcal{M} \to \Gamma(X, \mathcal{M})$. Then condition (iii) of §1.2 also holds.

Proof. First we show that if $U \subset X$ is open affine, then $R_U$ is flat as a right $R$-module.

Let $0 \to M_1 \to M_2$ be an exact sequence of $R$-modules. Put $\mathcal{M}_i = \mathcal{R} \otimes_R M_i$. The assumption implies that $0 \to \mathcal{M}_1 \to \mathcal{M}_2$ is exact in $\mathcal{R} Mod$. Hence $0 \to \Gamma(U, \mathcal{M}_1) \to \Gamma(U, \mathcal{M}_2)$ is an exact sequence of $R_U$-modules. But, as $\mathcal{R}$ is quasicoherent and $U$ is open affine, $\Gamma(U, \mathcal{M}_i) = R_U \otimes_R M_i$ (this is because the presheaf $\mathcal{R} \otimes M|_U$ is already a sheaf over $U$). Hence $0 \to R_U \otimes_R M_1 \to R_U \otimes_R M_2$ is exact, and $R_U$ is flat.

Suppose $(U_a)$ is any open affine cover for $X$. Put $R_a = \Gamma(U_a, \mathcal{R})$ and suppose $M$ is an $R$-module with $R_a \otimes_R M = 0$ for all $a$. As just remarked, $R_a \otimes_R M = \Gamma(U_a, \mathcal{R} \otimes M)$. So $\mathcal{R} \otimes M$ is locally zero, hence globally zero. The equivalence of categories implies that $M = \Gamma(X, \mathcal{R} \otimes M)$ is also zero. Hence $\mathcal{O} \otimes R_a$ is faithfully flat.

In the proof of the proposition the choice of the $U_a$ was arbitrary, so we have

Corollary. If $\mathcal{R}$ satisfies the conditions of §1.2, then condition (iii) will hold for any open affine cover $(U_a)$ of $X$. 
3. Comments and remarks.

3.1. We adopt here the notation of [BB]. The underlying topological space of the sheaf $\mathcal{D}_\lambda$ is $X = G/B$, the flag variety of the connected complex semisimple Lie group corresponding to the complex semisimple Lie algebra $\mathfrak{g}$. The cover $(U_a)$ in this case could be obtained by taking the translates of the large Bruhat cell under the action of the Weyl group; that is each $U_a = wBw_0B$ for some $w \in W$, the Weyl group, where $w_0$ is the long element of $W$. Each $R_\alpha$ is isomorphic to the Weyl algebra $A_n$, where $n = \dim X$. In [HS] it was shown that the restriction map $D_\lambda = \Gamma(X, \mathcal{D}_\lambda) \to \Gamma(Bw_0B, \mathcal{D}_\lambda) = A$ is (up to an automorphism of $\mathfrak{g}$ and an automorphism of $A$) the Conze embedding $D_{w_0\lambda} \to A$ obtained through the action of $\mathfrak{g}$ on the Verma module $M(w_0\lambda)$. In order to establish (iii) directly one would need to show (among other things) that the embedding $D_{w_0\lambda} \to A$ makes $A$ flat as a right $D_{w_0\lambda}$-module.

It is shown in [JS] for $\lambda$ dominant regular that $A$ is the union of the right $D_{w_0\lambda}$-modules $L(w_0\lambda, w_0\mu)$ (of $\mathfrak{g}$-finite maps from the Verma module $M(w_0\lambda)$ to $M(w_0\mu)$), where the union is taken over all dominant regular $\mu$ such that $\mu - \lambda$ is dominant integral. It already follows from [BG] that the $L(w_0\lambda, w_0\mu)$ are finitely generated projective right $D_{w_0\lambda}$-modules, and hence $A$ is flat as a right $D_{w_0\lambda}$-module.

3.2. The hypothesis that $X$ was irreducible was made only to avoid the possibility that some of the $U_{a_1 \cdots a_n}$ might be empty and therefore that some of the $R_{a_1 \cdots a_n}$ might be zero. If one assumes that $X$ has a finite number of irreducible components $X_i$, and the hypotheses in §1.2 are adjusted so that $\mathcal{R}|_{X_i}$ satisfies (i)–(iii), then one can still show that $\mathcal{R}$-Mod and $\mathcal{R}$-Mod are equivalent.

References


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