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The Minimal Nilpotent Orbit, the Joseph Ideal, and Differential Operators

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Fix a simple complex Lie algebra $\mathfrak{g}$, not of type G_2, F_4, or E_8. Let $\bar{\mathbf{O}}_{\min}$ denote the Zariski closure of the minimal non-zero nilpotent orbit in $\mathfrak{g}$, and let $\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^-$ be a triangular decomposition. We prove

THEOREM. (1) *If $\mathfrak{g}$ is not of type A_n then there exists an irreducible component $\bar{\mathbf{X}}$ of $\bar{\mathbf{O}}_{\min}\cap\mathfrak{n}^+$ such that $U(\mathfrak{g})/J_0=\mathscr{D}(\bar{\mathbf{X}})$, where J_0 is the Joseph ideal and $\mathscr{D}(\bar{\mathbf{X}})$ denotes the ring of differential operators on $\bar{\mathbf{X}}$.*

(2) *If $\mathfrak{g}$ is of type A_n then for $n-2$ of the n irreducible components $\bar{\mathbf{X}}_i$ of $\bar{\mathbf{O}}_{\min}\cap\mathfrak{n}^+$ there exist (distinct) maximal ideals J_i of $U(\mathfrak{g})$ such that $U(\mathfrak{g})/J_i=\mathscr{D}(\bar{\mathbf{X}}_i)$.* © 1988 Academic Press, Inc.

1. INTRODUCTION

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and let $\bar{\mathbf{O}}_{\min}$ denote the Zariski closure of the minimal (non-zero) nilpotent orbit in $\mathfrak{g}$. If $\mathfrak{g}$ is not of type A_n, there is a unique completely prime primitive ideal of $U(\mathfrak{g})$ with associated variety $\bar{\mathbf{O}}_{\min}$ [12]. This ideal is called the Joseph

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ideal, and is denoted J_0. Joseph introduces J_0 as the kernel of a certain $\mathbb{C}$-algebra homomorphism ψ from $U(\mathfrak{g})$ to a localisation of a Weyl algebra. Given an algebraic variety Y, write $\mathscr{D}(Y)$ for the ring of differential operators on Y. Thus Joseph's construction may be interpreted as $J_0 = \ker\psi$, where $\psi\colon U(\mathfrak{g}) \to \mathscr{D}(\mathbb{A}^n \backslash H)$ for a certain hyperplane $H \subset \mathbb{A}^n$. This homomorphism is never surjective.

If $\mathfrak{g}$ is of type B_n, C_n, D_n, E_6, or E_7 there is a second procedure for realising $U(\mathfrak{g})/J_0$ as differential operators on a suitable variety. One takes a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and a suitable irreducible component $\bar{\mathbf{X}}$ of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$. Then there exists a homomorphism $\psi\colon U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}})$ due to Goncharov [10]. It is not difficult to prove that $\mathcal{O}(\bar{\mathbf{X}})$ becomes a simple highest weight module and that $\ker\psi = J_0$ (see Proposition 3.5, and [16, Sect. 3] where this is done for $so(n)$).

The first main result of this paper is to *show that ψ is surjective, and hence that* $U(\mathfrak{g})/J_0 = \mathscr{D}(\bar{\mathbf{X}})$. For $\mathfrak{g}$ of type C_n this result is (implicitly) established in [15, Théorème 5, p. 170].

In outline, the theorem is proved as follows. One observes that $R := \psi(U(\mathfrak{g})) \subset \mathscr{D}(\bar{\mathbf{X}}) \subset \operatorname{Fract}(R)$. By a result of Gabber (see Lemma 5.2), and the fact that R is simple (Proposition 3.5), it then suffices to prove that $GK\dim(Rd + R/R) \leqslant GK\dim(R) - 2$ for all $d \in \mathscr{D}(\bar{\mathbf{X}})$. By passing to the associated graded ring of R this can be reduced to the problem of showing that $\dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) \leqslant \dim(\bar{\mathbf{O}}_{\min}) - 2$, where $\mathfrak{p}$ is a certain parabolic subalgebra of $\mathfrak{g}$ having an abelian nilpotent radical. This statement is proved in Section 4. The proof also requires a number of technical but standard results both on maps from Lie algebras to rings of differential operators (given in Section 2), and on the details of Goncharov's construction (given in Section 3)).

The second main result shows that for $\mathfrak{g}$ of type A_n, certain completely prime primitive ideals of $U(\mathfrak{g})$ associated to the minimal nilpotent orbit may be obtained as the kernel of a surjective map from $U(\mathfrak{g})$ to the ring of differential operators on an irreducible component of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$. Both Goncharov's construction (which works when there is a Hermitian symmetric space) and the proof of the main theorem, as outlined above, apply when $\mathfrak{g}$ is of type A_n. Of course, in this case the Joseph ideal is not defined; there is a one parameter family of completely prime primitive ideals having associated variety $\bar{\mathbf{O}}_{\min}$. If $\mathfrak{g} = sl(n+1)$ then $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$ is a union of n irreducible components $\bar{\mathbf{X}}_j$ $(1 \leqslant j \leqslant n)$, and for each j Goncharov's construction provides a homomorphism $\psi_j\colon U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}}_j)$. Let $(\alpha_j = \varepsilon_j - \varepsilon_{j+1} \mid 1 \leqslant j \leqslant n)$ be the simple roots for A_n as in Bourbaki [5, Planche 1, p. 250]. Let $\mathfrak{p}_j$ be the maximal parabolic obtained by deleting the simple root α_j. Then $\bar{\mathbf{X}}_j$ is contained in the nilpotent radical of $\mathfrak{p}_j$. If $j \neq 1$ and $j \neq n$, the proof outlined above goes through and ψ_j is surjective (Theorem 5.2); furthermore, if ω_j is the fundamental weight corresponding to the simple root α_j, and $L(\rho-\omega_j)$ is the simple module of highest weight $-\omega_j$ then $\ker\psi_j = \operatorname{Ann} L(\rho-\omega_j)$. This follows from the fact that $\mathcal{O}(\bar{\mathbf{X}}_j) \cong L(\rho-\omega_j)$. Note that the various $\ker\psi_j$ $(1<j<n)$ are distinct (Theorem 3.8). When $j=1$ or $j=n$, the outcome is quite different. In this case there is a 1-parameter family of maps $\psi\colon U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}}_j)$ and none of them is surjective (in fact, there can never be a surjective map $\psi\colon U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}}_j)$; see 3.9). In [20] Musson (using a technique different from ours) also shows that for $1<j<n$ there is a surjective map $U(sl(n+1)) \to \mathscr{D}(\bar{\mathbf{X}}_j)$.

In Section 6 we show that our results for $so(2n)$, $n \geqslant 5$, give the existence of two non-isomorphic (singular irreducible affine) varieties $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_n$, such that $\mathscr{D}(\bar{\mathbf{X}}_1)$ and $\mathscr{D}(\bar{\mathbf{X}}_n)$ are isomorphic. This illustrates that some information about a variety may be lost in passing to the ring of differential operators.

Since the proof of the Kazhdan–Lusztig conjectures [1] the connection between primitive ideals and differential operators has been vigorously investigated. See in particular [3, 4]. In those papers certain induced primitive ideals are realised as the kernel of the natural map $U(\mathfrak{g}) \to \mathscr{D}(G/P)$, where P is a parabolic of the connected semi-simple algeraic group G (where Lie $G=\mathfrak{g}$). In fact, $\mathscr{D}(G/P)$ is isomorphic to a certain primitive factor ring of $U(\mathfrak{g})$. However, the present paper differs in a number of respects. Our variety $\bar{\mathbf{X}}$ is affine (not projective), it is not smooth (whereas G/P is), $\bar{\mathbf{X}}$ is not a G-variety, the kernel of $\psi\colon U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}})$ (viz., J_0) is not an induced ideal (except when $\mathfrak{g} = sl(n+1)$), and $\mathfrak{g}$ does not act on $\bar{\mathbf{X}}$ as vector fields (some elements of $\mathfrak{g}$ act as second order operators, some as first, and some as zeroth order operators).

The results outlined above lead us to suspect that many other completely prime primitive ideals may be obtained as the kernel of a suitable action of $\mathfrak{g}$ as differential operators on certain components of $\bar{\mathbf{O}} \cap \mathfrak{n}^+$ (where $\bar{\mathbf{O}}$ is the associated variety of the primitive ideal). Apart from those examples already given, there is an example occurring in type G_2 (see 5.4). More recently, the first and third authors have obtained a number of other primitive ideals in this way [25].

2. Generalities

2.1. This section collects various facts about the action of a Lie algebra as differential operators on a variety. The following notation will be used, usually without comment, in the paper. Let Z be an irreducible algebraic variety. Write $\mathcal{O}_Z$ for the sheaf of regular functions on Z, with global sections $\mathcal{O}(Z)$. For each $p \in Z$, write $\mathcal{O}_{Z,p}$, or just $\mathcal{O}_p$, for the local ring at p, with maximal ideal $\boldsymbol{m}_p$. Write $\mathscr{D}_Z$ for the sheaf of differential operators, with global sections $\mathscr{D}(Z)$, and stalks $\mathscr{D}_{Z,p}$. For $p \in Z$, let T_pZ be the tangent space to Z at p, T_p^*Z the cotangent space to Z at p, and let $\operatorname{Der} \mathcal{O}_{Z,p}$ denote the module of derivations on $\mathcal{O}_{Z,p}$.

Throughout, G will be a connected complex semi-simple linear algebraic group, and $P \subset G$ a parabolic subgroup containing a fixed Borel subgroup B. Write $\mathfrak{g} = \operatorname{Lie}(G)$, $\mathfrak{p} = \operatorname{Lie}(P)$, $\mathfrak{b} = \operatorname{Lie}(B)$ for the corresponding Lie algebras. In general closed connected subgroups of G will be denoted by uppercase letters, and the corresponding Lie algebra will be denoted by the same letter in lowercase German script. Let $\mathfrak{h} \subset \mathfrak{b}$ be a fixed Cartan subalgebra, and let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the usual triangular decomposition relative to the choice of $\mathfrak{h}$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Write $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{r}^+$ as a direct sum of ad-$\mathfrak{h}$-modules, where $\mathfrak{r}^+$ is the nilpotent radical of $\mathfrak{p}$, and $\mathfrak{m}$ the reductive part. Thus $\mathfrak{g} = \mathfrak{r}^+ \oplus \mathfrak{m} \oplus \mathfrak{r}^-$, where $\mathfrak{r}^-$ is the ad-$\mathfrak{h}$-module complement to $\mathfrak{p}$. Write M for the closed connected subgroup of G with Lie $M = \mathfrak{m}$.

The Killing form on $\mathfrak{g}$ is denoted $B(X, Y)$ for $X, Y \in \mathfrak{g}$.

Write R for the set of roots of $\mathfrak{g}$, and R^+ for the roots of $\mathfrak{h}$ in $\mathfrak{n}^+$. If α is a root for $\mathfrak{g}$ then X_α denotes a root vector of weight α. Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots. Given $\lambda \in \mathfrak{h}^*$, write $M(\lambda)$ for the Verma module with highest weight $\lambda - \rho$. Let $L(\lambda)$ be the simple quotient of $M(\lambda)$, and write $J(\lambda) := \operatorname{Ann} L(\lambda)$ for its annihilator.

If Z is a variety on which there is a G-action and $p \in Z$, write $\operatorname{Stab}_G(p) := \{g \in G \mid g \cdot p = p\}$ for the *stabiliser* of p in G, and set $\mathfrak{g}(p) := \operatorname{Lie}(\operatorname{Stab}_G(p))$. Thus $\mathfrak{g}(p)$ is the kernel of the natural map $\mathfrak{g} \to T_pZ$.

Given a $\mathbb{C}$-vector space V, write $S(V)$ for the symmetric algebra on V. Write $S_n(V)$ for the nth symmetric power of V. Set $S(\mathfrak{r}^-)_q = \bigoplus_{n \leqslant q} S_n(\mathfrak{r}^-)$. Similarly, $U(\mathfrak{g})_p$ denote the elements in $U(\mathfrak{g})$ of degree $\leqslant p$.

2.2. Let V be a finite dimensional $\mathbb{C}$-vector space with basis $y_1, ..., y_d$ and let $x_1, ..., x_d$ be the dual basis for V^*. Thus $\mathcal{O}(V) = \mathbb{C}[x_1, ..., x_d] = S(V^*)$. For each $y \in V$ write ∂_y for the derivation on $S(V^*)$ defined by $\partial_y(x) = x(y)$, for $x \in V^*$. Write $\partial_j = \partial/\partial x_j$. The $\mathbb{C}$-linear map $V \to \operatorname{Der} S(V^*)$ given by $y_j \mapsto \partial_j = \partial/\partial x_j$ identifies $\operatorname{Der} S(V^*)$ with $S(V^*) \otimes_{\mathbb{C}} V$. The ring of differential operators on V is therefore $\mathscr{D}(V) = \mathscr{D}(S(V^*)) = \mathbb{C}[x_1, ..., x_d, \partial_1, ..., \partial_d] = S(V^*) \otimes_{\mathbb{C}} S(V)$.

The *Fourier transform* $\mathscr{F}: \mathscr{D}(V) \to \mathscr{D}(V^*)$ is defined to be the algebra isomorphism given by $\mathscr{F}(x_j) = i\partial/\partial y_j$, $\mathscr{F}(\partial/\partial x_j) = iy_j$, where $i = \sqrt{-1}$. The Fourier transform is independent of the choice of basis.

2.3. (Notation 2.1). Consider $\mathfrak{g}$ as vector fields on $P\backslash G$. Identify $\mathfrak{r}^-$ through the exponential map with (a translate of) the large Bruhat cell of $P\backslash G$. Restricting the $\mathfrak{g}$-action on $P\backslash G$ then gives a Lie algebra homomorphism $L: \mathfrak{g} \to \operatorname{Der} \mathcal{O}(\mathfrak{r}^-)$ which extends to an algebra homomorphism $L: U(\mathfrak{g}) \to \mathscr{D}(\mathfrak{r}^-)$.

Through the non-degeneracy of the Killing form identify $\mathfrak{r}^+$ with $(\mathfrak{r}^-)^*$. Fix a basis $x_1, ..., x_d$ for $\mathfrak{r}^+$ and let $y_1, ..., y_d$ be the dual basis for $\mathfrak{r}^-$. We may assume that the x_j are chosen to be ad-$\mathfrak{h}$-eigenvectors. As in (2.2), we have $\mathcal{O}(\mathfrak{r}^-) = S(\mathfrak{r}^+)$, $\operatorname{Der} \mathcal{O}(\mathfrak{r}^-) = S(\mathfrak{r}^+) \otimes_{\mathbb{C}} \mathfrak{r}^-$, and $\mathscr{D}(\mathfrak{r}^-) = S(\mathfrak{r}^+) \otimes_{\mathbb{C}} S(\mathfrak{r}^-)$.

2.4. Following Blattner [2], the action of $\mathfrak{g}$ on $S(\mathfrak{r}^+) = \mathcal{O}(\mathfrak{r}^-)$ obtained through L, and the image of $\mathfrak{g}$ in $\mathcal{O}(\mathfrak{r}^-)$, may be described as follows. First, set $F = \operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \mathbb{C})$. This has an algebra structure induced by the comultiplication of $U(\mathfrak{g})$, and $\mathfrak{g}$ acts as derivations on F by $(Xf)(u) = f(uX)$, for $f \in F$, $X \in \mathfrak{g}$, $u \in U(\mathfrak{g})$. Define $\tau\colon S(\mathfrak{r}^-) \to U(\mathfrak{g})$ by

$$\tau(z_1 \cdots z_n) = (1/n!) \sum (z_{\sigma 1} \cdots z_{\sigma n} \colon \sigma \in S_n)$$

for $z_1, ..., z_n \in \mathfrak{r}^-$, where S_n is the symmetric group. The map $\pi\colon F \to \operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{r}^-), \mathbb{C})$ given by $(\pi f)(a) = f(\tau(a))$ for $f \in F$, $a \in S(\mathfrak{r}^-)$, is a vector space isomorphism, and so allows us to give $\operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{r}-), \mathbb{C})$ an algebra structure.

There is a natural map $\gamma\colon S(\mathfrak{r}^+) \to S(\mathfrak{r}^-)^* = \operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{r}^-), \mathbb{C})$ which extends the identification of $\mathfrak{r}^+$ with $(\mathfrak{r}^-)^*$. If $K = (k_1, ..., k_d)$, $J = (j_1, ..., j_d)$ are multi-indices, then define $\delta_{KJ} = \prod_\lambda \delta_{kj}$ and $J! = \prod_\lambda (j_\lambda)!$. With the notation of (2.3), γ is defined by setting $x^K(y^J) = \delta_{KJ}(J!)$. The following proposition is now routine:

PROPOSITION (Blattner [2]). *Let F have the $\mathfrak{g}$-module structure defined above, and let $S(\mathfrak{r}^+)$ have the $\mathfrak{g}$-module structure obtained through L (as in (2.3)). Then γ: $S(\mathfrak{r}^+) \to F = \operatorname{Hom}(S(\mathfrak{r}^-), \mathbb{C})$ is*

(a) *a $\mathfrak{g}$-module homomorphism,*

(b) *an injective algebra homomorphism, and*

(c) *if $S(\mathfrak{r}^+) = \mathbb{C}[x_1, ..., x_d]$ is identified with its image in F, then $F = \mathbb{C}[[x_1, ..., x_d]]$, the power series completion of $S(\mathfrak{r}^+)$.*

2.5. Given $L\colon U(\mathfrak{g}) \to \mathscr{D}(\mathfrak{r}^-)$ as in (2.3), one may apply the Fourier transform to obtain $\mathscr{F} \circ L\colon U(\mathfrak{g}) \to \mathscr{D}(\mathfrak{r}^+)$. Alternatively, one may regard $\mathfrak{r}^+$ as an M-module, and differentiate the M-action to obtain an algebra homomorphism $\alpha\colon U(\mathfrak{m}) \to \mathscr{D}(\mathfrak{r}^+)$. Our immediate aim is to give an explicit formula relating α and the restriction of $\mathscr{F} \circ L$ to $U(\mathfrak{m})$. Although this is presumably well known we could not find it in the literature. We give it in slightly more generality than we will need.

Thus for the rest of (2.5), let M be any connected affine algebraic group over $\mathbb{C}$, and set $\mathfrak{m} = \operatorname{Lie}(M)$. Let V be a finite dimensional representation of M, and let $\beta\colon U(\mathfrak{m}) \to \mathscr{D}(V)$ be obtained by differentiating the M-action on V. Give V^* the contragredient representation, and let $\alpha\colon U(\mathfrak{m}) \to \mathscr{D}(V^*)$ be obtained by differentiating the M-action on V^*. Let $\mathscr{F}: \mathscr{D}(V) \to \mathscr{D}(V^*)$ be the Fourier transform, and let $(y_i \mid 1 \leqslant i \leqslant d)$ and $(x_j \mid 1 \leqslant j \leqslant d)$ be dual bases for V and V^*, respectively.

PROPOSITION. *For* $X \in \mathfrak{m}$, $\alpha(X) = (\mathscr{F} \circ \beta)(X) - \operatorname{trace}_V(X)$. *In particular*, $\alpha(U(\mathfrak{m})) = (\mathscr{F} \circ \beta)(U(\mathfrak{m}))$.

Proof. Adopt the notation of (2.2). Fix $X \in \mathfrak{m}$. Let $\alpha_{ij} \in \mathbb{C}$ $(1 \leqslant i, j \leqslant d)$ be such that $X \cdot y_j = \sum_{i=1}^d \alpha_{ij} y_i$ for $1 \leqslant j \leqslant d$. Then $X \cdot x_i = -\sum_{j=1}^d \alpha_{ij} x_j$ for $1 \leqslant i \leqslant d$. Therefore

$$\beta(X) = \sum_{i=1}^{d} \beta(X)(x_i)\,\partial/\partial x_i = \sum_{i=1}^{d} (X \cdot x_i)\,\partial/\partial x_i = -\sum_{i,j=1}^{d} \alpha_{ij} x_j \partial/\partial x_i,$$

and

$$(\mathscr{F} \circ \beta)(X) = \sum_{i,j=1}^{d} \alpha_{ij} (\partial/\partial y_j)\, y_i = \sum_{i,j=1}^{d} \alpha_{ij} (y_i \partial/\partial y_j + \delta_{ij}).$$

Similarly

$$\alpha(X) = \sum_{j=1}^{d} \alpha(X)(y_j)\,\partial/\partial y_j = \sum_{j=1}^{d} (X \cdot y_j)\,\partial/\partial y_j = \sum_{i,j=1}^{d} \alpha_{ij} y_i \partial/\partial y_j.$$

Hence $(\mathscr{F} \circ \beta)(X) - \alpha(X) = \sum_{i=1}^d \alpha_{ii} = \operatorname{trace}_V(X) \in \mathbb{C}$. In particular, $(\mathscr{F} \circ \beta)(\mathfrak{m} \oplus \mathbb{C}) = \alpha(\mathfrak{m} \oplus \mathbb{C})$, and the result follows. ∎

It is easy to check that, if L is as in (2.3), then its restriction $L: U(\mathfrak{m}) \to \mathscr{D}(\mathfrak{r}^-)$ coincides with that obtained by differentiating the M-action on $\mathfrak{r}^-$, say $\beta: U(\mathfrak{m}) \to \mathscr{D}(\mathfrak{r}^-)$. Hence one obtains

COROLLARY. *For* $X \in \mathfrak{m}$, $\alpha(X) = (\mathscr{F} \circ L)(X) - \operatorname{trace}_{\mathfrak{r}^-}(X)$. *In particular*, $\alpha(U(\mathfrak{m})) = (\mathscr{F} \circ L)(U(\mathfrak{m}))$.

2.6. (Notation 2.5). Let $X \subset V^*$ be an M-orbit. The map $\alpha: U(\mathfrak{m}) \to \mathscr{D}(V^*)$ restricts to give $\alpha: U(\mathfrak{m}) \to \mathscr{D}(X)$.

LEMMA. *For each* $p \in X$, *the image of* $\alpha: \mathfrak{m} \to \mathscr{D}(X)$ *generates* Der $\mathscr{O}_{X,p}$ *as an* $\mathscr{O}_{X,p}$*-module.*

Proof. (Notation 2.1). The map α gives rise to isomorphisms $\mathfrak{m}/\mathfrak{m}(p) \cong T_p X \cong (\mathscr{O}_{X,p}/\boldsymbol{m}_p) \otimes_{\mathscr{O}} \operatorname{Der} \mathscr{O}_{X,p}$. Hence by Nakayama's Lemma $\alpha(\mathfrak{m})$ generates Der $\mathscr{O}_{X,p}$. ∎

2.7. Consider the situation in (2.3) and apply the Fourier transform to obtain $\mathscr{F} \circ L: U(\mathfrak{g}) \to \mathscr{D}(\mathfrak{r}^+)$. Consider $\mathfrak{r}^+$ as an M-representation, and let $X \subset \mathfrak{r}^+$ be an M-orbit.

PROPOSITION. (a) *The map* $\mathscr{F} \circ L$ *restricts to give a map* $\chi: U(\mathfrak{m}) \to \mathscr{D}(X)$.

(b) *For each* $p \in X$, *the subalgebra* A *of* $\mathscr{D}_{X,p}$ *generated by* $\mathscr{O}_{X,p}$ *and* $\chi(U(\mathfrak{m}))$ *contains* Der $\mathscr{O}_{X,p}$, *and hence equals* $\mathscr{D}_{X,p}$.

Proof. (a) This follows from Corollary 2.5, and the first sentence of (2.6).

(b) It follows from (a), and Lemma 2.6, that A contains Der $\mathscr{O}_{X,p}$. However, X is non-singular at every point p, so $\mathscr{D}_{X,p}$ is generated by $\mathscr{O}_{X,p}$ and Der $\mathscr{O}_{X,p}$. ∎

3. GONCHAROV'S CONSTRUCTION

3.1. The first aim of Section 3 is to define Goncharov's map $\psi: U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}})$ for a suitable irreducible component $\bar{\mathbf{X}}$ of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$, and to identify $\ker \psi$ as a particular primitive ideal. Secondly, we compute the highest weight of $\mathscr{O}(\bar{\mathbf{X}})$ when $\mathfrak{g} = sl(n+1)$.

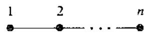
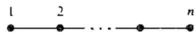
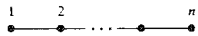
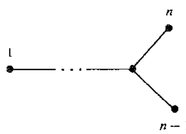
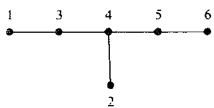
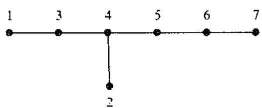
Retain the notation of Section 2. From now on assume that $\mathfrak{g}$ is simple and that $\mathfrak{r}^+$ is abelian. For completeness all occurrences of this situation are listed in Table 3.1. The extra notation in the Table is as follows. In each case $\mathfrak{p}$ is a maximal parabolic subalgebra of $\mathfrak{g}$, and so is completely determined by the simple root α such that $X_{-\alpha} \notin \mathfrak{p}$. Label the simple roots of $\mathfrak{g}$ by $\alpha_1, ..., \alpha_l$ (in Table 3.1 these have been abbreviated to $1, 2, ..., l$) and denote by $\mathfrak{p}_j$ the maximal parabolic subalgebra such that $X_{-\alpha_j} \notin \mathfrak{p}$. It follows from the fact that $\mathfrak{r}^+$ is abelian that $\mathfrak{r}^+$ is an irreducible $\mathfrak{m}$-module. The Dynkin diagram for $\mathfrak{m}_j$, the reductive part of $\mathfrak{p}_j$, is the subdiagram of that for $\mathfrak{g}$ obtained by deleting α_j. Our labelling of the roots is that of Bourbaki [5].

Let β be the highest root for $\mathfrak{g}$. Then X_β is a highest weight vector in $\mathfrak{r}^+$ for $\mathfrak{m}$. Define $\mathbf{X} = M \cdot X_\beta$, the cone of highest weight vectors. Let $\bar{\mathbf{X}}$ denote the Zariski closure. This is the variety on which $\mathfrak{g}$ will act as differential operators. Let I be the ideal in $\mathscr{O}(\mathfrak{r}^+) = S(\mathfrak{r}^-)$ defining $\bar{\mathbf{X}}$. As Goncharov remarks [10, Lemma 2], I is generated by $I^{(2)} := I \cap S^2(\mathfrak{r}^-)$. A proof of this may be found in [9]. It is straightforward to find the dimension of $\bar{\mathbf{X}}$ (which is given in the Table); a short description of $\bar{\mathbf{X}}$ is given in Remarks 3.2.

3.2. *Remarks.* (i) For type D_n (resp. E_6) the parabolics $\mathfrak{p}_n$ and $\mathfrak{p}_{n-1}$ (resp. $\mathfrak{p}_1$ and $\mathfrak{p}_6$) are isomorphic but not conjugate.

(ii) For $\mathfrak{g} = sl(n+1)$, $\mathfrak{r}^+$ may be identified with $M_{j,n-j+1}(\mathbb{C})$, the space of $j \times (n-j+1)$ matrices, and $\bar{\mathbf{X}}$ consists of those of rank $\leqslant 1$.

TABLE 3.1

Type of $\mathfrak{g}$	Dynkin diagram	Parabolics $\mathfrak{p}$ with abelian radical	Semi-simple part of $\mathfrak{m}$	$\dim \mathfrak{r}^+$	$\dim \bar{\mathbf{X}}$
$A_n\ (n \geqslant 1)$		$\mathfrak{p}_j\ (1 \leqslant j \leqslant n)$	$A_{j-1} \times A_{n-j}$	$j(n-j+1)$	n
$B_n\ (n \geqslant 3)$		$\mathfrak{p}_1$	B_{n-1}	$2n-1$	$2n-2$
$C_n\ (n \geqslant 2)$		$\mathfrak{p}_n$	A_{n-1}	$n(n+1)/2$	n
$D_n\ (n \geqslant 4)$		$\mathfrak{p}_1$	D_{n-1}	$2n-2$	$2n-3$
		$\mathfrak{p}_{n-1}, \mathfrak{p}_n$	A_{n-1}	$n(n-1)/2$	$2n-3$
E_6		$\mathfrak{p}_1$	D_5	16	11
		$\mathfrak{p}_6$	D_5	16	11
E_7		$\mathfrak{p}_7$	E_6	27	17

(iii) For type B_n, $M = \mathbb{C}^* \times SO(2n-1)$, $\mathfrak{r}^+ = \mathbb{C}^{2n-1}$ with its natural action, and $\bar{\mathbf{X}}$ is the space of isotropic vectors (including 0) in $\mathbb{C}^{2n-1}$.

(iv) For type C_n, $\mathfrak{r}^+$ identifies with the space of quadratic forms on $\mathbb{C}^n$, and $\bar{\mathbf{X}}$ consists of those of rank $\leqslant 1$.

(v) For type D_n, if $\mathfrak{p} = \mathfrak{p}_1$, then $\mathfrak{r}^+ = \mathbb{C}^{2(n-1)}$ and $M = \mathbb{C}^* \times SO(2(n-1))$, where $\bar{\mathbf{X}}$ is the space of isotropic vectors. When $\mathfrak{p} = \mathfrak{p}_n$, then $M = GL(n)$, $\mathfrak{r}^+ = \Lambda^2 \mathbb{C}^n =$ the set of alternating bilinear forms on $\mathbb{C}^n$, and $\bar{\mathbf{X}}$ consists of the alternating forms of rank $\leqslant 2$.

(vi) Since $\mathbf{X}$ is the orbit of a highest weight vector in $\mathfrak{r}^+$, [24, Theorem 3] ensures that $\bar{\mathbf{X}}$ is normal. In all cases except type A_n with $\mathfrak{p}$ equal $\mathfrak{p}_1$ or $\mathfrak{p}_n$, the singular locus $\operatorname{Sing}(\bar{\mathbf{X}}) = \{0\} = \bar{\mathbf{X}} \setminus \mathbf{X}$ [24], and $\operatorname{codim}_{\bar{\mathbf{X}}}(\mathbf{X}) \geqslant 2$. Thus $\mathscr{D}(\mathbf{X}) = \mathscr{D}(\bar{\mathbf{X}})$ by [15, Proposition 2, p. 167]. In particular, the results of Section 2 may be applied. If $\mathfrak{g}$ is of type A_n, with $\mathfrak{p}$ equal $\mathfrak{p}_1$ or $\mathfrak{p}_n$, then $\bar{\mathbf{X}} = \mathfrak{r}^+$ is non-singular.

(vii) There exists $H \in \mathfrak{h}^*$ such that $[H[\mathfrak{m}, \mathfrak{m}]] = 0$, $[H, X] = X$ for $X \in \mathfrak{r}^+$, and $[H, Y] = -Y$ for $Y \in \mathfrak{r}^-$. To see this note that the Levi decomposition of $\mathfrak{m}$ gives $H \in \mathfrak{h}^*$ with $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}] \oplus \mathbb{C}H$, and so $[H[\mathfrak{m}, \mathfrak{m}]] = 0$.

Further, as $\mathfrak{r}^+$ is a simple $[\mathfrak{m}, \mathfrak{m}]$-module, ad $H|_{\mathfrak{r}^+}$ must act as (non-zero) scalar multiplication by Schur's Lemma. Replacing H by a suitable scalar multiple establishes the claim.

(viii) Note that $\mathbf{X} = M \cdot X_\beta \subset G \cdot X_\beta = \mathbf{O}_{\min}$, and by comparing the Table with [11], dim $X = \dim(\mathbf{O}_{\min})/2$. It follows from [22] that $\bar{\mathbf{X}}$ is an irreducible component of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$.

3.3. In [10] Goncharov is able to realise $U(\mathfrak{g})$ acting as differential operators on the cone $\bar{\mathbf{X}} \subset \mathfrak{r}^+$. The first step in doing this is to consider $\mathscr{F} \circ L$: $U(\mathfrak{g}) \to \mathscr{D}(\mathfrak{r}^+)$ (see (2.3) and (2.7)). Let I be the ideal in $\mathcal{O}(\mathfrak{r}^-)$ defining $\bar{\mathbf{X}}$. It is well known that $\mathscr{D}(\bar{\mathbf{X}}) = \mathrm{II}(I\mathscr{D}(\mathfrak{r}^-))/I\mathscr{D}(\mathfrak{r}^-)$, where $\mathrm{II}(J) := \{d \in \mathscr{D}(\mathfrak{r}^-) \mid dJ \subset J\}$. Thus $\mathscr{F} \circ L$ will restrict to a map $U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}})$ if (and only if) I is stable under the action of $\mathfrak{g}$. Unfortunately I is not stable under $\mathfrak{g}$ (although it is stable under the action of $\mathfrak{m}$, as noted in (2.7), and under the action of $\mathfrak{r}^-$). The key point of Goncharov's construction is to replace L by a suitable "twisting," say L', such that $(\mathscr{F} \circ L')(\mathfrak{g})$ does leave I stable, and so obtain $(\mathscr{F} \circ L)(\mathfrak{g})$ acting as differential operators on $\bar{\mathbf{X}}$.

For $x \in \mathfrak{g}$, write L_x and L'_x rather than $L(x)$ and $L'(x)$.

THEOREM (Goncharov). *These exists $\lambda \in \mathbb{C}$, such that if L': $\mathfrak{g} \to \mathscr{D}(\mathfrak{r}^-)$ is defined by*

(i) $L'_y = L_y$ *for* $y \in \mathfrak{r}^-$,

(ii) $L'_x = L_x + \lambda x$ *for* $x \in \mathfrak{r}^+$,

(iii) $L'_{[x,y]} = L_{[x,y]} - \lambda B(x, y)$ *for* $y \in \mathfrak{r}^-$, *and* $x \in \mathfrak{r}^+$,

then I is stable under $(\mathscr{F} \circ L')(\mathfrak{g})$. Thus $\mathscr{F} \circ L'$ defines a map from $U(\mathfrak{g})$ to $\mathscr{D}(\bar{\mathbf{X}})$.

In all cases except when $\mathfrak{g}$ is of type A_n and $\mathfrak{p}$ equals $\mathfrak{p}_1$ or $\mathfrak{p}_n$, λ is unique (and non-zero). In these two exceptional cases $\bar{\mathbf{X}} = \mathfrak{r}^+$, and $I = 0$. Thus I is table under $(\mathscr{F} \circ L)(\mathfrak{g})$ and no twisting is required. Nevertheless, one can still twist the embedding by choosing any $\lambda \in \mathbb{C}$, and so obtain a one parameter family of maps $\mathscr{F} \circ L'$: $U(\mathfrak{g}) \to \mathscr{D}(\mathfrak{r}^+)$. See (3.9) for a discussion of this case.

Goncharov does not explicitly compute λ. We will need to compute λ when $\mathfrak{g} = sl(n+1)$, and $\mathfrak{p} = \mathfrak{p}_j$ for $1 < j < n$.

3.4. Let $\lambda \in \mathbb{C}$ be chosen as in Goncharov's Theorem and write $\varphi = \mathscr{F} \circ L'$: $U(\mathfrak{g}) \to \mathscr{D}(\mathfrak{r}^+)$, and ψ: $U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}})$ for the induced maps.

Remarks. (a) Recall that $\mathscr{D}(\mathfrak{r}^+) = S(\mathfrak{r}^-) \otimes_{\mathbb{C}} S(\mathfrak{r}^+)$ with $S(\mathfrak{r}^-) = \mathcal{O}(\mathfrak{r}^+)$.

Using the description of L given in (2.4), one may verify (as claimed in [10]) that

(i) $\varphi(\mathfrak{r}^-) = \mathfrak{r}^-$,

(ii) $\varphi(\mathfrak{m}) \subset (\mathfrak{r}^- \otimes \mathfrak{r}^+) \oplus \mathbb{C}$,

(iii) $\varphi(\mathfrak{r}^+) \subset (\mathfrak{r}^- \otimes S^2(\mathfrak{r}^+)) \oplus (1 \otimes \mathfrak{r}^+)$.

In particular, the image of $U(\mathfrak{r}^-)$ is $S(\mathfrak{r}^-)$. Therefore the elements of $\mathfrak{r}^-$ act on $S(\mathfrak{r}^-) = \mathcal{O}(\mathfrak{r}^+)$, and so on $\mathcal{O}(\bar{\mathbf{X}})$, as multiplication operators.

(b) (Notation 2.1). The observations in (a) show that the $U(\mathfrak{g})$ action on $S(\mathfrak{r}^-)$ obtained through φ satisfies $U(\mathfrak{g})_1 \cdot S(\mathfrak{r}^-)_q = S(\mathfrak{r}^-)_{q+1}$. Hence for all $p, q \in \mathbb{N}$, $U(\mathfrak{g})_p \cdot S(\mathfrak{r}^-)_q = S(\mathfrak{r}^-)_{p+q}$.

(c) It follows from (a) that $\mathcal{O}(\bar{\mathbf{X}}) \subset \psi(U(\mathfrak{g}))$. For $p \in \bar{\mathbf{X}}$, define $\psi(U(\mathfrak{g}))_p = \psi(U(\mathfrak{g})) \otimes_{\mathcal{O}(\bar{\mathbf{X}})} \mathcal{O}_{\bar{\mathbf{X}},p}$. Since $L'(\mathfrak{m} \oplus \mathbb{C}) = L(\mathfrak{m} \oplus \mathbb{C})$, it follows that $\varphi(U(\mathfrak{m})) = (\mathscr{F} \circ L)(U(\mathfrak{m}))$. Hence Proposition 2.7 shows that, for $p \in \mathbf{X}$, $\mathcal{O}_{\bar{\mathbf{X}},p}$ and $\psi(U(\mathfrak{m}))$ generate $\mathscr{D}_{\bar{\mathbf{X}},p}$. In particular, for $p \in \mathbf{X}$, $\psi(U(\mathfrak{g}))_p = \mathscr{D}_{\bar{\mathbf{X}},p}$, and one even has $\psi(U(\mathfrak{p}^-))_p = \mathscr{D}_{\bar{\mathbf{X}},p}$.

The problem in showing that $\psi\colon U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}})$ is surjective is to extend the equality $\psi(U(\mathfrak{g}))_p = \mathscr{D}_{\bar{\mathbf{X}},p}$ for $p \in \mathbf{X}$ to all $p \in \bar{\mathbf{X}}$. Since $\bar{\mathbf{X}} = \mathbf{X} \cup \{0\}$, the only problem occurs at $p = 0$. If $\mathbf{X}$ is smooth (i.e., $\mathfrak{g} = sl(n+1)$ and $\mathfrak{p} = \mathfrak{p}_1$ or $\mathfrak{p}_n$) then ψ is not surjective (3.9). However, in all other cases $\operatorname{Sing} \bar{\mathbf{X}} = \{0\}$, and ψ is surjective (Theorem 5.2). Of course, when $\bar{\mathbf{X}}$ is singular then $\mathscr{D}(\bar{\mathbf{X}})$ is not generated by $\mathcal{O}(\bar{\mathbf{X}})$ and $\operatorname{Der} \mathcal{O}(\bar{\mathbf{X}})$. The point is that the other generators of $\mathscr{D}(\bar{\mathbf{X}})$ come from $\psi(\mathfrak{r}^+)$.

(d) An explicit description of ψ for $\mathfrak{g} = so(7)$ is given in [17, Sect. 3.3].

3.5 Proposition. (a) $\operatorname{Fract}(U(\mathfrak{g})/\ker \psi) = \operatorname{Fract} \mathscr{D}(\bar{\mathbf{X}})$;

(b) $\ker \psi$ *is a completely prime primitive ideal*;

(c) $\mathcal{O}(\bar{\mathbf{X}})$ *is a highest weight module, with highest weight vector* 1;

(d) $\ker \psi = \operatorname{Ann} \mathcal{O}(\bar{\mathbf{X}})$;

(e) if $\mathfrak{g} \neq sl(n+1)$, *then* $\ker \psi = J_0$ (*the Joseph ideal*), *and* $\mathcal{O}(\bar{\mathbf{X}})$ *is a simple* $U(\mathfrak{g})$*-module.*

Proof. (a) is an immediate consequence of Remark 3.4(c). Since the centre of $\operatorname{Fract} \mathscr{D}(\bar{\mathbf{X}})$ is $\mathbb{C}$, certainly the centre of $U(\mathfrak{g})/\ker \psi$ equals $\mathbb{C}$. Thus (b) follows from [6]. Since $\varphi(U(\mathfrak{r}^-)) = S(\mathfrak{r}^-)$, it is clear that $\psi(U(\mathfrak{r}^-)) \cdot 1 = S(\mathfrak{r}^-) \cdot 1 = \mathcal{O}(\bar{\mathbf{X}})$. On the other hand, Remark 3.4(a) shows that $\varphi(\mathfrak{r}^+) \cdot 1 = 0$, and $\varphi(\mathfrak{m}) \cdot 1 \subset \mathbb{C} \cdot 1 = \mathbb{C}$. Hence $\mathbb{C} \cdot 1$ is a trivial $[\mathfrak{m}, \mathfrak{m}]$-module. Thus $\psi(\mathfrak{n}^+) \cdot 1 = 0$, and (c) holds. Clearly $\ker \psi \subset \operatorname{Ann} \mathcal{O}(\bar{\mathbf{X}})$. On the other hand, $\mathcal{O}(\bar{\mathbf{X}})$ is a faithful $\mathscr{D}(\bar{\mathbf{X}})$-module, whence $\operatorname{Ann} \mathcal{O}(\bar{\mathbf{X}}) \subset \ker \psi$. Hence (d) holds.

It remains to prove (e). By (d), $\dim_{\mathbb{C}} U(\mathfrak{g})/\ker \psi = \infty$. Since $\ker \psi$ is a completely prime ideal, [12, Lemma 8.8] implies that $GK\dim U(\mathfrak{g})/\ker \psi \geqslant \dim \bar{\mathbf{O}}_{\min}$ with equality if and only if $\ker \psi = J_0$. Pick $0 \neq f \in \mathcal{O}(\bar{\mathbf{X}})$ such that $\operatorname{Sing} \bar{\mathbf{X}} \subset f^{-1}(0)$. Write $\mathbf{X}_f = \bar{\mathbf{X}} \backslash f^{-1}(0)$. Then $\mathbf{X}_f$ is non-singular, and by [21, Corollary 2.3], $GK\dim \mathscr{D}(\mathbf{X}_f) = 2\dim \mathbf{X}_f = 2\dim \bar{\mathbf{X}}$. Thus by [12] $GK\dim \mathscr{D}(\bar{\mathbf{X}}) = 2\dim \bar{\mathbf{X}} = \dim \mathbf{O}_{\min}$. Since $U(\mathfrak{g})/\ker \psi \subset \mathscr{D}(\bar{\mathbf{X}}) \subset \mathscr{D}(\mathbf{X}_f)$ this implies that $GK\dim U(\mathfrak{g})/\ker \psi \leqslant \dim \mathbf{O}_{\min}$, and hence $\ker \psi = J_0$. If $\mathcal{O}(\bar{\mathbf{X}})$ has a proper factor, say N, as a $U(\mathfrak{g})$-module, then N has non-zero annihilator as an $\mathcal{O}(\bar{\mathbf{X}})$-module. Since $\mathcal{O}(\bar{\mathbf{X}}) \subset \psi(U(\mathfrak{g}))$, this implies that N has nonzero annihilator as a $U(\mathfrak{g})/\ker \psi$-module. This contradicts the fact that $\ker \psi = J_0$ is a maximal ideal. ∎

Remark. Not only is $\mathcal{O}(\bar{\mathbf{X}})$ a factor of a Verma module, but it is a factor of a generalised Verma module, induced from a 1-dimensional representation of $\mathfrak{p}$. This is because $\varphi(\mathfrak{m} \oplus \mathfrak{r}^+) \cdot 1 \subset \mathbb{C} \cdot 1$.

3.6. When $\mathfrak{g} = sl(n+1)$, there is no unique completely prime primitive ideal associated to the minimal orbit, and so a little extra work is required to identify $\ker \psi$ as a specific $J(\mu)$. The rest of Section 3 deals with this question.

For the rest of Section 3, let $\mathfrak{g} = sl(n+1)$, and let $\mathfrak{p}_j$ be the parabolic obtained by deleting α_j. Write $\mathfrak{g} = \mathfrak{r}_j^+ \oplus \mathfrak{m}_j \oplus \mathfrak{r}_j^-$, and let $\bar{\mathbf{X}}_j \subset \mathfrak{r}_j^+$ be the closure of the cone of highest weight vectors. Let $\psi_j \colon U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}}_j)$ be as in (3.4). To explicitly identify the various $\ker \psi_j$, and hence to show they are distinct, we will compute the highest weight of $\mathcal{O}(\bar{\mathbf{X}}_j)$. Write ω_j for the fundamental weight corresponding to the simple root α_j. For $1 < j < n$, it will be shown that the highest weight is $-\omega_j$ (Corollary 3.7), that $\mathcal{O}(\bar{\mathbf{X}}_j)$ is a simple module, and that $\ker \psi_j = J(\rho - \omega_j)$ is a maximal ideal. All these statements are false when $j = 1$ or n. See (3.9) for a discussion of these two special cases.

Fix j, $1 \leqslant j \leqslant n$, and drop the subscript j from $\mathfrak{p}_j$, $\mathbf{X}_j$, etc. Let η denote the highest weight of $\mathcal{O}(\bar{\mathbf{X}})$. Write $\gamma = 2(n+1)$. This constant will frequently appear because the Killing form on $sl(n+1)$ satisfies $B(e_{ab}, e_{ba}) = 2(n+1)$, where the elements e_{ab} $(1 \leqslant a,\ b \leqslant n+1)$ are the usual matrix units. Set $d = \dim \mathfrak{r}^+ = j(n-j+1)$, and let $\{x_1, ..., x_d\} = \{e_{ab} \mid 1 \leqslant a \leqslant j,\ j+1 \leqslant b \leqslant n+1\}$ be a basis for $\mathfrak{r}^+$. As in (2.3), $\{y_1, ..., y_d\}$ is the dual basis for $\mathfrak{r}^-$. Hence if $x_i = e_{ab}$, then $y_i = \gamma^{-1} e_{ba}$. We shall sometimes write $y_{ab} := \gamma^{-1} e_{ab}$.

LEMMA. *The highest weight η of $\mathcal{O}(\bar{\mathbf{X}})$ is given by $\eta(H) = \operatorname{trace}_{\mathfrak{r}^-}(H) - \lambda B(x, y)$, where $H \in \mathfrak{h}$ satisfies $H = [x, y]$ for $x \in \mathfrak{r}^+$, $y \in \mathfrak{r}^-$ (and λ is chosen as in Theorem 3.3). In particular, $\eta(H_{\alpha_i}) = 0$ for $i \neq j$, and $\eta(H_{\alpha_j}) = -\gamma(\lambda + \frac{1}{2})$.*

Proof. Recall Corollary 2.5, with $V=\mathfrak{r}^-$. With the notation of (2.5), $(\mathscr{F}\circ L)(H)=\alpha(H)+\operatorname{trace}_{\mathfrak{r}^-}(H)$. However, $\alpha(H)=\sum_{i=1}^d [H, y_i]\partial/\partial y_i$, and in particular $\alpha(H)\cdot 1=0$. Thus $\mathscr{F}(L_H)\cdot 1=\operatorname{trace}_{\mathfrak{r}^-}(H)$, and by (3.5), $1\in\mathscr{C}(\bar{\mathbf{X}})$ is the highest weight vector. By Theorem 3.3, the action of $\mathfrak{g}$ on $\mathscr{C}(\bar{\mathbf{X}})$ defined through ψ is such that

$$\psi(H)\cdot 1=\mathscr{F}(L'_H)\cdot 1=\mathscr{F}(L_H)\cdot 1-\lambda B(x,y)\cdot 1=\operatorname{trace}_{\mathfrak{r}^-}(H)-\lambda B(x,y),$$

where $H\in\mathfrak{h}$ satisfies $H=[x,y]$ with $x\in\mathfrak{r}^+$, $y\in\mathfrak{r}^-$. In particular, for $H=H_{\alpha_j}$, note that $\operatorname{trace}_{\mathfrak{r}^-}(H)=-(n+1)$, whence $\eta(H)=-(n+1)-\lambda\gamma=-\gamma(\lambda+\frac{1}{2})$. On the other hand, we observed in (3.5) that $[\mathfrak{m},\mathfrak{m}]\cdot 1=0$. Hence $\eta(H_{\alpha_i})=0$ if $i\neq j$. ∎

3.7 (Notation 3.6). By Lemma 3.6, in order to calculate $\eta(H)$ we need to know the value of λ and thus, by (3.3) we need to compute $\mathscr{F}(L_X)(I)$ for $X\in\mathfrak{r}^+$, where I is the ideal of $\mathscr{C}(\mathfrak{r}^+)=S(\mathfrak{r}^-)$ that defines $\bar{\mathbf{X}}$. By (3.2(ii)), I is generated by the polynomials $\{y_{ab}y_{cd}-y_{ad}y_{cb}\mid 1\leqslant a,\ c\leqslant j,\ j+1\leqslant b,\ d\leqslant n+1\}$. Theorem 3.3 says that to find λ one must look at the action of $L_X+\lambda X$ on the various $y_{ab}y_{cd}-y_{ad}y_{cv}$ when $X\in\mathfrak{r}^+$. We shall adopt the notation and conventions outlined in Section 2.

PROPOSITION. *Let $X=e_{ab}\in\mathfrak{r}^+$ with $1\leqslant a\leqslant j$, $j+1\leqslant b\leqslant n+1$. Then (with $i=(-1)^{1/2}$)*

(a) $L_X=-\gamma^{-1}\Sigma\{e_{cb}e_{ad}(\partial/\partial e_{cd})\mid 1\leqslant c\leqslant j,\ j+1\leqslant d\leqslant n+1\}$;

(b) $\mathscr{F}(L_X)=i\gamma^{-1}\{\Sigma(y_{cd}(\partial/\partial y_{cb})(\partial/\partial y_{ad})\mid 1\leqslant c\leqslant j,\ j+1\leqslant d\leqslant n+1\}+(n+1)(\partial/\partial y_{ab})\}$;

(c) *if $a\neq l$, $b\neq k$ and $1\leqslant l\leqslant j$, $j+1\leqslant k\leqslant n+1$ then* $\mathscr{F}(L_X)(y_{ab}y_{lk}-y_{ak}y_{lb})=i\gamma^{-1}by_{lk}$.

Proof. Recall from (2.4) that $L_X=\Sigma\{L_X(e_{cd})(\partial/\partial e_{cd})\mid 1\leqslant c\leqslant j,\ j+1\leqslant d\leqslant n+1\}$, and for $z\in\mathfrak{r}^+$ $L_X(z)=\sum_J z(y^JX)x^J/J!$. Fix $z=e_{cd}$. Since X is of weight $\varepsilon_a-\varepsilon_b$, and z is of weight $\varepsilon_c-\varepsilon_d$, it follows that if $z(y^JX)\neq 0$, then $|J|=2$, and $y^J=y_ly_k$ with the sum of the weights of y_l and y_k equal to $(\varepsilon_b-\varepsilon_a)+(\varepsilon_d-\varepsilon_c)=(\varepsilon_b-\varepsilon_c)+(\varepsilon_d-\varepsilon_a)$. In that case $z(y^JX)=z([y_l[y_k,X]])=B(z,[y_l[y_k,X]])$.

If $a\neq c$ and $b\neq d$, then y^J is either $\gamma^{-2}e_{ba}e_{dc}$ or $\gamma^{-2}e_{bc}e_{da}$. Note that $B(e_{cd},[e_{ba}[e_{dc},e_{ab}]])=0$, and therefore $L_X(e_{cd})=-\gamma^{-1}e_{cb}e_{ad}$. If $a=c$ and $b\neq d$, then $y^J=\gamma^{-2}e_{ba}e_{da}$, and therefore $L_X(e_{cd})=-\gamma^{-1}e_{ab}e_{ad}$. If $a\neq c$ and $b=d$, then $y^J=\gamma^{-2}e_{ba}e_{bc}$, and $L_X(e_{cd})=-\gamma^{-1}e_{cb}e_{ad}$. If $a=c$ and $b=d$, then $y^J=\gamma^{-2}e_{ba}^2$, and $L_X(e_{cd})=-\gamma^{-1}e_{cd}e_{ad}$. This proves (a).

(b) Note that $\mathscr{F}(e_{cb}e_{ad}(\partial/\partial e_{cd}))=-i(\partial/\partial y_{cb})(\partial/\partial y_{ad})y_{cd}$. If $a\neq c$ and $b\neq d$, this equals $-iy_{cd}(\partial/\partial y_{cb})(\partial/\partial y_{ad})$. If $a=c$ and $b\neq d$, this equals $-i(y_{cd}(\partial/\partial y_{ad})+1)(\partial/\partial y_{cb})$. If $a\neq c$ and $b=d$, this equals $-i(y_{cd}(\partial/\partial y_{cb})$

$+1)(\partial/\partial y_{ad})$. If $a=c$ and $b=d$, this equals $-i(y_{cd}(\partial/\partial y_{cb})+2)(\partial/\partial y_{ad})$. Summing completes the proof of (b).

(c) This follows easily from (b). ∎

The condition in (c) is vacuous if $j=1$ or n.

COROLLARY. *Let* $1<j<n$. *If* λ *is as in Goncharov's Theorem* (3.3) *then* (a) $\lambda=-n\gamma^{-1}$, *and* (b) *the highest weight of* $\mathcal{O}(\bar{\mathbf{X}}_j)$ *is* $-\omega_j$.

Proof. (a) Let $X=e_{ab}\in \mathfrak{r}^+$, with $1\leqslant a\leqslant j$, and $j+1\leqslant b\leqslant n+1$. Note that $\mathcal{F}(\lambda X)(y_{ab}y_{lk}-y_{ak}y_{lb})=i\lambda(\partial/\partial y_{ab})(y_{ab}y_{lk}-y_{ak}y_{lb})=i\lambda y_{lk}$. Thus by Proposition 3.7(c), λ must satisfy $(in\gamma^{-1}+i\lambda)\,y_{lk}=0$.

(b) By part (a) and Lemma 3.6, $\eta(H_{\alpha_j})=-\gamma(\lambda+\frac{1}{2})=-\gamma(-n\gamma^{-1}+\frac{1}{2})$ $=n-(n+1)=-1$. Since $\eta(H_{\alpha_i})=0$ for $i\neq j$, this gives the result. ∎

3.8 THEOREM. *Let* $\mathfrak{g}=sl(n+1)$, *and* $\mathfrak{p}=\mathfrak{p}_j$ *with* $1<j<n$. *Then*

(a) $\mathcal{O}(\bar{\mathbf{X}}_j)\cong L(\rho-\omega_j)$,

(b) $\ker\psi_j=J(\rho-\omega_j)$.

(c) *In particular, for* $1<j<n$, *the* $\ker\psi_j$ *are distinct maximal ideals of* $U(sl(n+1))$.

Proof. Corollary 3.7 implies that $L(\rho-\omega_j)$ is a quotient of $\mathcal{O}(\bar{\mathbf{X}})$. Now by Remark 3.4(b), for any $U(\mathfrak{g})$-module factor module N of $\mathcal{O}(\bar{\mathbf{X}})$, one has $GK\dim_{\mathcal{O}(\bar{\mathbf{X}})}N=GK\dim_{U(\mathfrak{g})}N$. In particular, since $\mathcal{O}(\bar{\mathbf{X}})$ is a domain $GK\dim_{U(\mathfrak{g})}N<GK\dim\mathcal{O}(\bar{\mathbf{X}})$ whenever N is a proper factor of $\mathcal{O}(\bar{\mathbf{X}})$.

But $GK\dim\mathcal{O}(\bar{\mathbf{X}})=\dim\bar{\mathbf{X}}=(\frac{1}{2})\dim\bar{\mathbf{O}}_{\min}$. In particular, either $\mathcal{O}(\bar{\mathbf{X}})=L(\rho-\omega_j)$, or $GK\dim L(\rho-\omega_j)<(\frac{1}{2})\dim\bar{\mathbf{O}}_{\min}$, in which case $L(\rho-\omega_j)$ is finite dimensional (by [12, 13]). But the latter cannot occur as $\rho-\omega_j$ is not dominant integral. Hence $\mathcal{O}(\bar{\mathbf{X}})=L(\rho-\omega_j)$. Part (b) now follows from Proposition 3.5(d).

For distinct j $(1<j<n)$, the orbits of the $\rho-\omega_j$ under the Weyl group are distinct (the $\rho-\omega_j$ are all in the dominant chamber and are distinct). Hence the $J(\rho-\omega_j)$ are distinct. The maximality of the $J(\rho-\omega_j)$ follows, for example from [6]. ∎

3.9. It is clear from our earlier remarks that the case when $\mathfrak{g}=sl(n+1)$ and $\mathfrak{p}=\mathfrak{p}_1$ or $\mathfrak{p}_n$ is "exceptional." This is essentially because in that case $\bar{\mathbf{X}}=\mathfrak{r}^+$, and $I=0$. Thus Theorem 3.3 holds for every $\lambda\in\mathbb{C}$, and so for each λ there is a ring homomorphism ψ_λ: $U(\mathfrak{g})\to\mathcal{D}(\mathfrak{r}^+)$. Set $J_\lambda=\ker\psi_\lambda$. Let $j=1$ or n. By (3.4)–(3.6) $\mathcal{O}(\bar{\mathbf{X}}_j)$ is the generalized Verma module $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}\mathbb{C}$ of highest weight $-2(n+1)(\lambda+\frac{1}{2})\omega_j$. Thus, sometimes $\mathcal{O}(\bar{\mathbf{X}}_j)$ is simple, in which case $J_\lambda=J(\rho-2(n+1)(\lambda+\frac{1}{2})\omega_j)$.

To show that ψ_λ is not surjective in either of these cases, we use the following argument shown to us by A. Joseph. In fact, if $\mathfrak{g}$ is any semi-simple Lie algebra then a factor ring $R = U(\mathfrak{g})/J$ cannot contain elements x, y such that $xy - yx = 1$. To see this note that, as an ad-$\mathfrak{g}$ module, R decomposes as $R = \mathbb{C} \cdot 1 \oplus V$ for some submodule V. For any $a, b, c \in R$ one has $[ab, c] = [a, bc] + [b, ca]$; hence $[R, R] = [\mathfrak{g}, R] = [\mathfrak{g}, V] \subset V$. So $1 \notin [R, R]$.

4. The Minimal Orbit

4.1. In Section 4, $\mathfrak{g}$ will denote one of the Lie algebras with parabolic subalgebra $\mathfrak{p}$ described in Table 3.1. One of the key steps in proving that Goncharov's map ψ: $U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}})$ is surjective is to prove that $\dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) \leqslant \dim(\bar{\mathbf{O}}_{\min}) - 2$. In this section we prove that this inequality holds except in the case when $\mathfrak{g} = sl(n+1)$, and $\mathfrak{p} = \mathfrak{p}_1$ or $\mathfrak{p}_n$ (Notation 3.1). It is only at this point that the proof of the surjectivity of ψ will fail for $(\mathfrak{g}, \mathfrak{p}) = (sl(n+1), \mathfrak{p}_1$ or $\mathfrak{p}_n)$. Since ψ is not surjective in this case (see (3.9)), one is forced to conclude that $\dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) = \dim(\bar{\mathbf{O}}_{\min}) - 1$. A direct proof of this fact is given in (4.8).

4.2. Given an orbit $\mathbf{O}$ in $\mathfrak{g}$, and $W \in \mathbf{O}$, then the tangent space to $\mathbf{O}$ at W is $T_W\mathbf{O} = [W, \mathfrak{g}]$. Given $X \in \mathfrak{g}$, view X as a function on $\mathfrak{g}$ via the Killing form B. Then the differential dX is an element of $T^*\mathfrak{g} = \mathfrak{g}^*$, and we may again identify dX with X via $dX = B(X, -) \in \mathfrak{g}^*$. Thus, if $[W, Z] \in T_W\mathbf{O}$, then $dX \cdot [W, Z] = B(X, [W, Z]) = B([X, W], Z)$. In particular, if we set $\mathfrak{g}(W) = \{X \in \mathfrak{g} \mid [X, W] = 0\}$ then $\{X \in \mathfrak{g} \mid dX = 0 \text{ on } T_W\mathbf{O}\} = \mathfrak{g}(W)$.

4.3. The strategy is to use the observation in (4.2) to obtain an upper bound on the dimension of the local rings $\mathscr{O}(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)_Y$, for $Y \in \mathbf{O}_{\min} \cap \mathfrak{p}^-$, and so obtain a bound on $\dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)$.

Lemma. *Let $W \in \mathbf{O}_{\min} \cap \mathfrak{p}^-$. Suppose that $Y_1, ..., Y_S$ are linearly independent elements of $\mathfrak{r}^-$ such that $(\sum \mathbb{C} Y_j) \cap \mathfrak{g}(W) = 0$. Then $\dim \mathscr{O}(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)_W \leqslant \dim(\bar{\mathbf{O}}_{\min}) - s$.*

Proof. Observe that $\mathscr{O}(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) = \mathscr{O}(\bar{\mathbf{O}}_{\min})/\mathfrak{r}^- \mathscr{O}(\bar{\mathbf{O}}_{\min})$, since the ideal in $S(\mathfrak{g}) = \mathscr{O}(\mathfrak{g})$ of functions vanishing on $\mathfrak{p}^-$ is generated by $\mathfrak{r}^-$. Let $W \in \mathbf{O}_{\min} \cap \mathfrak{p}^-$, and write R for the local ring $\mathscr{O}(\bar{\mathbf{O}}_{\min})_W$, with maximal ideal $\boldsymbol{m}_W$. If $Y_1, ..., Y_S$ are as in the statement of the lemma, then certainly each $Y_i \in \boldsymbol{m}_W$. Furthermore, by (4.2) $dY_1, ..., dY_S$ are linearly independent in $T^*_W\mathbf{O}_{\min}$. Since $T^*_W\mathbf{O}_{\min} = \boldsymbol{m}_W/(\boldsymbol{m}_W)^2$ and R is regular, this is equivalent to saying that $Y_1, ..., Y_S$ form part of a system of parameters for R. Thus $\dim(R/\mathfrak{r}^- R) \leqslant \dim(R/\sum Y_i R) = \dim R - s$, as required. ∎

4.4. To apply Lemma 4.3 we must consider some special cases. Let $W \in \bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-$ and write $W = W_0 + W_-$, where $W_0 \in \mathfrak{m}$ and $W_- \in \mathfrak{r}^-$. A standard density argument (given in detail in Theorem 4.7) will show that we only need an upper bound on $\dim \mathcal{O}(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)_W$ for $W_0 \neq 0$, so we will assume for the rest of (4.4) that $W_0 \neq 0$.

As in (3.1(vii)), write $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}] \oplus \mathbb{C}H$ with $H \in \mathfrak{h}$. If $W_0 \in \mathbb{C}H$ then, since $\mathfrak{r}^-$ is abelian, W acts ad-semi-simply on $\mathfrak{r}^-$, contradicting the fact that $W \in \bar{\mathbf{O}}_{\min}$. Thus W has a non-zero component in $[\mathfrak{m}, \mathfrak{m}]$. If $\mathfrak{g} = sl(n+1)$, and $\mathfrak{p} = \mathfrak{p}_j$ for $1 < j < n$, then $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with each $\mathfrak{m}_i$ simple. Order the $\mathfrak{m}_i$ such that W has a non-zero component in $\mathfrak{m}_i$, and set $\mathfrak{a} = \mathfrak{m}_2 \oplus \mathbb{C}H$. If $\mathfrak{g}$ is not of type A_n then $\mathfrak{m}_1 := [\mathfrak{m}, \mathfrak{m}]$ is simple, and we set $\mathfrak{a} = \mathbb{C}H$.

CLAIM. *With the above notation and assumptions, we may assume, in applying Lemma* 4.3 *to determine* $\dim \mathcal{O}(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)_W$, *that* $W_0 = bX_\mu + Z$, *where μ is the highest root of* $\mathfrak{m}_1$, $0 \neq b \in \mathbb{C}$, *and Z is a sum of root vectors in* $\mathfrak{m}$, *having weights distinct from* μ.

Proof. Write $W_0 = W_1 + W_2$ with $0 \neq W_1 \in \mathfrak{m}_1$ and $W_2 \in \mathfrak{a}$. Since $\mathfrak{m}_1$ is simple, $M_1 \cdot W_1$ spans $\mathfrak{m}_1$. Thus there exists $g \in M_1$ such that $g \cdot W_0 = bX_\mu + Z$ is as in the claim. But multiplication by $g \in M_1$ acts as an automorphism on $\mathcal{O}(\bar{\mathbf{O}}_{\min})$ that preserves $\mathfrak{r}^- \mathcal{O}(\bar{\mathbf{O}}_{\min})$. Thus $\mathcal{O}(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)_W = \mathcal{O}(\bar{\mathbf{O}}_{\min})_W / \mathfrak{r}^- \mathcal{O}(\bar{\mathbf{O}}_{\min})_W \cong \mathcal{O}(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)_{g \cdot W}$ and so these algebras certainly have the same dimension. ∎

4.5. Assume that $W = W_0 + W_- \in \bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-$ with W_0 as in the claim above (so $W_0 \neq 0$). We intend to find elements that satisfy the hypotheses of Lemma 4.3. To do this choose positive roots $\gamma(1), ..., \gamma(s)$ of $\mathfrak{g}$ such that

(4.5.1) $X_{-\gamma(1)}, ..., X_{-\gamma(s)} \in \mathfrak{r}^-$;

(4.5.2) for $1 \leqslant i \leqslant s$, $\mu - \gamma(i)$ is a root not equal to any $-\gamma(j)$ for $1 \leqslant j \leqslant s$;

(4.5.3) $[\mathfrak{a}, \sum \mathbb{C}X_{-\gamma(j)}] \subset \sum \mathbb{C}X_{-\gamma(j)}$, where $\mathfrak{a}$ is defined as in (4.4);

(4.5.4) s is maximal with respect to (4.5.1)–(4.5.3).

We remark that if $\mathfrak{g} \neq sl(n+1)$, then (4.5.3) is trivially satisfied since $\operatorname{ad} H|_{\mathfrak{r}} = 1$.

LEMMA. *If the roots $\gamma(j)$ are as above, then* $(\sum \mathbb{C}X_{-\gamma(j)}) \cap \mathfrak{g}(W) = 0$.

Proof. This is a routine highest weight argument. Partially order the roots R of $\mathfrak{g}$, such that $\alpha \leqslant \beta$ if and only if $\beta - \alpha \in R^+$. Write $W_0 = bX_\mu + W' + T$, where $W' \in \mathfrak{m}_1$ has no component of weight μ, and $T \in \mathfrak{a}$. Suppose that $0 \neq Y \in (\sum \mathbb{C}X_{-\gamma(j)}) \cap \mathfrak{g}(W)$. Write $Y = \sum a_\alpha X_{-\alpha} + \sum a_\beta X_{-\beta}$, where $a_\alpha, a_\beta \in \mathbb{C} \backslash \{0\}$, and the α's are the minimal elements among the $\gamma(j)$ under the partial ordering on R. Since $\mathfrak{r}^-$ is abelian and $Y \in \mathfrak{g}(W)$,

$$0 = [Y, W] = [Y, W_0] = b \sum a_\alpha [X_{-\alpha}, X_\mu] + b \sum a_\beta [X_{-\beta}, X_\mu]$$
$$+ [Y, T] + [Y, W']$$

By (4.5.2), the $[X_{-\alpha}, X_\mu]$ are distinct non-zero roots vectors, so they must cancel with terms from one of the other three sums. Since the α's are minimal they cannot cancel with the $[X_{-\beta}, X_\mu]$ terms. By (4.5.2) and (4.5.3) they cannot cancel with the $[Y, T]$ terms. Hence $[X_{-\alpha}, X_\mu]$ must have the same weight as some $[X_{-\gamma(j)}, X_\nu]$ with X_ν a component of W'. But then $\alpha - \gamma(j) = \mu - \nu > 0$, by the choice of μ. This contradicts the minimality of the α's. ∎

4.6. A case by case examination is needed to determine the s of (4.5). We adopt Bourbaki's notation [5, Planches I–VI, pp. 250–266] for the root system of $\mathfrak{g}$. We also retain the notation of (3.1) and (4.5), except that we write γ_j rather than $\gamma(j)$.

(4.6.1) ($\mathfrak{g}$ of type A_n, $\mathfrak{p} = \mathfrak{p}_j$, $1 \leqslant j \leqslant n$). There are two possibilities for $\mathfrak{m}_1$. If $\mathfrak{m}_1$ has simple roots $\alpha_1, ..., \alpha_{j-1}$ then $\mu = \varepsilon_1 - \varepsilon_j$ and $\{\gamma_1, ..., \gamma_{n-j+1}\} = \{\varepsilon_1 - \varepsilon_k \mid j+1 \leqslant k \leqslant n+1\}$. If $\mathfrak{m}_1$ has simple roots $\alpha_{j+1}, ..., \alpha_n$ then $\mu = \varepsilon_{j+1} - \varepsilon_{n+1}$ and $\{\gamma_1, ..., \gamma_j\} = \{\varepsilon_k - \varepsilon_{n+1} \mid 1 \leqslant k \leqslant j\}$. Thus $s = \min\{j, n-j+1\}$.

(4.6.2) ($\mathfrak{g}$ of type B_n, $n \geqslant 3$). $\mu = \varepsilon_2 + \varepsilon_3$, $\gamma_1 = \varepsilon_1 + \varepsilon_2$, $\gamma_2 = \varepsilon_1 + \varepsilon_3$. Thus $s = 2$.

(4.6.3) ($\mathfrak{g}$ of type C_n, $n \geqslant 2$). $\mu = \varepsilon_1 - \varepsilon_n$, $\gamma_1 = 2\varepsilon_1$ and $\gamma_i = \varepsilon_1 + \varepsilon_i$ for $2 \leqslant i \leqslant n$. Thus $s = n$.

(4.6.4) ($\mathfrak{g}$ of type D_n, $n \geqslant 4$). There are two cases. First if $\mathfrak{p} = \mathfrak{p}_1$, then $\mu = \varepsilon_2 + \varepsilon_3$, $\gamma_1 = \varepsilon_1 + \varepsilon_2$, $\gamma_2 = \varepsilon_1 + \varepsilon_3$, and $s = 2$. Secondly, if $\mathfrak{p} = \mathfrak{p}_n$, then $\mu = \varepsilon_1 - \varepsilon_n$, and $\gamma_i = \varepsilon_{i+1} + \varepsilon_n$, for $1 \leqslant i \leqslant n-2$. Thus $s = n-2$.

(4.6.5) ($\mathfrak{g}$ of type E_6). We only need to consider the case $\mathfrak{p} = \mathfrak{p}_1$. Then

$$\mu = \begin{smallmatrix}01221,\\1\end{smallmatrix} \quad \gamma_1 = \begin{smallmatrix}11221,\\1\end{smallmatrix} \quad \gamma_2 = \begin{smallmatrix}12221,\\1\end{smallmatrix} \quad \gamma_3 = \begin{smallmatrix}12321,\\1\end{smallmatrix} \quad \gamma_4 = \begin{smallmatrix}12321.\\2\end{smallmatrix}$$

Thus $s = 4$.

(4.6.6) ($\mathfrak{g}$ of type E_7). We have

$$\mu = \begin{smallmatrix}123210,\\2\end{smallmatrix} \quad \gamma_1 = \begin{smallmatrix}123211,\\2\end{smallmatrix} \quad \gamma_2 = \begin{smallmatrix}123221,\\2\end{smallmatrix} \quad \gamma_3 = \begin{smallmatrix}123321,\\2\end{smallmatrix}$$

$$\gamma_4 = \begin{smallmatrix}124321,\\2\end{smallmatrix} \quad \gamma_5 = \begin{smallmatrix}134321,\\2\end{smallmatrix} \quad \gamma_6 = \begin{smallmatrix}234321.\\2\end{smallmatrix}$$

Thus $s = 6$.

4.7. The peceding results are now combined to give

THEOREM. *Let* $(\mathfrak{g}, \mathfrak{p})$ *be as in Table* 3.1, *and suppose that* $(\mathfrak{g}, \mathfrak{p}) \neq (sl(n+1), \mathfrak{p}_1)$ *or* $(sl(n+1), \mathfrak{p}_n)$. *Let* s *be as in* (4.6). *Then*

$$\dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) \leqslant \dim(\bar{\mathbf{O}}_{\min}) - s \leqslant \dim(\bar{\mathbf{O}}_{\min}) - 2.$$

Remark. For the classical groups, one of us (the first author) can prove that these inequalities are actually equalities.

Proof. The second inequality has been proved in (4.6). It is enough to prove that $\dim \mathcal{O}(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)_W \leqslant \dim(\bar{\mathbf{O}}_{\min}) - s$ for all W in some dense subset of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-$. As in (4.4) write $W = W_0 + W_-$. We have already proved in (4.4) and (4.5) that the inequality holds if $W_0 \neq 0$. Since $\mathfrak{r}^-$ is closed in $\mathfrak{p}^-$, the subset $U := \{W \in \bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^- \mid W_0 \neq 0\}$ is open in $\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-$. Let V be an irreducible component of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-$. Either $V \subset \bar{\mathbf{O}}_{\min} \cap \mathfrak{r}^- = \bar{\mathbf{X}}$, or $V \cap U \neq \varnothing$. In the first case $\dim V \leqslant \dim \bar{\mathbf{X}} \leqslant \dim(\bar{\mathbf{O}}_{\min}) - s$, by Table 3.1, and (4.6). In the second case $\dim V \leqslant \dim U$, and we have already seen that $\dim U \leqslant \dim(\bar{\mathbf{O}}_{\min}) - s$. ∎

4.8. For $\mathfrak{g} = sl(n+1)$ and $\mathfrak{p} = \mathfrak{p}_1$ or $\mathfrak{p}_n$, we have $s = 1$. One can show that in this case $\dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) = \dim(\bar{\mathbf{O}}_{\min}) - 1$. One roundabout proof is to note that, although the conclusion of Theorem 5.2 fails in this case (as was shown in Section 3.9), the only step in its proof which fails is Theorem 4.7. A direct proof is as follows:

PROPOSITION. *Let* $\mathfrak{g} = sl(n+1)$, *and* $\mathfrak{p} = \mathfrak{p}_1$ *or* $\mathfrak{p}_n$. *Then*

$$\dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) = \dim(\bar{\mathbf{O}}_{\min}) - 1.$$

Proof. We consider only the case $\mathfrak{p} = \mathfrak{p}_n$. Consider the natural representation of $\mathfrak{g}$ on $V = \mathbb{C}^{n+1}$. Then $\bar{\mathbf{O}}_{\min} = \{X \in \mathfrak{g} \mid \operatorname{rank}_V(X) \leqslant 1\}$. It is easy to see that $\dim \bar{\mathbf{O}}_{\min} = 2n$. Note that $\mathfrak{m} \cong sl(n) \oplus \mathbb{C}H$, where $sl(n) \subset sl(n+1)$ is identified with those matrices whose final row and column are zero.

Let $\pi: \mathfrak{p}^- \to \mathfrak{m}$ be the projection, with kernel $\mathfrak{r}^-$, and consider the restriction of π to $\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-$. The image is clearly $\bar{\mathbf{O}}_{\min} \cap \mathfrak{m}$, which coincides with the minimal nilpotent orbit in $\mathfrak{m}$. Hence $\dim \pi(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) = 2(n-1)$. Let $0 \neq x \in \pi(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)$, and let $y \in \pi^{-1}(x)$. Thus the component of y in $\mathfrak{m}$ equals x, and the last row of y is a scalar multiple of some row of x, since $\operatorname{rank}(x) = \operatorname{rank}(y) = 1$. In particular, $\dim \pi^{-1}(x) = 1$. Thus $\dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-) = 2(n-1) + 1 = 2n - 1$, as required. ∎

4.9 *Remark.* The fact that Theorem 4.7 does not hold for $sl(n+1)$ when $\mathfrak{p} = \mathfrak{p}_1$ or $\mathfrak{p}_n$ was a surprise to us. In fact, we once believed the opposite. Our error was in part due to our not believing the following fact which is a consequence of Proposition 4.8, and still strikes us as rather a surprise.

Let $S = \mathscr{D}(\mathbb{A}^n)$ be the nth Weyl algebra; say, $S = \mathbb{C}[t_1, ..., t_n, \partial_1, ..., \partial_n]$. Then there exists a finitely generated noetherian $\mathbb{C}$-algebra $R \subset S$ (in fact R is a homomorphic image of $U(sl(n+1))$) such that $GK \dim R = GK \dim S = 2n$, $\operatorname{Fract} R = \operatorname{Fract} S$, $t_1, ..., t_n \in R$ and $GK \dim(R/Rt_1 + \cdots + Rt_n) = GK \dim R - 1 = 2n - 1$. In contrast $GK \dim(S/St_1 + \cdots + St_n) = GK \dim S - n = n$.

The algebra R is obtained as $\varphi(U(sl(n+1)))$ when $\mathfrak{p} = \mathfrak{p}_1$ or $\mathfrak{p}_n$. R is generated by the elements t_i, $t_i \partial_j$, and $(t_1 \partial_1 + \cdots t_n \partial_n) \partial_i$, for $1 \leqslant i, j \leqslant n$.

5. The Main Theorem

5.1. In this section we prove the main theorem of the introduction. This follows very easily by combining the earlier results with the following lemma due to O. Gabber. First, however, recall that a finitely generated left $U(\mathfrak{g})$-module M is called *d-homogeneous* if $GK \dim M = GK \dim N = d$ for all non-zero submodules N of M (see [14, p. 68]).

Lemma. *Let $\mathfrak{g}$ be any finite dimensional Lie algebra over $\mathbb{C}$. Let M be a finitely generated d-homogeneous left $U(\mathfrak{g})$-module. Let Q be any $U(\mathfrak{g})$-module (not necessarily finitely generated) that contains M as an essential submodule. Then the set of left $U(\mathfrak{g})$-modules*

$$\mathscr{S} := \{M' \mid M \subset M' \subset Q, \text{ with } M' \text{ finitely generated and } GK \dim(M'/M) \leqslant d-2\}$$

contains a unique maximal element.

Proof. The argument of the third step in the proof of [8, Théorème 4.2.1] may be used unaltered. ∎

5.2 Theorem. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra not of type G_2, F_4, or E_8. Let $\mathfrak{p}$ be a parabolic subalgebra with abelian nilpotent radical (but exclude the parabolics $\mathfrak{p}_1$ and $\mathfrak{p}_n$ for $\mathfrak{g} = sl(n+1)$). Let $\bar{\mathbf{X}}$ be the irreducible component of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$ contained in the nilpotent radical of $\mathfrak{p}$. Let $\psi: U(\mathfrak{g}) \to \mathscr{D}(\bar{\mathbf{X}})$ be the map defined by Goncharov (see 3.3). Then ψ is surjective.*

Proof. We will apply Lemma 5.1 with $M = R = \psi(U(\mathfrak{g}))$ and $Q = \mathscr{D}(\bar{\mathbf{X}})$. Proposition 3.5(a) says that R is an essential submodule of Q as an R-module, and hence as a $U(\mathfrak{g})$-module. Furthermore R is d-homogeneous, since it is a domain.

We first show that, for all $q \in Q = \mathscr{D}(\mathbf{X})$, one has $Rq + R \in \mathscr{S}$, the set defined in Lemma 5.1. Let $q \in Q$, and set $K = \{a \in \mathcal{O}(\bar{\mathbf{X}}) \mid aq \in R\}$. By (3.4(c)), $K \subset R$, and there is a surjection $R/RK \to (Rq + R)/R$. Hence $GK \dim(Rq + R/R) \leqslant GK \dim(R/RK)$. Let $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}(\bar{\mathbf{X}})$, not equal to $\boldsymbol{m}$ the maximal ideal of the singular point $0 \in \bar{\mathbf{X}}$. The local ring $\mathcal{O}_{\bar{\mathbf{X}},\mathfrak{p}}$ is regular, so $\mathscr{D}_{\bar{\mathbf{X}},\mathfrak{p}}$ is generated by $\mathcal{O}_{\bar{\mathbf{X}},\mathfrak{p}}$ and $\operatorname{Der} \mathcal{O}_{\bar{\mathbf{X}},\mathfrak{p}}$. By Remark 3.4(c), $\mathscr{D}_{\bar{\mathbf{X}},\mathfrak{p}} = R_{\mathfrak{p}}$. Hence $q \in R_{\mathfrak{p}}$, and $K_{\mathfrak{p}} = \mathcal{O}_{\bar{\mathbf{X}},\mathfrak{p}}$. Thus $\boldsymbol{m}^r \subset K$ for some $r \geqslant 0$, and in order to prove that $GK \dim(R/RK) \leqslant GK \dim R - 2$ it suffices to prove that $GK \dim(R/R\boldsymbol{m}) \leqslant GK \dim R - 2$. To prove this inequality we may pass to the associated graded algebras; therefore it suffices to prove that $GK \dim(gr(R/R\boldsymbol{m})) \leqslant GK \dim(grR) - 2$. The GK-dimension of these factor rings of $S(\mathfrak{g}) = grU(\mathfrak{g})$ equals the dimension of the corresponding subvarieties of $\mathfrak{g}$. Hence $GK \dim(grR) = \dim \bar{\mathbf{O}}_{\min}$, and since $\boldsymbol{m}$ is generated by $\mathfrak{r}^-$, it follows that $GK \dim(gr(R/R\boldsymbol{m})) = \dim(\bar{\mathbf{O}}_{\min} \cap \mathfrak{p}^-)$ (because the subvariety of $\mathfrak{g}$ defined by the vanishing of $\mathfrak{r}^- \subset \mathcal{O}(\mathfrak{g})$ is $\mathfrak{p}^-$). Now Theorem 4.7 gives the desired inequality, and therefore $Rq + R \in \mathscr{S}$.

Since $Rq + R \in \mathscr{S}$ for all $q \in \mathscr{D}(\bar{\mathbf{X}})$, the only possible maximal element of $\mathscr{S}$ is $\mathscr{D}(\bar{\mathbf{X}})$ itself. Thus Lemma 5.1 implies that $\mathscr{D}(\bar{\mathbf{X}})$ is a finitely generated left R-module. By Proposition 3.5(a), $R \subset \mathscr{D}(\bar{\mathbf{X}}) \subset \operatorname{Fract} R$, and therefore we may write these generators as $d_i = e_i f^{-1}$ for some $e_i, f \in R$. Thus $\mathscr{D}(\bar{\mathbf{X}}) f \subset R$ and the right annihilator, $r\text{-}\operatorname{Ann}_R(\mathscr{D}(\bar{\mathbf{X}})/R)$, is a non-zero two-sided ideal of R. However, by Proposition 3.5(e) and Theorem 3.8(b), R is a simple ring. Thus $r\text{-}\operatorname{Ann}(\mathscr{D}(\bar{\mathbf{X}})/R) = R$ and $\mathscr{D}(\bar{\mathbf{X}}) = \psi(U(\mathfrak{g}))$. ∎

5.3. We emphasise the two contrasting special cases of the Theorm.

Corollary A. *Let* $\mathfrak{g}$ *be a simple of type* B_n, C_n, D_n, E_6, *or* E_7, *and let* J_0 *be the Joseph ideal of* $U(\mathfrak{g})$. *Then there exists an irreducible component* $\bar{\mathbf{X}}$ *of* $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$ *such that* $U(\mathfrak{g})/J_0 \cong \mathscr{D}(\bar{\mathbf{X}})$.

Corollary B. *Let* $\mathfrak{g}$ *be of type* A_n. *Then for* $n-2$ *of the* n *irreducible components* $\bar{X}_j$ *of* $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$, *there exist (distinct) maximal ideals* J_j *of* $U(\mathfrak{g})$ *such that* $U(\mathfrak{g})/J_j \cong \mathscr{D}(\bar{\mathbf{X}}_j)$.

5.4. *Remarks.* (a) It is a consequence of Theorem 5.2 that (with the notation of 5.2) $\operatorname{Der} \mathcal{O}(\bar{\mathbf{X}}) \subset \psi(U(\mathfrak{g}))$. This is satisfying to observe, because prior to the Theorem all one knows is that $\operatorname{Der} \mathcal{O}_{\bar{\mathbf{X}},p} \subset \psi(U(\mathfrak{g}))_p$ for $p \in \mathbf{X} = \bar{\mathbf{X}} \setminus \{0\}$.

(b) Theorem 5.2 also gives some information about $\mathfrak{g}_2$, the simple Lie algebra of type G_2. Let J_0 be the Joseph ideal in $U(so(7))$. By [17, Theorem 3.9] there is a completely prime ideal J_1 in $U(\mathfrak{g}_2)$ associated to $\mathbf{O}_8$, the 8-dimensional nilpotent orbit in $\mathfrak{g}_2$, such that $U(\mathfrak{g}_2)/J_1 \cong U(so(7))/J_0$. Hence if $\bar{\mathbf{X}}$ is the variety occurring in the Theorem for $so(7)$, then $U(\mathfrak{g}_2)/J_1 \cong \mathscr{D}(\bar{\mathbf{X}})$. However, it is also shown in [17, Sect. 5] that $\bar{\mathbf{X}}$ is isomorphic to an irreducible component of $\bar{\mathbf{O}}_8 \cap \mathfrak{n}_2^+$, where $\mathfrak{n}_2^+$ is the upper triangular part of $\mathfrak{g}_2$. This naturally leads one to ask whether there is a more extensive version of the theorem, realising other primitive factor rings as differential operators on certain components of $\bar{\mathbf{O}} \cap \mathfrak{n}^+$, where $\bar{\mathbf{O}}$ is the associated variety of the given primitive ideal.

(c) The main result of Moeglin in [19] together with [3] shows that every completely prime primitive factor ring of $U(sl(n+1))$ is of the form $\mathscr{D}_\mu(G/P)$, the global sections of a sheaf of twisted differential operators on G/P, for a suitable parabolic P, and a suitable twisting μ. Dimension arguments show that if the primitive ideal is associated to the minimal orbit then Lie P must be either $\mathfrak{p}_1$ or $\mathfrak{p}_n$, whence $G/P \cong \mathbb{P}^n$, projective n-space. Hence we have the curious fact that for the singular affine varieties $\bar{\mathbf{X}}_j$ occurring in Corollary B, $\mathscr{D}(\bar{\mathbf{X}}_j) \cong \mathscr{D}_\mu(\mathbb{P}^n)$ for a suitable choice of μ. It would be interesting if there were an a priori explanation of this isomorphism.

6. The Ring of Differential Operators Does Not Determine the Variety

6.1. If $\mathbf{X}$ is an irreducible affine variety it is interesting to see how the geometric properties of $\mathbf{X}$ are reflected in the algebraic structure of $\mathscr{D}(\mathbf{X})$. For example, Makar–Limanov [26] has recently shown that if $\mathbf{X}$ and $\mathbf{Y}$ are non-isomorphic curves of genus >0, then $\mathscr{D}(\mathbf{X})$ and $\mathscr{D}(\mathbf{Y})$ are non-isomorphic rings. In fact, he shows how the curve $\mathbf{X}$ may be recovered from $\mathscr{D}(\mathbf{X})$. The following proposition shows that for higher dimensional varieties, it is no longer true that $\mathbf{X}$ is completely determined by $\mathscr{D}(\mathbf{X})$. Our example involves a singular variety. As far as we know it is not known whether for smooth $\mathbf{X}$, $\mathscr{D}(\mathbf{X})$ completely determines $\mathbf{X}$.

Propostion. *Consider the parabolics $\mathfrak{p}_1$ and $\mathfrak{p}_n$ in $so(2n)$, for $n \geqslant 5$, and the corresponding components $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_n$ of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}^+$. Then $\mathscr{D}(\bar{\mathbf{X}}_1) \cong \mathscr{D}(\bar{\mathbf{X}}_n)$, but $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_n$ are non-isomorphic varieties. In fact $H^*(\mathbf{X}_1, \mathbb{Z}) \neq H^*(\mathbf{X}_n, \mathbb{Z})$.*

Proof. The fact that the rings of differential operators are isomorphic follows from Theorem 5.2, since both are isomorphic to $U(so(2n))/J_0$.

Recall the explicit descriptions of $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_n$ in (3.2). It is easy to see that $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_n$ are non-isomorphic varieties. To see this, let $\boldsymbol{m}_i$ be the maximal ideal of the unique singular point on $\bar{\mathbf{X}}_i$ $(i = 1, n)$. It is generated by the image in $\mathcal{O}(\bar{\mathbf{X}}_i)$ of $\mathfrak{r}_i \subset \mathcal{O}(\mathfrak{r}_i^+) = S(\mathfrak{r}_i^-)$. The ideal in $\mathcal{O}(\mathfrak{r}_i^+)$ defining $\bar{\mathbf{X}}_i$ is generated by homogeneous polynomials of degree 2, hence $\dim_{\mathbb{C}} \boldsymbol{m}_i/\boldsymbol{m}_i^2 = \dim \mathfrak{r}_i^+$. But $\dim \mathfrak{r}_1^+ = 2(n-1)$, and $\dim \mathfrak{r}_n^+ = n(n-1)/2$, whence $\bar{\mathbf{X}}_1 \not\cong \bar{\mathbf{X}}_n$.

Write $L_i = [M_i, M_i]$, and note that $\mathbf{X}_i = L_i \cdot x_i$, where $x_i \in \mathfrak{r}_i^+$ is a highest weight vector. Write $Q_i = \mathrm{Stab}_{L_i}(\mathbb{C}x_i)$, and $B_i = L_i/Q_i$. There is a fibration $\mathbb{C}\backslash\{0\} = \mathbb{C}^* \to \mathbf{X}_i \to B_i$. Let $Y_i \subset \mathbf{X}_i$ be the S^1-bundle over B_i obtained by collapsing each fibre $\mathbb{C}^*$ to the unit circle. Then $H^*(Y_i, \mathbb{Z}) = H^*(\mathbf{X}_i, \mathbb{Z})$.

Viewing Y_1 as a real manifold, it is a Steiffel manifold $S^{2n-4} \to Y_1 \to S^{2n-3}$. To see this recall that $\mathbf{X}_1 = \{(t_1, ..., t_{2n-2}) \in \mathbb{C}^{2n-2}\backslash\{0\} \mid \sum t_i^2 = 0\}$. If $t \in Y_1 \subset \mathbf{X}_1 \subset \mathbb{C}^{2n-2} = \mathbb{R}^{2n-2} \oplus i\mathbb{R}^{2n-2}$, is written as $t = (x, y) = (x_1, ..., x_{2n-2}, y_1, ..., y_{2n-2})$ then $\sum x_k^2 = \sum y_k^2 = \frac{1}{2}$ and $\sum x_k y_k = 0$. The projection $Y_1 \to \mathbb{R}^{2n-2}$, onto the first component, fibres Y_1 over S^{2n-3}, with fibres the spheres S^{2n-4} living in the tangent space to S^{2n-3}. Hence by [18] Y_1 is $(2n-5)$-connected, and $H^{2n-4}(Y_1, \mathbb{Z}) = H^{2n-3}(Y_1, \mathbb{Z}) = \mathbb{Z}$. In particular, since $n \geqslant 5$, $H^4(Y_1, \mathbb{Z}) = 0$.

Since B_n is the Grassmanian of 2-planes in $\mathbb{C}^n$, [18] gives for $n \geqslant 5$, $H^2(B_n, \mathbb{Z}) = \mathbb{Z}$ and $H^4(B_n, \mathbb{Z}) = \mathbb{Z}^2$. The cohomology of B_n is concentrated in even degree, so the Gysin sequence [23] for the fibration $S^1 \to Y_n \to B_n$ collapses to give short exact sequences

$$0 \to H^{2r-1}(Y_n, \mathbb{Z}) \to H^{2r-2}(B_n, \mathbb{Z}) \xrightarrow{c} H^{2r}(B_n, \mathbb{Z}) \to H^{2r}(Y_n, \mathbb{Z}) \to 0.$$

In particular for $r = 2$, c cannot be surjective, whence $H^4(Y_n, Z) \neq 0$.

Hence $H^4(\mathbf{X}_1, \mathbb{Z}) \neq H^4(\mathbf{X}_n, \mathbb{Z})$, and the proof is complete. ∎

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