JOURNAL OF ALGEBRA 71, 189–194 (1981)

Krull Dimension of the Enveloping Algebra of $sl(2, \mathbb{C})$

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Communicated by A. W. Goldie

Received September 30, 1980

1. INTRODUCTION

Let U denote the enveloping algebra of the simple Lie algebra $sl(2, \mathbb{C})$. In this paper it is shown that the Krull dimension of U (denoted $|U|$) is two.

If $U(g)$ is the enveloping algebra of a finite-dimensional solvable Lie algebra g then it is straightforward to show that $|U(g)| = \dim g$ [5, 3.8.11]. The problem as to the Krull dimension of U was first mentioned by Gabriel and Nouazé [9] — they show that U has a chain of prime ideals of length two, and none of length greater than two. From this they conclude that the Krull dimension of U is two, although the correct conclusion is only that $|U| \geqslant 2$. Subsequent to [9], both Arnal and Pinczon [1] and Roos [10] established that if R were a non-artinian simple primitive factor ring of U then $|R| = 1$. More recently the author [11] proved that if R were a non-artinian primitive factor ring of U which was not simple then again $|R| = 1$. The result in the present paper implies those in [1, 10, 11].

The fundamental tool in the proof that $|U| = 2$ is Gelfand–Kirillov dimension (GK-dimension). The proof is in two parts. In Section 2 a number of preliminary results (already known) concerning GK-dimension are recalled. In particular, Lemma 2.3 provides the basic connection between GK-dimension and Krull dimension. The more detailed analysis of U is carried out in Section 3. The crucial result is that any finitely generated U module of Krull dimension 1 has GK-dimension 2 — the result then quickly follows from Lemma 2.3.

The author would like to thank J. C. McConnell both for bringing this problem to his attention, and for many helpful conversations.

0021-8693/81/070189–06$02.00/0

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2. Gelfand–Kirillov Dimension

For the basic definitions and properties concerning GK-dimension the reader is referred to [3, 8]. We present here only those properties which are essential for our purposes. For the rest of Section 2 let R denote a factor ring of the enveloping algebra of a finite dimensional Lie algebra.

As in [8], given a finitely generated R-module M (all modules will be left modules), we can associate a polynomial $q(M)$ (the Hilbert–Samuel polynomial) with M such that the degree of q is precisely the GK-dimension of M (denoted by $GK(M)$). Set $e(M)$ to be $GK(M)! \times$ (leading coefficient of $q(M)$). Recall that $e(M)$ is a positive integer.

Lemma 2.1 [8, Lemma 2.2]. *Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of finitely generated R-modules. Then one of the following holds*:

(i) $GK(M_1) < GK(M)$ *and* $GK(M_2) = GK(M)$ *and* $e(M_2) = e(M)$;

(ii) $GK(M_1) = GK(M) = GK(M_2)$ *and* $e(M) = e(M_1) + e(M_2)$;

(iii) $GK(M_2) < GK(M)$ *and* $GK(M_1) = GK(M)$ *and* $e(M_1) = e(M)$.

Corollary 2.2. (i) *For any submodule N of a finitely generated R-module, M, $GK(M) = \max\{GK(M/N), GK(N)\}$.* (ii) *If M is an R-module with $GK(M) = d$, and $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is a chain of submodules satisfying $GK(M_i/M_{i+1}) = d$, then the chain has at most $e(M)$ terms.*

Proof. (i) follows immediately from the lemma. By (i) and induction, $GK(M_i) = d$ for each i. Then (ii) follows by repeatedly applying (ii) of the lemma.

The following lemma is implicit in [7, 2.2] but does not appear to have been stated explicitly anywhere.

Lemma 2.3. *Suppose that for all finitely generated R-modules M' with $|M'| = \alpha$ that $GK(M') \geqslant \alpha + r$ ($\alpha, r \in \mathbb{N}$). Let M be a finitely generated R-module with $|M| \geqslant \alpha$. Then $GK(M) \geqslant |M| + r$.*

Proof. By induction on $|M|$. It is true by the hypothesis when $|M| = \alpha$. Suppose the result is true for modules with Krull dimension strictly less than β, and let $|M| = \beta$. Then there exists a chain $M = M_0 \supseteq M_1 \supseteq M_2 \cdots$ of submodules such that $|M_i/M_{i+1}| = \beta - 1$ for $i = 0, 1, 2, \ldots$. By the induction hypothesis, $GK(M_i/M_{i+1}) \geqslant \beta - 1 + r$. It follows from Corollary 2.2 that $GK(M) > \beta - 1 + r$; that is $GK(M) \geqslant \beta + r$.

Two interesting consequences of this lemma are worth mentioning:

(1) [7, 2.2]. Bernstein [2] has shown for the Weyl algebra A_n that any simple A_n-module has GK-dimension at least n. So applying the lemma with $\alpha = 0$, $r = n$ it follows in particular that $|A_n| \leqslant n$ (because $GK(A_n) = 2n$).

(2) If R is a simple non-artinian factor ring of an enveloping algebra with $GK(R)=2$, then $|R|=1$. To see this let M be a simple R-module. Then M is not finite dimensional (otherwise R itself is finite dimensional and then $GK(R)=0$), so $GK(M) \geqslant 1$. Hence the lemma implies, with $\alpha=0$, $r=1$, $|R| \leqslant 1$. In particular this argument gives a brief proof of the fact that the simple primitive factor rings of U which are not artinian have Krull dimension 1 — see [1, 10].

An unpublished result of the author actually shows that such a ring R (i.e., a factor of an enveloping algebra), if it is primitive, cannot have GK-dimension 1.

Finally we give a particularly easy lemma which we will need.

LEMMA 2.4. *Let I be a left ideal of R, and let S be a subring of R such that S is a finitely generated algebra over $\mathbb{C}$, and such that $I \cap S = 0$. Then $GK(R/I) \geqslant GK(S)$.*

Proof. Let $V \supseteq \mathbb{C}$ be a finite-dimensional generating subspace of S, and let $W \supseteq V$ be a finite-dimensional generating subspace of R. Then

$$GK(R/I) = \limsup_{n\to\infty} \frac{\log \dim((I+W^n)/I)}{\log n}$$
$$\geqslant \limsup_{n\to\infty} \frac{\log \dim((I+V^n)/I)}{\log n}$$

but as $V^n \subseteq S$ and $I \cap S = 0$, $\dim((I+V^n)/I) = \dim V^n$. So

$$GK(R/I) \geqslant \limsup_{n\to\infty} \frac{\log \dim V^n}{\log n} = GK(S).$$

3. MAIN RESULT

We begin with some notation and elementary facts about $sl(2, \mathbb{C})$. More detail may be found in Dixmier [4]. We take as a basis for $sl(2, \mathbb{C})$ the elements e, f, h subject to the relations

$$[e,f]=h, \qquad [h,e]=2e, \qquad [h,f]=-2f.$$

The element $Q = 4ef + h^2 - 2h = 4fe + h^2 + 2h$ is central in U. Given $n \in \mathbb{N}$, U has a unique finite-dimensional simple module of dimension $(n+1)$. This module is annihilated by the central element $Q - n(n+2)$.

Let $n \in \mathbb{N}$. The simple module of dimension $(n+1)$ may be thought of as a $\mathbb{C}$-vector space with basis $1, f, f^2, \ldots, f^n$ where the action of $sl(2, \mathbb{C})$ on these basis elements is given as follows:

$$\begin{aligned} e \cdot f^j &= j(n-j+1)f^{j-1} &\quad \text{with} \quad e \cdot 1 = 0; \\ f \cdot f^j &= f^{j+1} &\quad \text{with} \quad f \cdot f^n = 0; \\ h \cdot f^j &= (n-2j)f^j. \end{aligned}$$

This may be deduced from (for example) [5, 7.2.7].

LEMMA 3.1. *Let U/J be an artinian U-module such that each composition factor is isomorphic to the same finite-dimensional simple module S, say. Then U/J is of length at most* $\dim_{\mathbb{C}} S$.

Proof. Let $P = \text{ann}(S)$. Then U/P is simple artinian with simple module S, so U/P is of length at most $\dim_{\mathbb{C}} S$. Because $sl(2, \mathbb{C})$ is semi-simple and U/J is finite dimensional, U/J splits as a sum of simple modules each of which is isomorphic to S by hypothesis. Thus $U/J \cong S^{(n)}$ for some $n \in \mathbb{N}$, and consequently $P \cdot (U/J) = 0$. That is, $P \subseteq J$, and the conclusion follows.

LEMMA 3.2. *Let M be a finitely generated U-module of Krull dimension* 1. *Then $GK(M) \geqslant 2$.*

Proof. By [6, Chap. 2] M has a 1-critical factor module (i.e., a factor module of Krull dimension 1, any proper factor of which is artinian), and now by Corollary 2.2(i) it is enough to prove the result when M is 1-critical. Suppose M is 1-critical. If there exists an infinite chain $M = M_0 \supseteq M_1 \supseteq \cdots$ of non-zero submodules such that each M_i/M_{i+1} is infinite dimensional (thus of GK-dimension at least 1) then $GK(M) \geqslant 2$ by Corollary 2.2(ii) and we are finished. Suppose this is not the case. Then there exists a non-zero submodule M' of M such that every proper factor module of M' is finite dimensional. Furthermore M' may be chosen cyclic, and it is enough to prove that $GK(M') \geqslant 2$ in order for the lemma to hold. This is what will be proved. Let I be a left ideal such that U/I is 1-critical and every proper factor of U/I is finite dimensional.

We show first that U/I has simple factor modules of arbitrarily large finite dimension. Suppose, to the contrary, that there is $n \in \mathbb{N}$ such that every simple module of U/I has dimension $\leqslant n$. There are of course (up to isomorphism) only finitely many simple modules of dimension $\leqslant n$. Pick a chain $U = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I$ of left ideals such that each factor I_j/I_{j+1} is simple. It follows that for some sufficiently large j, the composition series for U/I_{j+1} contains at least $(n+1)$ distinct copies of the same simple module, S, say. Now the fact that U/I_{j+1} is semi-simple (being finite dimensional) implies the existence of some left ideal $J \supseteq I_{j+1}$ such that U/J is of length at least $(n+1)$ and each simple module appearing in the composition series for U/J is isomorphic to S. This contradicts Lemma 3.1 as $\dim_{\mathbb{C}} S \leqslant n$. Thus, the claim holds.

We will now show that either $I \cap \mathbb{C}[e, Q] = 0$ or $I \cap \mathbb{C}[f, Q] = 0$, whence the result will follow by Lemma 2.4. Suppose to the contrary that there are non-zero elements $p_1 = p_1(e, Q) \in I \cap \mathbb{C}[e, Q]$ and $p_2 = p_2(f, Q) \in I \cap \mathbb{C}[f, Q]$. Let n_1 denote the degree of p_1 as a polynomial in e, and let n_2 denote the degree of p_2 as a polynomial in f. It is easy to see that there exists an integer m, such that if $n \in N$ and $n \geqslant m$ then $p_1(e, n(n+2))$ and $p_2(f, n(n+2))$ have degree n_1 and n_2, respectively. Let K be a maximal left ideal of U such that $I \subseteq K$ and $\dim_{\mathbb{C}}(U/K) > m$. Put $n+1 = \dim_{\mathbb{C}}(U/K)$. Now, by the comments at the beginning of Section 3, $Q - n(n+2)$ annihilates the simple module U/K, and so $Q - n(n+2) \in K$. Because p_1, $p_2 \in I$ it follows that both $q_1 = p_1(e, n(n+2))$ and $q_2 = p_2(f, n(n+2))$ are elements of K, and non-zero. Looking at U/K as $\mathbb{C} \oplus \mathbb{C}f \oplus \cdots \oplus \mathbb{C}f^n$, there is a non-zero element $a \in U/K$, $a = \alpha_s f^s + \cdots + \alpha_t f^t$ with $s \leqslant t$, $\alpha_s \neq 0$, $\alpha_t \neq 0$ and $K = \operatorname{ann}(a)$. Consequently, $q_1 \cdot a = q_2 \cdot a = 0$. By considering the lowest degree term in $q_2 \cdot a$ it is clear that $s + \deg q_2 \geqslant n+1$. By considering the highest degree term in $q_1 \cdot a$ it is clear that $t \leqslant \deg q_1$. Hence

$$\deg q_1 + \deg q_2 \geqslant t - s + (n+1) \geqslant n+1.$$

But $\deg q_1 + \deg q_2 = n_1 + n_2$, and so $n_1 + n_2 \geqslant n+1$. But n_1 and n_2 are fixed while n can be arbitrarily large — this contradiction completes the proof of the lemma.

THEOREM 3.3. *The Krull dimension of $U(sl(2))$ is two.*

Proof. The result of Nouazé–Gabriel shows that $|U| \geqslant 2$, and the reverse inequality is obtained from Lemma 2.3 and Lemma 3.2 (because $GK(U) = 3$).

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