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Integral Non-commutative Spaces

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A non-commutative space X is a Grothendieck category $\operatorname{Mod} X$. We say X is integral if there is an indecomposable injective X-module $\mathcal{E}_X$ such that its endomorphism ring is a division ring and every X-module is a subquotient of a direct sum of copies of $\mathcal{E}_X$. A noetherian scheme is integral in this sense if and only if it is integral in the usual sense. We show that several classes of non-commutative spaces are integral. We also define the function field and generic point of an integral space and show that these notions behave as one might expect.

1. INTRODUCTION

We follow Rosenberg and Van den Bergh in taking a Grothendieck category as our basic non-commutative geometric object. We think of a Grothendieck category $\operatorname{Mod} X$ as "the quasi-coherent sheaves on an imaginary non-commutative space X." The commutative model is the category $\operatorname{Qcoh} X$ of quasi-coherent sheaves on a quasi-separated, quasi-compact scheme X. The two non-commutative models are $\operatorname{Mod} R$, the category of right modules over a ring, and $\operatorname{Proj} A$, the non-commutative projective spaces defined by Verevkin [13] and Artin and Zhang [2].

This paper defines X to be *integral* if $\operatorname{Mod} X$ is locally noetherian and there is an indecomposable injective X-module $\mathcal{E}_X$ such that $\operatorname{End} \mathcal{E}_X$ is a division ring and every X-module is a subquotient of a direct sum of copies of $\mathcal{E}_X$ (Definition 3.1). If X is integral, then up to isomorphism there is

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only one indecomposable injective with these properties. The *function field* of an integral space is the division ring End $\mathcal{E}_X$. We also define the generic point of an integral space. Corollary 4.2 shows that a noetherian scheme is integral in the usual sense if and only if Qcoh X is integral in our sense. In that case $\mathcal{E}_X$ is the constant sheaf with sections equal to the function field of X, and the function field in our sense coincides with the usual function field of X.

Goldie's theorem implies that an affine space having a prime right noetherian coordinate ring is integral. However, we give a categorical definition of integrality so that it can be applied to those non-commutative spaces that are not defined in terms of a ringed space. The non-commutative projective planes defined by Artin *et al.* [1] are integral. The non-commutative analogues of $\mathbb{P}^n$ associated with enveloping algebras of Lie algebras [6], and the analogues of $\mathbb{P}^n$ arising from the Sklyanin algebras [11] are integral. The exceptional fiber in Van den Bergh's blowup of a non-commutative surface at a point [12] is always integral.

Section 5 shows that non-commutative integral spaces enjoy some of the properties of integral schemes.

2. PRELIMINARIES

Throughout we work over a fixed commutative base ring k. All categories are assumed to be k-linear, and so are all functors between them.

We adopt the framework for non-commutative algebraic geometry originated by Rosenberg [8] and further developed by Van den Bergh [12]. Definitions of terms we do not define can be found in [12].

Definition 2.1. A *non-commutative space* X is a Grothendieck category Mod X. Objects in Mod X are called X-modules. We say X is *locally noetherian* if Mod X is locally noetherian (that is, if it has a set of noetherian generators).

Definition 2.2. If X and Y are non-commutative spaces, a weak map $f\colon Y \to X$ is a natural equivalence class of left exact functors $f_*\colon \operatorname{Mod}Y \to \operatorname{Mod}X$. A weak map $f\colon Y \to X$ is a map if f_* has a left adjoint. A left adjoint to f_* will be denoted by f^*, and a right adjoint will be denoted by $f^!$ if it exists.

We say X is *affine* if Mod X has a progenerator, and in this case any ring R for which Mod X is equivalent to Mod R is called a *coordinate ring* of X.

If $(X, \mathcal{O}_X)$ is a scheme then the category Mod $\mathcal{O}_X$ of all sheaves of $\mathcal{O}_X$-modules is a Grothendieck category. If X is quasi-compact and quasi-separated (for example, if X is a noetherian scheme) the full subcategory of $\operatorname{Mod} \mathcal{O}_X$ consisting of the quasi-coherent $\mathcal{O}_X$-modules is a Grothendieck category [5, p. 186]. We denote this category by $\operatorname{Qcoh} X$. Whenever X is a quasi-compact and quasi-separated scheme we will speak of it as a space in our sense with the tacit understanding that $\operatorname{Mod} X$ is synonymous with $\operatorname{Qcoh} X$.

3. INTEGRAL SPACES, GENERIC POINTS, AND FUNCTION FIELDS

Throughout this section we fix a locally noetherian space X. We denote the injective envelope of an X-module M by $E(M)$.

DEFINITION 3.1. A locally noetherian space X is *integral* if there is an indecomposable injective $\mathcal{E}_X$ such that $\operatorname{End} \mathcal{E}_X$ is a division ring and every X-module is a subquotient of a direct sum of copies of $\mathcal{E}_X$. We call $\mathcal{E}_X$ the *big injective* in $\operatorname{Mod} X$.

Remarks. The endomorphism ring of an indecomposable injective $\mathcal{E}$ is a division ring if and only if $\operatorname{Hom}_X(\mathcal{E}/N, \mathcal{E}) = 0$ for all non-zero submodules N of $\mathcal{E}$.

When X is locally noetherian the following conditions on an X-module $\mathcal{E}$ are equivalent: (a) every X-module is a subquotient of a direct sum of copies of $\mathcal{E}$; (b) every noetherian X-module is a subquotient of a finite direct sum of copies of $\mathcal{E}$.

Corollary 3.7 shows that the big injective is unique up to isomorphism, thus justifying the use of the definite article. Therefore the rank of a module, the generic point, and the function field of X, all of which are defined below in terms of $\mathcal{E}_X$, are unambiguously defined.

DEFINITION 3.2. Let X be an integral locally noetherian space. An X-module M is *torsion* if $\operatorname{Hom}(M, \mathcal{E}_X) = 0$. A module is *torsion-free* if the only submodule of it that is torsion is the zero submodule.

The torsion modules form a localizing subcategory of $\operatorname{Mod} X$.

DEFINITION 3.3. Let X be an integral locally noetherian space. The *rank* of an X-module M is the length of $\operatorname{Hom}_X(M, \mathcal{E}_X)$ as a left $\operatorname{End} \mathcal{E}_X$-module. We denote it by rank M.

Thus an X-module is torsion if and only if its rank is zero.

Because $\mathcal{E}_X$ is injective, rank is additive on short exact sequences.

The hypotheses on $\mathcal{E}_X$ ensure that it has rank one, and every proper quotient of it has rank zero. Hence every non-zero submodule of $\mathcal{E}_X$ has rank one.

Because a noetherian X-module is a subquotient of a finite direct sum of copies of $\mathscr{E}_X$, its rank is finite.

If rank $M \geq 1$, then M has a quotient of rank one, namely $M/\ker f$, where f is a non-zero element of $\mathrm{Hom}_X(M, \mathscr{E}_X)$.

If M is a noetherian torsion-free module of rank $n \geq 1$, then there is a finite chain $M = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = 0$ such that each M_i/M_{i+1} is torsion-free of rank one. To see this begin by choosing M_1 to be maximal subject to the condition that $\mathrm{rank}(M_0/M_1) = 1$; the maximality ensures that M_0/M_1 is torsion-free; then argue by induction on n.

Since rank is additive on exact sequences, it induces a group homomorphism rank: $K_0(X) \to \mathbb{Z}$.

LEMMA 3.4. *Let X be an integral locally noetherian space. Let M be a noetherian X-module. There exist noetherian submodules $L_1, \ldots, L_n$ of $\mathscr{E}_X$, a submodule $L \subset L_1 \oplus \cdots \oplus L_n$, and an epimorphism φ: $L \to M$ such that $\varphi(L \cap L_i) \neq 0$ for all i.*

Furthermore, the rank of L is n.

Proof. By the definition of integrality there are noetherian submodules $L_1, \ldots, L_n$ of $\mathscr{E}$, a submodule $L \subset L_1 \oplus \cdots \oplus L_n$, and an epimorphism φ: $L \to M$. Choose this data so that n is as small as possible. If $\varphi(L \cap L_i)$ were equal to zero, then there would be an epimorphism $L/L \cap L_i \to M$, and since $L/L \cap L_i$ is isomorphic to a submodule of $L_1 \oplus \cdots \oplus L_n/L_i$ this would contradict the minimality of n. So we conclude that $\varphi(L \cap L_i) \neq 0$ for all i.

Since the rank of each L_i is one, $\mathrm{rank}(L_1 \oplus \cdots \oplus L_n) = n$. Thus rank $L \leq n$. However, $L \cap L_i \neq 0$ for all i, whence rank $L = n$. ∎

PROPOSITION 3.5. *Let X be an integral locally noetherian space. If J is a non-zero injective, then* $\mathrm{Hom}_X(\mathscr{E}_X, J) \neq 0$.

Proof. If J is a non-zero injective X-module, then it contains a non-zero noetherian submodule, say N. Let φ: $L \to N$ be an epimorphism as in Lemma 3.4. The restriction of φ to $L \cap L_1$, which is a submodule of $\mathscr{E}_X$, extends to a non-zero map map from $\mathscr{E}_X$ to J. ∎

PROPOSITION 3.6. *Let X be an integral locally noetherian space. An essential extension of a torsion module is torsion.*

Proof. Let $P \subset M$ be an essential extension of a torsion module P. It suffices to prove the result when M is noetherian because every M is a directed union of noetherian submodules M_i each of which is an essential extension of $M_i \cap P$.

Choose an epimorphism φ: $L \to M$ as in Lemma 3.4. Since $\varphi(L \cap L_i) \neq 0$, $P \cap \varphi(L \cap L_i) \neq 0$. But P is torsion and $L \cap L_i$ is torsion-free, so the restriction of φ to $L \cap L_i$ is not monic. Thus $\ker \varphi \cap L_i \neq 0$. Since L_i is torsion-free of rank one, $L_i/\ker\varphi\cap L_i$ is torsion. Since M is a subquotient of $\bigoplus_{i=1}^n L_i/\ker\varphi\cap L_i$ it is also a torsion module. ∎

COROLLARY 3.7. *If X is an integral locally noetherian space, there is only one indecomposable injective up to isomorphism having the properties in Definition* 3.1.

Proof. Let $\mathscr{E}_X$ be the injective in Definition 3.1, and let $\mathscr{E}$ be another indecomposable injective such that its endomorphism ring is a division ring and every X-module is a subquotient of a direct sum of copies of $\mathscr{E}$.

By Proposition 3.5, there is a non-zero map α: $\mathscr{E}_X \to \mathscr{E}$. If α is monic, then its image would be a direct summand of $\mathscr{E}$, so would equal $\mathscr{E}$ because $\mathscr{E}$ is indecomposable; hence the result is true if α is monic. Suppose to the contrary that α is not monic. Then its image is a proper quotient of $\mathscr{E}_X$ so is torsion. Therefore $\mathscr{E}$ is the injective envelope of a torsion module, so is itself torsion by Proposition 3.6. That is, $\operatorname{Hom}_X(\mathscr{E}, \mathscr{E}_X) = 0$. It follows that $\operatorname{Hom}_X(-, \mathscr{E}_X)$ vanishes on all X-modules. This is absurd, so we conclude that α is monic. ∎

DEFINITION 3.8. Let X be an integral locally noetherian space. The *function field* of X is the division algebra

$$k(X) := \operatorname{Hom}_X(\mathscr{E}_X, \mathscr{E}_X).$$

The *generic point* of X is the space η defined by

$$\operatorname{Mod}\eta = \operatorname{Mod} X/\mathsf{T},$$

where T is the full subcategory consisting of the torsion modules.

Since T is a localizing subcategory of $\operatorname{Mod} X$, there is an adjoint pair of functors (j^*, j_*), where j^*: $\operatorname{Mod} X \to \operatorname{Mod}\eta := \operatorname{Mod} X/\mathsf{T}$ is the quotient functor, and j_* its right adjoint. This defines a map of spaces

$$j\colon \eta \to X.$$

For the rest of this section j will denote this map.

PROPOSITION 3.9. *Let X be an integral locally noetherian space. If η is its generic point, then* $\operatorname{Mod}\eta$ *is equivalent to* $\operatorname{Mod} k(X)$.

Proof. Since $\mathscr{E}_X$ is torsion-free and every proper quotient of it is torsion, $j^*\mathscr{E} \cong j^*M$ for every non-zero submodule M of $\mathscr{E}_X$. It follows that $j^*\mathscr{E}_X$ is a simple module in $\operatorname{Mod}\eta$.

If M is an X-module, then $E(M)/M$ is torsion by Proposition 3.11, so $j^*M \cong j^*E(M)$. Since $E(M)$ is a direct sum of indecomposable injectives, and j^* commutes with direct sums, and an indecomposable injective is either torsion or isomorphic to $\mathscr{E}_X$, j^*M is isomorphic to a direct sum of copies of $j^*\mathscr{E}_X$. Therefore every η-module is isomorphic to a direct sum of copies of $j^*\mathscr{E}_X$. Thus $\operatorname{Mod}\eta$ is equivalent to $\operatorname{Mod}D$, where D is the endomorphism ring of $j^*\mathscr{E}_X$.

Since $\mathscr{E}_X$ is torsion-free and injective, $j_*j^*\mathscr{E}_X \cong \mathscr{E}_X$, whence

$$D = \operatorname{Hom}_\eta(j^*\mathscr{E}_X, j^*\mathscr{E}_X) \cong \operatorname{Hom}_X(\mathscr{E}_X, j_*j^*\mathscr{E}_X) \cong k(X).$$

This completes the proof. ∎

Remark. The rank of an X-module M is equal to the length of j^*M as a right $k(X)$-module. To see this, first observe that this length is equal to the length of the left $k(X)$-module $\operatorname{Hom}_\eta(j^*M, j^*\mathscr{E}_X)$; second, observe that we have following natural isomorphisms:

$$\begin{aligned}\operatorname{Hom}_\eta(j^*M, j^*\mathscr{E}_X) &\cong \operatorname{Hom}_\eta(j^*j_*j^*M, j^*\mathscr{E}_X)\\ &\cong \operatorname{Hom}_X(j_*j^*M, j_*j^*\mathscr{E}_X)\\ &\cong \operatorname{Hom}_X(j_*j^*M, \mathscr{E}_X).\end{aligned}$$

It follows that the length of j^*M is equal to the rank of j_*j^*M. However, there is an exact sequence $0 \to A \to M \to j_*j^*M \to B \to 0$ where A and B are torsion modules, so $\operatorname{rank} M = \operatorname{rank} j_*j^*M$.

THEOREM 3.10 (Zhang). *Let X be an integral locally noetherian space. Then*

1. *every torsion-free module has a non-zero submodule that is isomorphic to a submodule of $\mathscr{E}_X$*;
2. *a uniform torsion-free module has rank one*;
3. *the injective envelope of every torsion-free module of rank one is isomorphic to $\mathscr{E}_X$*;
4. *$\mathscr{E}_X$ is the unique indecomposable injective of rank one*;
5. *every simple X-module is a subquotient of $\mathscr{E}_X$.*

Proof. (1) It suffices to prove this for a noetherian torsion-free module M. Choose an epimorphism $\varphi\colon L \to M$ as in Lemma 3.4. Since M is torsion-free and every proper quotient of $L \cap L_1$ is torsion, $\varphi(L \cap L_1) \cong L \cap L_1$, which is a non-zero submodule of $\mathscr{E}_X$.

(2) It suffices to prove this for a noetherian torsion-free uniform module M. Choose an epimorphism $\varphi\colon L \to M$ as in Lemma 3.4 and set $M_i = \varphi(L \cap L_i)$. Thus M_i is torsion-free of rank one. Since M is uniform, $\bigcap_{i=1}^n M_i \neq 0$. An induction argument shows that the rank of $M_1 + \cdots + M_n$ is one: certainly $\operatorname{rank}(M_j) = 1$ for all j, and

$$\begin{aligned}\operatorname{rank}(M_1 + \cdots + M_{i+1}) &= \operatorname{rank}(M_1 + \cdots + M_i) + \operatorname{rank}(M_{i+1})\\ &\quad -\operatorname{rank}(M_1 + \cdots + M_i) \cap M_{i+1}\\ &= \operatorname{rank}(M_1 + \cdots + M_i) + 1 - 1\\ &= \operatorname{rank}(M_1 + \cdots + M_i).\end{aligned}$$

But the rank of $L/\bigoplus_{i=1}^{n}(L \cap L_i)$ is zero, so the rank of its quotient $M/\sum_{i=1}^{n} M_i$ is also zero. Hence rank $M = 1$.

(3) By (1) a rank one torsion-free module is an essential extension of a non-zero submodule of $\mathscr{E}_X$, so embeds in $\mathscr{E}_X$.

(4) Let E' be an indecomposable injective of rank one. Since E' is the injective envelope of all its non-zero submodules, it follows from Proposition 3.6 that E' is torsion-free. Hence by (1) E' and $\mathscr{E}_X$ have a common submodule, whence $E' \cong \mathscr{E}_X$.

(5) Let S be a simple module. There is an epimorphism $\varphi: L \to S$ as in Lemma 3.4. There is a finite descending chain $L = K_0 \supset K_1 \supset \cdots \supset K_r = 0$ of submodules such that each K_i/K_{i+1} is torsion-free of rank one. Since $\operatorname{Hom}_X(L, S) \neq 0$, $\operatorname{Hom}_X(K_i/K_{i+1}, S) \neq 0$ for some i. Since S is simple, this provides the required epimorphism.

The next result improves on Proposition 3.6.

PROPOSITION 3.11. *Let X be a locally noetherian integral space. If $L \subset M$ is an essential extension of X-modules, then* rank $L =$ rank M.

Proof. It is sufficient to prove the result when M is the injective envelope of L. In that case, we can write M as a direct sum of indecomposable injectives, say $M = \bigoplus_i M_i$. Then $L \cap M_i \neq 0$ for all i, and M/L is a quotient of $\bigoplus M_i/L \cap M_i$. Since M_i is an indecomposable injective, either its rank is zero or it is isomorphic to $\mathscr{E}_X$; in either case, rank $M_i/L \cap M_i = 0$. Hence rank $M/L = 0$, and the result follows. ∎

4. EXAMPLES OF INTEGRAL SPACES

A scheme X is integral in the usual sense of algebraic geometry if and only if $\mathscr{O}_X(U)$ is an integral domain for all open subsets $U \subset X$. Corollary 4.2 shows that a noetherian scheme is integral in our sense if and only if it is integral in the usual sense.

We show that an affine space having a prime right noetherian coordinate ring is integral. We give other examples which indicate that our notion of integral is reasonable. In particular, Theorem 4.5 implies that the non-commutative analogues of $\mathbb{P}^2$ discovered by Artin *et al.* are integral spaces, as are the Sklyanin analogues of $\mathbb{P}^n$.

PROPOSITION 4.1. *Let X be an integral noetherian scheme. Let $\mathscr{K}$ denote the constant sheaf having sections the function field of X. If $\mathscr{M}$ is a coherent $\mathscr{O}_X$-module, then there is a coherent $\mathscr{O}_X$-submodule, $\mathscr{L}$ say, of a finite direct sum of copies of $\mathscr{K}$ and an epimorphism ψ: $\mathscr{L} \to \mathscr{M}$.*

Proof. For the purposes of this proof we call a coherent $\mathcal{O}_X$-module $\mathcal{M}$ good if there is such an epimorphism. Clearly a finite direct sum of good modules is good, a submodule of a good module is good, and a quotient of a good module is good.

Let $E(\mathcal{M})$ denote the injective envelope in $\operatorname{Qcoh} X$ of an $\mathcal{O}_X$-module. This is a direct sum of indecomposable injectives. Each indecomposable injective is isomorphic to $E(\mathcal{O})_Z$ for some closed reduced and irreducible subscheme Z of X [3, Théorème 1, p. 443]. It therefore suffices to show that every coherent submodule of each $E(\mathcal{O}_Z)$ is good.

Fix a closed reduced and irreducible subscheme $Z \subset X$, and a coherent $\mathcal{O}_X$-submodule $\mathcal{M} \subset E(\mathcal{O}_Z)$. Let z denote the generic point of Z, and let $\mathcal{O}_z$ denote the stalk of $\mathcal{O}_X$ at z. There is a morphism $f\colon \operatorname{Spec} \mathcal{O}_z \to X$ with the following properties: the inverse image functor f^* is exact; the direct image functor f_* is fully faithful and exact and has a right adjoint $f^!$. Because f_* is fully faithful the counit $f^*f_* \to \mathrm{id}_{\operatorname{Spec} \mathcal{O}_z}$ is an isomorphism.

Let $i\colon Z \to X$ be the inclusion. Let $\mathcal{E}$ be the constant sheaf on Z having sections the function field of Z. Then $i_*\mathcal{E}$ is an essential extension of $\mathcal{O}_Z$, so $E(\mathcal{O}_Z) = E(i_*\mathcal{E})$. But $i_*\mathcal{E}$ is also gotten by applying f_* to the residue field of $\mathcal{O}_z$, so the unit $i_*\mathcal{E} \to f_*f^*(i_*\mathcal{E})$ is an isomorphism. However, f_* sends injectives to injectives because it is right adjoint to an exact functor, so if $\mathcal{F}$ is an injective envelope of $f^*(i_*\mathcal{E})$ in $\operatorname{Mod} \mathcal{O}_z$, $f_*\mathcal{F}$ is an injective quasi-coherent $\mathcal{O}_X$-module containing a copy of $i_*\mathcal{E}$. Thus $E(\mathcal{O}_Z) \cong f_*\mathcal{F}$. There is a surjective map $\mathcal{O}_z^{(I)} \to \mathcal{F}$ from a suitably large direct sum of copies of $\mathcal{O}_z$, and therefore an epimorphism $f_*(\mathcal{O}_z^{(I)}) \to f_*\mathcal{F}$. Since f_* has a right adjoint it commutes with direct sums, so we obtain an epimorphism $(f_*\mathcal{O}_z)^{(I)} \to f_*\mathcal{F}$. Because $\operatorname{Qcoh} X$ is locally noetherian, every coherent $\mathcal{O}_X$-submodule of $f_*\mathcal{F}$ is therefore an epimorphic image of a coherent submodule of $f_*\mathcal{O}_z^{(I)}$. However, $f_*\mathcal{O}_z$ is an $\mathcal{O}_X$-submodule of $\mathcal{K}$, so every coherent $\mathcal{O}_X$-submodule of it is good. It follows that every coherent submodule of $f_*\mathcal{F}$ is good. Hence $\mathcal{M}$ is good. ∎

COROLLARY 4.2. *Let X be a noetherian scheme. Then X is integral in the usual sense if and only if it is integral in the sense of Definition* 3.1. *In that case, $\mathcal{E}_X$ is isomorphic to the constant sheaf $\mathcal{K}$ with sections the function field of X.*

Proof. Let X be integral in the usual sense of algebraic geometry. By [3, Chap. VI], $\mathcal{K}$ is an indecomposable injective. It is also clear from Gabriel's classification of the indecomposable injectives in $\operatorname{Qcoh} X$ that $\mathcal{K}$ is the only indecomposable injective of Krull dimension equal to $\dim X$. It therefore follows from Proposition 4.1 that X is integral in our sense. Furthermore, $\mathcal{E}_X = \mathcal{K}$, and the endomorphism ring of $\mathcal{K}$ is $k(X)$, so function field and generic point in our sense agree with the usual notions.

Conversely, suppose that X is integral in the sense of Definition 3.1. By [3], $\mathscr{E}_X \cong E(\mathscr{O}_Z)$ for some closed reduced and irreducible subscheme Z of X. In particular, rank $\mathscr{O}_Z = 1$.

We will show that every coherent $\mathscr{O}_X$-submodule of $\mathscr{E}_X$ is an $\mathscr{O}_Z$-module. It will then follow that the same is true of every coherent subquotient of a finite direct sum of copies of $\mathscr{E}_X$. In particular, $\mathscr{O}_X$ will be an $\mathscr{O}_Z$-module, whence $Z = X$, and the proof is complete.

It suffices to prove that every coherent submodule of $E(\mathscr{O}_Z)$ containing $\mathscr{O}_Z$ is an $\mathscr{O}_Z$-module. Let M be such a submodule. If W denotes the support of $M/\mathscr{O}_Z$, then $M/\mathscr{O}_Z$ is annihilated by some power of $\mathscr{I}_W$, the ideal cutting out W. Hence $M\mathscr{I}_W^n\mathscr{I}_Z = 0$ for $n \gg 0$. If $M\mathscr{I}_Z = 0$, then M is an $\mathscr{O}_Z$-module, so we may suppose that $M\mathscr{I}_Z$ is non-zero. Hence $M\mathscr{I}_Z$ has non-zero intersection with the essential submodule $\mathscr{O}_Z$ of $E(\mathscr{O}_Z)$, so $\mathscr{I}_W^n$ annihilates a non-zero ideal of $\mathscr{O}_Z$. But Z is integral, so this can only happen if $\mathscr{I}_W^n \subset \mathscr{I}_Z$; hence $Z \subset W$.

On the other hand the injective envelope of $M/\mathscr{O}_Z$ is a direct sum of indecomposable injectives, so a direct sum of copies of $E(\mathscr{O}_{W_i})$ for various closed integral subschemes W_i of X. Since Z is contained in the support of $M/\mathscr{O}_Z$, and every nonzero coherent submodule of $E(\mathscr{O}_{W_i})$ has support equal to W_i, Z is contained in the union of the W_is. Since Z is integral it must be contained in one of the W_is. Hence $\mathscr{O}_Z$ is a quotient of $\mathscr{O}_{W_i}$ for some i, and we deduce that rank $\mathscr{O}_{W_i} \geq 1$. It follows that the rank of $E(M/\mathscr{O}_Z) = 1$. Hence by Proposition 3.11, rank $M/\mathscr{O}_Z = 1$. This contradicts the fact that rank $\mathscr{E}_X/\mathscr{O}_Z = 0$, so we conclude that $M\mathscr{I}_Z = 0$. Hence M is an $\mathscr{O}_Z$-module, as required. ∎

PROPOSITION 4.3. *Let R be a right noetherian ring and let X be the affine space with coordinate ring R. If R is prime, then X is integral.*

Proof. By Goldie's theorem [4], the ring of fractions of R is a matrix ring over a division ring, say D. Furthermore, that matrix ring is an injective envelope of R as a right R-module. Let $\mathscr{E}$ be a simple right ideal of that matrix ring. The endomorphism ring of $\mathscr{E}$ as an R-module is the same as its endomorphism ring as a module over the matrix ring, so is equal to the division ring D. Since R embeds in a finite direct sum of copies of $\mathscr{E}$, and is a generator in $\operatorname{Mod} R$, every noetherian right R-module is a subquotient of a direct sum of copies of $\mathscr{E}$.

In Proposition 4.3, the function field of X is the division ring D that appears in Goldie's theorem.

It is not the case that a right noetherian ring R is prime if and only if $\operatorname{Mod} R$ is integral. For example, the ring of upper triangular matrices over a field is integral in our sense. However, it is easy to see that if X is affine and integral, then its coordinate ring is prime if and only if $\mathscr{E}_X$ is a prime

X-module in the sense of [9, Definition 4.3]. Proposition 6.4 also gives a criterion which implies that the coordinate ring of an integral affine space is prime.

Our notion of integral is not an invariant of the derived category. For example, let A be the path algebra of the quiver $\bullet \to \bullet \to \bullet$ and let A' be the path algebra of the quiver $\bullet \leftarrow \bullet \to \bullet$. The derived categories of modules over A and A' are equivalent. By listing the three indecomposable injectives over each algebra it is clear that $\operatorname{Mod} A$ is integral, but $\operatorname{Mod} A'$ is not. In fact, the path algebra of a quiver without loops is integral if and only if it has a unique sink. We are grateful to D. Happel for these observations.

We now show that a non-commutative analogue of a projective scheme is integral if it has a homogeneous coordinate ring that is prime and noetherian.

DEFINITION 4.4 (Verevkin [13]; Artin and Zhang [2]). Let A be an $\mathbb{N}$-graded k-algebra such that $\dim_k A_n < \infty$ for all n. Define $\operatorname{GrMod} A$ to be the category of $\mathbb{Z}$-graded A-modules with morphisms the A-module homomorphisms of degree zero. We write $\operatorname{Fdim} A$ for the full subcategory of direct limits of finite dimensional modules. We define the quotient category

$$\operatorname{Tails} A = \operatorname{GrMod} A / \operatorname{Fdim} A,$$

and denote by π and ω the quotient functor and its right adjoint. The *projective space* X with *homogeneous coordinate ring* A is defined by $\operatorname{Mod} X :=$ $\operatorname{Tails} A$.

THEOREM 4.5. *Let A be prime noetherian locally finite $\mathbb{N}$-graded k-algebra. Suppose that* $\dim_k A = \infty$. *Suppose further that the graded ring of fractions* $\operatorname{Fract}_{\mathrm{gr}} A$ *contains an isomorphic copy of $A(n)$ for every integer n. Then the projective space with homogeneous coordinate ring A is locally noetherian and integral. Its function field is the degree zero component of* $\operatorname{Fract}_{\mathrm{gr}} A$.

Proof. Define X by $\operatorname{Mod} X = \operatorname{Tails} A$. Since $\operatorname{Mod} X$ is a quotient of a locally noetherian category it is locally noetherian.

It is well known that the injective envelope of A in $\operatorname{GrMod} A$ is its graded ring of fractions, say $E = \operatorname{Fract}_{\mathrm{gr}} A$. Let $\mathscr{E} = \pi E$ be its image in $\operatorname{Mod} X$. Since A is prime and has infinite dimension, zero is the only finite dimensional graded submodule of it. The same is true of E, so $\mathscr{E}$ is injective in $\operatorname{Mod} X$.

To show that X is integral it only remains to show that every noetherian X-module is a subquotient of a finite direct sum of copies of $\mathscr{E}$. If $\mathscr{M}$ is a noetherian X-module, then $\mathscr{M} \cong \pi M$ for some noetherian A-module M. Now M is a quotient of a finite direct sum of shifts $A(n)$ for various integers n, so $\mathscr{M}$ is a quotient of a finite direct sum of various twists $\mathscr{O}_X(n) = \pi A(n)$.

However, each $A(n)$ embeds in E, so each $\mathcal{O}_X(n)$ embeds in $\mathcal{E}$. Thus $\mathcal{M}$ is a subquotient of a finite direct sum of copies of $\mathcal{E}$.

Finally,

$$k(X) = \mathrm{Hom}_X(\mathcal{E}, \mathcal{E}) = \mathrm{Hom}_X(\pi E, \pi E) \cong \mathrm{Hom}_{\mathrm{Gr}A}(E, \omega\pi E).$$

However, since zero is the only finite dimensional submodule of E and E is injective, $\omega\pi E \cong E$. Hence

$$k(X) \cong \mathrm{Hom}_{\mathrm{Gr}A}(E, E) \cong (\mathrm{Fract}_{\mathrm{gr}} A)_0,$$

as claimed. ∎

The hypothesis in Theorem 4.5 that $\mathrm{Fract}_{\mathrm{gr}} A$ contain a copy of each $A(n)$ is necessary because if $A = k[x^2]$ with $\deg x = 1$, then $X \cong \mathrm{Spec} k^2$. This hypothesis holds if A has a regular element in all sufficiently high degrees. In particular, if A is a domain generated in degree one, then X is integral. Thus, the quantum planes of Artin *et al.* are integral, as are the other standard non-commutative analogues of the projective spaces $\mathbb{P}^n$.

Van den Bergh has defined the notion of the blowup at a closed point on a non-commutative surface [12]. The exceptional fiber, E say, is sometimes, but not always, a projective line. Nevertheless it is always integral. For example, when $\mathrm{Mod} E = \mathrm{GrMod}\, k[x]$ its big injective is $k[x, x^{-1}]$ and its function field is k. In the other cases $\mathrm{Mod}\, E$ is of the form $\mathrm{Tails}\, k[x, y]$, where $k[x, y]$ is the commutative polynomial ring with $\deg x = 1$ and $\deg y = n < \infty$, and its integrality is guaranteed by Theorem 4.5. In these cases the function field of E is the rational function field $k(y/x^n)$.

5. PROPERTIES OF INTEGRAL SPACES

An integral scheme has several properties that we might expect a non-commutative integral space to have. For example, every non-empty open subscheme of a noetherian integral scheme is dense because it contains the generic point. To get a non-commutative version of this we must first introduce analogues of "open subspace" and "closure." This is done in [9], but we recall the definition here.

DEFINITION 5.1. Let X be a non-commutative space. A *weakly open subspace*, say U, of X is a full subcategory $\mathrm{Mod}\, U$ of $\mathrm{Mod}\, X$ such that the inclusion functor α_*: $\mathrm{Mod}\, U \to \mathrm{Mod}\, X$ has an exact left adjoint α^*.

For example, the generic point of an integral space is a weakly open subspace.

DEFINITION 5.2. A *weakly closed subspace* W of a non-commutative space X is a full subcategory $\operatorname{Mod} W$ of $\operatorname{Mod} X$ that is closed under subquotients and isomorphisms, and for which the inclusion functor α_*: $\operatorname{Mod} W \to \operatorname{Mod} X$ has a right adjoint. We write α: $W \to X$ for the weak map corresponding to α_*.

Let α: $W \to X$ be the inclusion of a weakly closed subspace. Then $\operatorname{Mod} W$ is a Grothendieck category and is locally noetherian if $\operatorname{Mod} X$ is. Because $\operatorname{Mod} W$ is closed under subquotients, α_* is an exact functor. Because α_* has a right adjoint it commutes with direct sums. Further information about weakly closed subspaces can be found in [9].

The requirement in the definition of an integral space that every X-module be a subquotient of a direct sum of copies of $\mathscr{E}_X$ is equivalent to the requirement that X is the only weakly closed subspace having $\mathscr{E}_X$ as a module over it.

Let U and Z be respectively a weakly open and a weakly closed subspace of X. We say that Z *contains* U if $\operatorname{Mod} U$ is contained in $\operatorname{Mod} Z$. In other words, if α: $U \to X$ and δ: $Z \to X$ are the inclusions, then U is contained in Z if and only if there is a weak map ε: $U \to Z$ such that $\delta\varepsilon = \alpha$. In this case, U becomes a weakly open subspace of Z because $\alpha^*\delta_*$ is an exact left adjoint to ε_*: if $M \in \operatorname{Mod} Z$ and $N \in \operatorname{Mod} U$, then

$$\begin{aligned}\operatorname{Hom}_Z(M, \varepsilon_* N) &= \operatorname{Hom}_X(\delta_* M, \delta_*\varepsilon_* N)\\ &= \operatorname{Hom}_X(\delta_* M, \alpha_* N) \cong \operatorname{Hom}_U(\alpha^*\delta_* M, N).\end{aligned}$$

DEFINITION 5.3 [9]. If U is a weakly open subspace of a locally noetherian space X its *weak closure*, denoted $\overline{U}$, is the smallest weakly closed subspace of X that contains U.

This makes sense because the intersection of two weakly closed subspaces is a weakly closed subspace. If α: $U \to X$ is the inclusion, then $\operatorname{Mod} \overline{U}$ consists of all subquotients of X-modules of the form $\alpha_* N$ as N ranges over $\operatorname{Mod} U$. More details about weak closure can be found in [9].

LEMMA 5.4. *If η is the generic point of an integral space X, then $\bar{\eta} = X$.*

Proof. If Z is a weakly closed subspace of X containing η, then $\mathscr{E}_X$ belongs to $\operatorname{Mod} Z$. Since $\operatorname{Mod} Z$ is closed under subquotients and direct sums, every X-module belongs to $\operatorname{Mod} Z$, showing that $Z = X$. ∎

LEMMA 5.5. *Let p be a weakly open point in a locally noetherian space X. That is, p is a weakly open subspace of X and* $\operatorname{Mod} p = \operatorname{Mod} D$ *for some division ring D. If $\bar{p} = X$, then X is integral, p is its generic point, and $k(X) = D$.*

Proof. Let $\alpha\colon p \to X$ denote the inclusion. The big injective in $\operatorname{Mod} p$ is D. Since α_* is right adjoint to an exact functor, $\mathscr{E} := \alpha_* D$ is an injective X-module. Using the adjoint pair (α^*, α_*) it is easy to see that $\mathscr{E}$ is indecomposable because D is, and its endomorphism ring is the same as that of D, namely D. Furthermore, if M is an X-module, it is a subquotient of $\alpha_* N$ for some p-module N because $\bar{p} = X$. But N is a direct sum of copies of D, and α_* commutes with direct sums [3, Cor. 1, p. 379], so M is a subquotient of a direct sum of copies of $\mathscr{E}$. Hence X is integral.

To see that p is the generic point of X it suffices to show that α^* vanishes on the torsion modules. However, if M is torsion, then $0 = \operatorname{Hom}_X(M, \mathscr{E}) \cong \operatorname{Hom}_p(\alpha^* M, D)$, whence $\alpha^* M = 0$. ∎

PROPOSITION 5.6. *Let U be a weakly open subspace of a locally noetherian space X. Suppose that U is integral and $\overline{U} = X$. If the inclusion $U \to X$ is an affine map, then X is integral and $k(X) = k(U)$.*

Proof. The notion of an affine map is defined in [9]; the important point here is that if $\alpha\colon U \to X$ denotes the inclusion, then α_* is exact. Let $\mathscr{E}_U$ be the big injective in $\operatorname{Mod} U$. Since α_* is right adjoint to an exact functor, $\alpha_* \mathscr{E}_U$ is an injective X-module. It is also indecomposable, and its endomorphism ring is equal to $\operatorname{End}_U \mathscr{E}_U$.

It remains to show that every X-module is a subquotient of a direct sum of copies of $\alpha_* \mathscr{E}_U$. Let $P \in \operatorname{Mod} U$. Since U is integral, $P \cong B/A$ for some U-submodules $A \subset B \subset \mathscr{E}_U^{(I)}$ and some index set I. Since α_* is exact, $\alpha_* P \cong (\alpha_* B)/(\alpha_* A)$; since α_* commutes with direct sums we have X-submodules $\alpha_* A \subset \alpha_* B \subset (\alpha_* \mathscr{E}_U)^{(I)}$; thus $\alpha_* P$ is a subquotient of a direct sum of copies of $\alpha_* \mathscr{E}_U$. But $\overline{U} = X$, so every X-module is a subquotient of $\alpha_* P$ for some $P \in \operatorname{Mod} U$. The result now follows. ∎

Proposition 5.6 applies to the situation where one has an affine space and embeds it in a projective space by adding an effective divisor at infinity (see [9, Sec. 8]); if the affine space is integral, so is the projective space, and their function fields coincide.

Let W be a weakly closed subspace of a locally noetherian space X. Its complement $X \backslash W$ is defined in [9, Sect. 6]. In particular, $X \backslash W$ is a weakly open subspace of X, and every weakly open subspace arises as such a complement.

PROPOSITION 5.7. *Let X be an integral space and W a weakly closed subspace. Suppose that $W \neq X$. If $\mathscr{E}_X$ does not contain a non-zero W-submodule, then*

1. *η belongs to $X \backslash W$ and $\overline{X \backslash W} = X$;*
2. *$X \backslash W$ is integral and $k(X \backslash W) = k(X)$.*

Proof. Let $\alpha\colon X\backslash W \to X$ denote the inclusion. Let $\tau\colon \operatorname{Mod} X \to \operatorname{Mod} X$ denote the functor that is the kernel of the natural transformation $\mathrm{id}_X \to \alpha_*\alpha^*$. There is an exact sequence

$$0 \to \tau\mathscr{E}_X \to \mathscr{E}_X \to \alpha_*\alpha^*\mathscr{E}_X \to R^1\tau\mathscr{E}_X \to 0.$$

By hypothesis, $\tau\mathscr{E}_X = 0$. Since $\mathscr{E}_X$ is injective, $R^1\tau\mathscr{E}_X = 0$. Hence $\mathscr{E}_X \cong \alpha_*\alpha^*\mathscr{E}_X$. It follows that the generic point of X belongs to $X\backslash W$. More formally, if $j\colon \eta \to X$ is the inclusion, then there is a map $\gamma\colon \eta \to X\backslash W$ such that $j = \alpha\gamma$ (this is straightforward, though it can also be seen as a special case of [9, Proposition 6.1]). By Lemma 5.4, the weak closure of $X\backslash W$ is X. This proves (1).

Because $\tau\mathscr{E}_X = 0$, $\alpha^*\mathscr{E}_X$ is an injective $X\backslash W$-module. It is an indecomposable injective because

$$\operatorname{Hom}_{X\backslash W}(\alpha^*\mathscr{E}_X, \alpha^*\mathscr{E}_X) \cong \operatorname{Hom}_X(\mathscr{E}_X, \alpha_*\alpha^*\mathscr{E}_X) = \operatorname{Hom}_X(\mathscr{E}_X, \mathscr{E}_X)$$

is a division ring. If $\mathscr{M}$ is a noetherian $X\backslash W$-module, then $\mathscr{M} = \alpha^*M$ for some noetherian X-module M. There is a noetherian submodule L of $\mathscr{E}_X^{\oplus n}$ and an epimorphism $L \to M$. Hence α^*L is a noetherian submodule of $\alpha^*\mathscr{E}_X^{\oplus n}$ and there is an epimorphism $\alpha^*L \to \alpha^*M$. Thus $X\backslash W$ is integral. ∎

We define the *empty space* ϕ by declaring $\operatorname{Mod}\phi$ to be the zero category, that is, the abelian category having only one object and one morphism. Part (1) of Proposition 5.7 can now be rephrased as follows. If W_1 and W_2 are non-empty weakly closed subspaces of an integral space X such that $\mathscr{E}_X$ contains neither a non-zero W_1-module nor a non-zero W_2-module, then $(X\backslash W_1) \cap (X\backslash W_2) \neq \phi$. By [9, Sect. 6], this intersection is equal to $X\backslash(W_1 \cup W_2)$, so we deduce that $W_1 \cup W_2 \neq X$.

6. DIMENSION FUNCTIONS

Van den Bergh has suggested that a dimension function should play a prominent role in non-commutative geometry.

In an earlier version of this paper our definition of integrality required the big injective to be critical with respect to a dimension function. We are grateful to the referee for suggesting that this was unnecessary. Nevertheless, since dimension functions play an important role in non-commutative algebra and geometry it is useful to examine the connection.

DEFINITION 6.1. Let X be a locally noetherian space. A *dimension function* on X is a function $\delta\colon \operatorname{Mod}X \to \mathbb{R}_{\geq 0} \cup \{-\infty, \infty\}$ satisfying the following conditions:

- $\delta(0) = -\infty$;

- if $0 \to L \to M \to N \to 0$ is exact, then $\delta(M) = \max\{\delta(L),\ \delta(N)\}$;
- $\delta(M) = \max\{\delta(N) | N$ is a noetherian submodule of $M\}$;
- if σ is an auto-equivalence of $\operatorname{Mod} X$, then $\delta(M^\sigma) = \delta(M)$.

We define the dimension of X, $\dim X$, to be the maximum of $\delta(M)$ as M ranges over all X-modules.

Remarks. 1. We will make no use in this paper of the condition that δ is invariant under auto-equivalences.

2. A dimension function δ determines various localizing subcategories of X. If $d \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, we write $\operatorname{Mod}_{\leq d} X$ and $\operatorname{Mod}_{<d} X$ for the full subcategories of $\operatorname{Mod} X$ consisting of those M such that $\delta(M) \leq d$ and $\delta(M) < d$ respectively. These are localizing subcategories because $\delta(\sum_j N_j) = \max_j \delta(N_j)$. One can specify the dimension function simply by specifying these localizing subcategories.

3. The notion of Krull dimension as defined by Gabriel in [3] is a dimension function. It is defined inductively: $\operatorname{Mod}_{<0} X$ consists of only the zero module, and for each integer $n \geq 0$, $\operatorname{Mod}_{\leq n} X / \operatorname{Mod}_{\leq n} X$ consists of all direct limits of artinian modules in $\operatorname{Mod} X / \operatorname{Mod}_{<n} X$.
The version of Krull dimension defined using posets that appears in [7, Chap. 6] does *not* satisfy our definition of dimension function. In fact, it is not even defined for all modules and does not lead to an ascending chain of localization subcategories. Thus, we always use Gabriel's version of Krull dimension.

4. If X is a noetherian scheme, then the Krull dimension of a coherent $\mathcal{O}_X$-module is equal to the dimension of its support.

5. Each of the localizing subcategories described above determines a subgroup of $K_0(X)$, and in this way one obtains a filtration of $K_0(X)$.

6. If X is a locally noetherian space with a dimension function δ, then every weakly closed subspace of X is locally noetherian, and it inherits the dimension function. The dimension of such a subspace is the maximum of the dimensions of its noetherian modules.

DEFINITION 6.2. An X-module M is *d-critical* if $\delta(M) = d$ and $\delta(M/N) < d$ for all non-zero submodules N contained in M. We say that M is *d-pure* if $\delta(N) = d$ for all its non-zero submodules N. The *d-length* of an X-module M is its length in $\operatorname{Mod} X / \operatorname{Mod}_{<d} X$. It is denoted by $l_d(M)$, and it may take the value ∞.

Let X be a noetherian scheme with Krull dimension as the dimension function. If Z is a closed subscheme of X, then $\mathcal{O}_Z$ is critical in $\operatorname{Qcoh} X$ if and only if Z is an integral subscheme of X.

The function $l_d(-)$ is additive on short exact sequences. One sees this by passing to the quotient category $\operatorname{Mod} X/\operatorname{Mod}_{<d} X$ and using the fact that the usual notion of length is additive. Because l_d is additive, a d-critical module is uniform (i.e., two non-zero submodules of it have non-zero intersection). Hence an injective envelope of a d-critical module is indecomposable.

If M is a noetherian module of dimension d, then M has a d-critical quotient module, namely M/N, where N is a submodule of M maximal subject to the condition that $\delta(M/N) = d$.

A d-critical module is d-pure. A d-pure module is critical if and only if its d-length is one.

PROPOSITION 6.3. *Let X be a locally noetherian space. Suppose that $\mathscr{E}$ is an indecomposable injective such that every X-module is a subquotient of a direct sum of copies of $\mathscr{E}$. If $\mathscr{E}$ is d-critical with respect to some dimension function, then*

1. *X is integral of dimension d and $\mathscr{E}$ is the big injective*;
2. *M is torsion if and only if $\delta(M) < d$*;
3. $l_d(M) = \operatorname{rank} M$.

Proof. (1) If M is a non-zero submodule of $\mathscr{E}$, then $\delta(\mathscr{E}/M) < \delta(\mathscr{E})$, whence $\operatorname{Hom}_X(\mathscr{E}/M, \mathscr{E}) = 0$. It follows that the endomorphism ring of $\mathscr{E}$ is a division ring. Hence X is integral. Since an X-module is a subquotient of a direct sum of copies of $\mathscr{E}$ its dimension is at most d. Hence $\dim X = d$.

(2) If $\delta(M) < d$, then $\operatorname{Hom}_X(M, \mathscr{E}) = 0$ because $\mathscr{E}$ is d-critical, and M is torsion. To prove the converse it suffices to show if M is a noetherian module such that $\delta(M) = d$, then M is not torsion. Suppose to the contrary that there is such an M which is torsion. Then M has a d-critical quotient $\overline{M}$. This is also torsion, and so is its injective envelope $E(\overline{M})$ by Proposition 3.6. By Proposition 3.5, there is a non-zero map $\varphi\colon \mathscr{E} \to E(\overline{M})$. Since $E(\overline{M})$ is torsion, φ is not monic. Since $\mathscr{E}$ is d-critical, $\delta(\operatorname{im}\varphi) < d$. Hence $\delta(\operatorname{im}\varphi \cap \overline{M}) < d$. But $\operatorname{im}\varphi \cap \overline{M} \neq 0$, so this contradicts the fact that $\overline{M}$ is d-critical. We conclude that M can not be torsion.

(3) By (2), $\operatorname{Mod}_{<d} X$ consists of the torsion modules, whence $\operatorname{Mod} X/\operatorname{Mod}_{<d} X = \operatorname{Mod} \eta$, where η is the generic point of X. The remark after Proposition 3.9 implies that $\operatorname{rank} M = l_d(M)$. ∎

PROPOSITION 6.4. *Let X be an integral locally noetherian affine space with coordinate ring R. Suppose there is a dimension function δ such that $\delta(M \otimes_R I) \leq \delta(M)$ for all noetherian modules M and all two-sided ideals I. If $\mathscr{E}_X$ is critical with respect to δ, then R is prime.*

Proof. Since X is locally noetherian, R is right noetherian. By [10, Prop. 3.9], the condition on δ ensures that the annihilator of a critical right R-module is a prime ideal. In particular, Ann $\mathscr{E}_X$ is a prime ideal. But R itself is a subquotient of a finite direct sum of copies of $\mathscr{E}_X$, so the annihilator of $\mathscr{E}_X$ is zero. Hence R is prime. ∎

We expect there is a dimension function for right noetherian rings satisfying the hypothesis in Proposition 6.4. For many two-sided noetherian rings, such as factors of enveloping algebras, Gelfand–Kirillov dimension satisfies the hypothesis.

Every proper closed subscheme of an integral noetherian scheme X has strictly smaller dimension than X. For non-commutative spaces Krull dimension does not necessarily have this property—for example, take the ring of upper triangular matrices over a field.

We now pick out a better behaved class of weakly closed subspaces.

DEFINITION 6.5. Let δ be a dimension function on X. A weakly closed subspace W of X is *good* if whenever $0 \to L \to M \to N \to 0$ is an essential extension of a W-module L by an X-module N such that $\delta(N) < \delta(L)$, then $M \in \operatorname{Mod} W$.

A subspace can be good with respect to one dimension function but not good with respect to another.

If X is integral and $W \subset X$ is a proper weakly closed subspace, then $\dim W < \dim X$ if W is good. Hence we have the following result.

LEMMA 6.6. *Let X be an integral space, and suppose that $\delta(M) \in \mathbb{N}$ for all $M \neq 0$. If*

$$\phi \neq W_0 \subset W_1 \subset \cdots \subset W_d$$

is a chain of distinct good integral subspaces of X, then $d \leq \dim X$.

EXAMPLE 6.7. Let R be the ring of lower triangular 2×2 matrices over a field. Let $\mathscr{O}_p$ and $\mathscr{O}_q$ be the two simple right R-modules with $\mathscr{O}_p$ the projective one. There are closed points, p and q, defined by Mod p consists of all direct sums of copies of $\mathscr{O}_p$; Mod$q$ is defined similarly (closed points are defined in [9]). There is a non-split exact sequence $0 \to \mathscr{O}_p \to \mathfrak{p} \to \mathscr{O}_q \to 0$, where $\mathfrak{p}$ is the annihilator of $\mathscr{O}_p$. The indecomposable injectives are $\mathscr{O}_q$ and $\mathfrak{p}$.

Since $\operatorname{End}_R(\mathfrak{p}) \cong k$ and $R \cong \mathfrak{p} \oplus \mathfrak{p}/\mathscr{O}_p$, every R-module is a subquotient of a direct sum of copies of $\mathfrak{p}$. Therefore X is integral, $\mathfrak{p}$ is the big injective, and the function field of X is k. If j: $\eta \to X$ is the inclusion of the generic point, then $j_*(\operatorname{Mod} \eta)$ consists of all direct sums of copies of $\mathfrak{p}$. We also note that $\eta = X \backslash q$.

There are several ways in which X does not behave like an integral scheme. The inclusion $X \backslash p \to X$ sends $X \backslash p$ isomorphically onto q, so $X \backslash p$ is both open and closed in X. In particular, $\overline{X \backslash p} \neq X$. Furthermore, if we view η as an open subspace of X, then $\eta \cap (X \backslash p) = \phi$. Finally p is a proper closed subspace of X having the same Krull dimension as X.

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REFERENCES

1. M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), 335–388.
2. M. Artin and J. Zhang, Non-commutative projective schemes, *Adv. Math.* **109** (1994), 228–287.
3. P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 328–448.
4. A. W. Goldie, The structure of prime rings under ascending chain conditions, *Proc. London Math. Soc.* **8** (1958), 589–608.
5. A. Grothendieck (with P. Berthelot and L. Illusie), "Théorie des intersections et Théorème de Riemann–Roch (SGA 6)," Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, Berlin/New York, 1971.
6. L. Le Bruyn and M. Van den Bergh, On quantum spaces of Lie algebras, *Proc. Amer. Math. Soc.* **119** (1993), 407–414.
7. J. C. McConnell and J. C. Robson, "Noncommutative Noetherian Rings," Wiley, New York, 1987.
8. A. L. Rosenberg, "Non-commutative Algebraic Geometry and Representations of Quantized Algebras," Vol. 330, Kluwer Academic, Dordrecht/Norwell, MA, 1995.
9. S. P. Smith, Subspaces of non-commutative spaces, *Trans. Amer. Math. Soc.*, to appear.
10. J. T. Stafford, Stable structure of non-commutative noetherian rings, II, *J. Algebra* **52** (1978), 218–235.
11. J. Tate and M. Van den Bergh, Homological properties of Sklyanin algebras, *Invent. Math.* **124** (1996), 619–647.
12. M. Van den Bergh, "Blowing Up of Non-commutative Smooth Surfaces," Memoirs of the American Mathematical Society, Vol. 154, No. 734, Amer. Math. Soc., Providence, RI, 2001.
13. A. B. Verevkin, On a non-commutative analogue of the category of coherent sheaves on a projective scheme, *Amer. Math. Soc. Trans.* **151** (1992), 41–53.