PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 3, November 1989

# CAN THE WEYL ALGEBRA BE A FIXED RING?

S. P. SMITH

(Communicated by Donald S. Passman)

ABSTRACT. If a finite soluble group acts as automorphisms of a domain, then the invariant subring is not isomorphic to the first Weyl algebra $\mathbf{C}[t, d/dt]$.

Let $R = \mathbf{C}[t, d/dt]$ be the first Weyl algebra. We prove the following result.

**Theorem.** *Let $G$ be a finite solvable group. Let $S \supset R$ be a $\mathbf{C}$-algebra such that*

(a) *$S_R$ and ${}_RS$ are finitely generated,*

(b) *$S$ is a domain,*

(c) *$G$ acts as automorphisms of $S$, and $S^G = R$.*

*Then $S = R$.*

We will prove a rather more general result, from which the theorem follows. The original proof was improved by comments of T. J. Hodges. I would like to thank him for his interest, and for allowing his improvements to be included here.

Let $B$ be an $R$-$R$-bimodule. We call $B$ an *invertible bimodule*, if there exists another bimodule, $C$ say, such that $B \otimes_R C$ is isomorphic to $R$ as a bimodule. The invertible bimodules form a group under the operation of tensor product $\otimes_R$; this group is called the Picard group, denoted $\mathrm{Pic}(R)$. If $\sigma$, $\tau \in \mathrm{Aut}(R)$ are $\mathbf{C}$-linear algebra automorphisms of $R$, then we write ${}_\sigma R_\tau$ for the invertible bimodule which is $R$ as an abelian group, and for which the right $R$-module action is given by

$$b \cdot x = b\tau(x) \qquad \text{for } x \in R, b \in {}_\sigma R_\tau$$

and the left $R$-module action is given by

$$x \cdot b = \sigma(x)b \qquad \text{for } x \in R, b \in {}_\sigma R_\tau.$$

There is a map $\mathrm{Aut}(R) \to \mathrm{Pic}(R)$ given by $\sigma \mapsto {}_\sigma R_1$. This is a group homomorphism. A key point in our analysis is the following result of J. T. Stafford.

Received by the editors September 28, 1988 and, in revised form, October 10, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A72.

**Theorem** [3, Corollary 4.5]. *The map* $\operatorname{Aut}(R) \to \operatorname{Pic}(R)$ *is an isomorphism.*

Hence if $B$ is an invertible $R$-bimodule, there exists $e \in B$ and $\sigma \in \operatorname{Aut}(R)$ such that $x \cdot e = e \cdot \sigma(x)$ for all $x \in R$ (just take $e$ to be the image of 1 under the isomorphism ${}_\sigma R_1 \to B$ ).

**Proposition.** *Let* $S \supset R$ *be a* $\mathbf{C}$*-algebra satisfying conditions* (a) *and* (b) *of the theorem. Then the only invertible* $R$*-bimodule contained in* $S$ *is* $R$ *itself.*

*Proof.* Let $B \subset S$ be an invertible bimodule. Choose $e \in B$ and $\sigma \in \operatorname{Aut}(R)$ such that $B = Re = eR$ and $x \cdot e = e \cdot \sigma(x)$ for all $x \in R$ (here $\cdot$ denotes multiplication in $S$ ). The multiplication in $S$ is an $R$-bimodule map, so $B^n \cong_{\sigma^n} R_1$ . If $\sigma$ has infinite order (or equivalently, if $B$ has infinite order in $\operatorname{Pic} R$ ), then all the bimodules $B^n$ are non-isomorphic, and their sum in $S$ would be direct. However, since $S_R$ is finitely generated, $S$ has finite length as an $R$-bimodule. Therefore, $\sigma^n = 1$ for some $n$. Hence for all $x \in R$, $xe^n = e^n\sigma^n(x) = e^n x$.

Therefore there is a surjective algebra homomorphism $R \otimes_{\mathbf{C}} \mathbf{C}[X] \to R[e^n]$, with $X \mapsto e^n$, where $X$ is an indeterminate commuting with $R$. By [1, 4.5.1], the ideals of $R \otimes_{\mathbf{C}} \mathbf{C}[X]$ are of the form $R \otimes_{\mathbf{C}} I$ where $I$ is an ideal of $\mathbf{C}[X]$. For $R[e^n] \cong R \otimes_{\mathbf{C}} \mathbf{C}[X]/R \otimes_{\mathbf{C}} I \cong R \otimes_{\mathbf{C}} \mathbf{C}[X]/I$ to be a domain it is necessary that $I = \langle X - \alpha \rangle$ for some $\alpha \in \mathbf{C}$. Thus $e^n = \alpha$. But $\mathbf{C}[e] \subset S$ is a domain, so $n = 1$. Therefore $B = R$. ■

If $M$ is a left $R$-module, then the *rank* of $M$ is the dimension of $\operatorname{Fract} R \otimes_R M$ as a left $\operatorname{Fract} R$-module. It is clear that an invertible bimodule is of rank 1.

*Proof of the theorem.* First we prove it for $G$ abelian. In that case write $S = \bigoplus_\chi S_\chi$ where the sum is over the irreducible characters of $G$, and $S_\chi$ is the $\mathbf{C}G$-submodule of $S$ which is the sum of the $\chi$-isotypical components. Therefore $S_1 = R$, $S_\chi S_\xi = S_{\chi\xi}$, and each $S_\chi$ is an $R$-bimodule.

Suppose that $\chi\xi = 1$, and let $0 \neq a \in S_\xi$. Then $S_\chi a \subset R$, and is isomorphic to $S_\chi$ as a left $R$-module since $S$ is a domain. In particular, $S_\chi$ is of rank 1 as a left $R$-module. Similarly, $S_\chi$ is of rank 1 as a right $R$-module. The multiplication map on $S$ gives an $R$-bimodule homomorphism $S_\chi \otimes_R S_\xi \to S_\chi S_\xi$. The image is a non-zero subbimodule of $R$, hence equals $R$. Because all the ranks are 1, the map is injective. Therefore $S_\chi$ is an invertible bimodule. By the proposition, this forces $S_\chi = R$. Hence $S = R$ as required.

Now let $G$ be any finite solvable group, and set $H = [G, G]$. Then there is an action of $G/H$ as automorphisms of $S^H$, and $R = S^G = (S^H)^{G/H}$. But $G/H$ is abelian, and the first part of the argument applied to $S^H$ shows that $S^H = R$. Now by induction on $|G|$, the theorem follows. ■

*Remarks.* 1. It would be very nice to have the same result for an arbitrary finite group $G$, but a new idea is necessary. Not much is known about finitely generated $R$-bimodules which are not invertible, and that is probably a prerequisite.

2. I do not know of any domain $S \supsetneq R$ such that (a) and (b) hold. It would be very interesting to know whether or not such an $S$ could exist. I expect not.

3. More generally I think it would be an interesting question to look at some other well understood non-commutative algebras, and ask if they can occur as the fixed ring of some reasonable extension ring. See [2] for an example concerning primitive factor rings of $U(\mathrm{sl}(2))$.

## REFERENCES


1. J. Dixmier, *Enveloping algebras*, North-Holland Publishing Co., Amsterdam, 1977.
2. S. P. Smith, *Overrings of primitive factor rings of* $U(\mathrm{sl}(2))$, J. Pure and Appl. Alg., (to appear).
3. J. T. Stafford, *Endomorphisms of right ideals of the Weyl algebra*, Trans. Amer. Math. Soc. **299** (1987), 623–639.


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195