CURVES, DIFFERENTIAL OPERATORS AND FINITE DIMENSIONAL ALGEBRAS

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§ 1. INTRODUCTION.

This talk is mainly a report on some joint work with J.T.Stafford which appears in [6]. That paper examines the structure of $\mathcal{D}(X)$, the ring of differential operators on an irreducible affine curve X, defined over an algebraically closed field k of characteristic zero. When X is non-singular the structure of $\mathcal{D}(X)$ is well understood, and is but a particular case of a structure theory which applies to non-singular affine varieties X of any dimension. However, when X is singular the structure of $\mathcal{D}(X)$ is not well understood, and [6] examines the easiest case viz. X is a (singular) curve. In all that follows X will denote an irreducible affine curve defined over an algebraically closed field k of characteristic zero.

This paper begins by recalling in § 2 some of the main results of [6] concerning the structure of $\mathcal{D}(X)$. On the positive side, $\mathcal{D}(X)$ is a finitely generated k-algebra and right and left noetherian. However, in contrast to the non-singular case, $\mathcal{D}(X)$ need not be a simple ring if X is singular. In Theorem 2.3 it is seen that the simplicity of $\mathcal{D}(X)$ is equivalent to a number of other properties. In particular, $\mathcal{D}(X)$ is simple if and only if the natural projection $\pi : \tilde{X} \to X$ from the normalisation is bijective. When $\mathcal{D}(X)$ is not simple, there is a unique minimal non-zero ideal $J(X)$, and $H(X) := \mathcal{D}(X)/J(X)$ is a finite dimensional k-algebra. The ring of regular functions $\mathcal{O}(X)$ need not be a simple $\mathcal{D}(X)$-module, but it has a unique simple submodule $J(X).\mathcal{O}(X)$, and $C(X) := \mathcal{O}(X)/J(X).\mathcal{O}(X)$ is a finite dimensional k-algebra. Both $H(X)$ and $C(X)$ split as a direct sum of finite dimensional algebras, H_x and C_x, one for each singular point $x \in \operatorname{Sing} X$. The algebras H_x and C_x depend only on the local ring $\mathcal{O}_{X,x}$, and § 3 examines how the structure of H_x and C_x depends on that of $\mathcal{O}_{X,x}$. We have no general theorem, and it is clearly a key question to understand how the nature of the singularity at x is reflected in the structure of H_x and C_x.

In Section 4 we provide some light relief and show how some of the results in § 2 may be used to describe the space of polynomial solutions of a (very restricted) class of differential equations. For example, if $D = \partial_y^2 - \partial_x^3$ is viewed as a differential operator on $k[x,y]$ and $S = \{f \in k[x,y] \mid D(f) = 0\}$ we show that S is a simple $\mathcal{D}(X)$-module where X is the curve in $\mathbb{A}^2$ defined by $y^2 = x^3$. Knowing generators of $\mathcal{D}(X)$ as a k-algebra, allows one to produce a basis for S in an extremely simple way.

Section 5, shows how the results of § 2 may be used when $\pi : \tilde{X} \to X$ is injective to solve the following problem. Let $R = k[x,y]$ be the polynomial ring in two variables, and let $0 \neq f \in R$ be an irreducible polynomial defining the curve C. It is well known that the $\mathcal{D}(R)$-action on R extends to the localisation R_f, and that R_f/R is a $\mathcal{D}(R)$-module of finite length with a unique simple submodule. When C is non-singular, it is not hard to show that R_f/R is itself a simple $\mathcal{D}(R)$-module (the proof of this is given in § 5) ; this is well-known, but when C is singular it is difficult to describe the simple submodule of R_f/R. We prove that when $\pi : \tilde{C} \to C$ is injective then R_f/R is a simple $\mathcal{D}(R)$-module.

§ 2. STRUCTURE OF $\mathcal{D}(X)$.

Let A be a commutative k-algebra and let M and N be A-modules. The space $\mathcal{D}_A(M,N)$ of k-linear differential operators from M to N is defined to be $\mathcal{D}_A(M,N) =$

$$\cup_{n=0}^{\infty}\{\theta \in \mathrm{Hom}_k(M,N) \mid [a_n[\ldots[a_1\ a_0,\theta]]\ldots]] = 0 \text{ for all } a_0,a_1,\ldots,a_n \in A\}$$

where $[a,\theta] = a\theta - \theta a$.

We are interested in $\mathcal{D}(A) = \mathcal{D}_A(A,A)$, the ring of differential operators on A, when A is either $\mathcal{O}(X)$, the co-ordinate ring of the curve X, or $\mathcal{O}_{X,x}$, the local ring at the point $x \in X$. We denote $\mathcal{D}(A)$ by $\mathcal{D}(X)$ and $\mathcal{D}_{X,x}$ in these two cases.

When X is a non-singular curve, $\mathcal{D}(X)$ is a finitely generated k-algebra, (right and left) noetherian [4, § 6], and a simple ring of global homological dimension 1. For non-singular X, $\mathcal{D}(X)$ is generated by $\mathcal{O}(X)$ and $\mathrm{Der}_k X$, the module of k-linear derivations on $\mathcal{O}(X)$. Unfortunately when X is singular this is not true.

EXAMPLE 1. Let X be the curve in $\mathbb{A}^3$ defined by $x^5 = y^3$, $x^7 = z^3$. There is a unique singular point at $(0,0,0)$. The normalisation $\tilde{X}$, is isomorphic to $\mathbb{A}^1$ and the natural projection $\pi : \tilde{X} \to X$ is given by $\pi(\alpha) = (\alpha^3,\alpha^5,\alpha^7)$. Write $\mathcal{O}(\tilde{X}) = k[t]$ and $\mathcal{O}(X) = k[t^3,t^5,t^7]$. Since $\operatorname{Der} \tilde{X}$ is the free $k[t]$-module generated by $\partial = d/dt$, and any derivation on $\mathcal{O}(X)$ extends to $\mathcal{O}(\tilde{X})$ [5], it is easy to see that $\operatorname{Der} X$ is the subspace of $k[t]\partial$ with basis $\{t\partial, t^3\partial\} \cup \{t^n\partial \mid n > 4\}$. Set $D = t^{-1}(t\partial - 2)(t\partial - 7)\partial$. This is a differential operator on $k(t)$, and it leaves $\mathcal{O}(X)$ stable. Hence $D \in \mathcal{D}(X)$, but D is <u>not</u> in the subalgebra of $\operatorname{End}_k \mathcal{O}(X)$ generated by $\mathcal{O}(X)$ and $\operatorname{Der} X$.

This example illustrates the difficulty in trying to decide whether $\mathcal{D}(X)$ is a finitely generated k-algebra. In fact, if one takes Z to be the surface in $\mathbb{C}^3$ defined by $X_1^3 + X_2^3 + X_3^3 = 0$, then $\mathcal{D}(Z)$ is not finitely generated [2]. However, for curves one has the following.

THEOREM 2 [6]. *Let X be a curve. Then $\mathcal{D}(X)$ is a finitely generated k-algebra and is right and left noetherian.*

Although $\mathcal{D}(X)$ need not be a simple ring we have the following (recall that $\pi : \tilde{X} \to X$ is the natural projection from the normalisation).

THEOREM 3 [6]. *The following are equivalent* :

(a) $\mathcal{D}(X)$ *is a simple ring* ;

(b) $\pi : \tilde{X} \to X$ *is bijection* ;

(c) $\mathcal{O}(X)$ *is a simple* $\mathcal{D}(X)$-*module* ;

(d) $\operatorname{gl.dim} \mathcal{D}(X) = 1$;

(e) $\mathcal{D}(X)$ *is Morita equivalent to* $\mathcal{D}(\tilde{X})$.

As, perhaps, suggested by (e), the key to understanding $\mathcal{D}(A)$ for $A = \mathcal{O}(X)$, or $A = \mathcal{O}_{X,x}$ is to compare $\mathcal{D}(A)$ and $\mathcal{D}(\bar{A})$ where $\bar{A}$ denotes the integral closure of A in Fract A, its field of fractions. Define

$$\mathcal{D}(\bar{A},A) = \{D \in \mathcal{D}(\bar{A}) \mid D(f) \in A \text{ for all } f \in \bar{A}\}.$$

This is a non-zero right ideal of $\mathcal{D}(\bar{A})$ and a left ideal of $\mathcal{D}(A)$. Since $\mathcal{D}(A)$ is a simple, hereditary ring $\mathcal{D}(\bar{A},A)$ is necessarily a progenerator in Mod-$\mathcal{D}(\bar{A})$. Thus we have $\mathcal{D}(A) \subseteq \operatorname{End}_{\mathcal{D}(\bar{A})}\mathcal{D}(\bar{A},A) = T$, where T is Morita equivalent to $\mathcal{D}(\bar{A})$. The relation between $\mathcal{D}(A)$ and T depends on the fact that they have a common left ideal, namely $\mathcal{D}(\bar{A},A)$. A key lemma is that $\mathcal{D}(A) = T$ if and only if $\mathcal{D}(\bar{A},A)*\bar{A}=A$, where $\mathcal{D}(\bar{A},A)\star\bar{A}$ denotes the linear span of all $D(f)$ such that $D\in\mathcal{D}(\bar{A},A)$ and $f\in\bar{A}$. It is these observations which are exploited to obtain the above results.

Through part (e) of Theorem 3 we can, in a sense, say that we understand $\mathcal{D}(X)$ completely when $\mathcal{D}(X)$ is simple. So from now on we concentrate on what happens when $\mathcal{D}(X)$ is not simple. However, there is one question still of interest when $\mathcal{D}(X)$ is simple ; give a procedure for obtaining generators for $\mathcal{D}(X)$, or find the least n such that $\mathcal{D}(X)$ is generated by differential operators of order $\leq n$.

To understand $\mathcal{D}(X)$ when $\pi : \tilde{X}\to X$ is not injective one is led to prove.

THEOREM 4 [6]. $\mathcal{D}(X)$ constains a unique minimal non-zero ideal $J(X)$. The factor $H(X) := \mathcal{D}(X)/J(X)$ is a finite dimensional k-algebra, and $H(X) = \oplus_{x\in \mathrm{Sing}X} H_x$ is a direct sum of algebras H_x one for each singular point x . The structure of H_x depends only on the local ring $\mathcal{O}_{X,x}$. In fact $\mathcal{D}_{X,x}$ has a unique minimal non-zero ideal $J_{X,x}$ and $H_x = \mathcal{D}_{X,x}/J_{X,x}$.

The relationship between the ideal structure of $\mathcal{D}(X)$ and the submodule structure of $\mathcal{O}(X)$ is illustrated by

THEOREM 5 [6]. Consider $\mathcal{O}(X)$ as a $\mathcal{D}(X)$-module. Then

(a) $\mathcal{O}(X)$ has finite length ;

(b) $\mathcal{O}(X)$ has a unique simple submodule, namely

$$J(X).\mathcal{O}(X) = \mathcal{D}(\tilde{X},X)\star\mathcal{O}(\tilde{X}) \ ;$$

(c) If $C(X) = \mathcal{O}(X)/J(X).\mathcal{O}(X)$ then $C(X)$ is a faithful $H(X)$-module;

(d) $C(X) \simeq \oplus_{x\,\mathrm{Sing}X} C_x$ is a direct sum of local algebras, one for each singular point of X ;

(e) $C_x \simeq \mathcal{O}_{X,x}/J_{X,x}.\mathcal{O}_{X,x}$ and is a faithful H_x-module .

Clearly one would like to understand the structure of the finite dimensional algebras H_x and C_x , and so $H(X)$ and $C(X)$. First note that, since H_x and C_x depend only on $\mathcal{O}_{X,x}$, it will follow from Theorem 3 that H_x and C_x are zero precisely when $\#\,\pi^{-1}(x)=1$. It is not difficult to observe that if $\mathcal{O}(X) = k[t_1,\dots,t_n]/(f_1,\dots,f_r)$ then $C(X)$ is a homomorphic image of

$k[t_1,\dots,t_n]/(f_1,\dots,f_r,\partial f_i/\partial t_j)$ because $J(X).\mathcal{O}(X)$ contains the conductor of $\mathcal{O}(X)$ in $\mathcal{O}(\tilde{X})$ and the image of each $\partial f_i/\partial t_j$ belongs to the conductor

§ 3. THE ALGEBRAS H_x AND C_x .

In this section X is a curve with a unique singular point x , and we set $A = \mathcal{O}_{X,x}$ and $B = \bar{A}$. This section is a collection of examples illustrating some of the possibilities for H_x and C_x . We will give examples where H_x may be either 0 , or $M_n(k)$, the ring of $n \times n$ matrices over k , or $\begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ the ring of lower triangular 2×2 matrices , or $\begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}$. In these examples C_x is respectively 0, $k[t]/(t^n)$, $k[t]/(t^2)$, and $k[s,t]/(s,t)^2$. We have no general result, but these examples do give some clues as to what should be expected in general.

We denote the maximal ideal of A by $\underline{m}$. B is a semi-local ring with Jacobson radical denoted $\underline{r}$. The maximal ideals of B correspond to the points $\pi^{-1}(x)$. Since $H_x = 0$ if and only if $\# \pi^{-1}(x) = 1$, we may rephrase this as

PROPOSITION 1. $H_x = 0$ <u>if and only if</u> $\bar{\mathcal{O}}_{X,x}$ <u>is a local ring</u>.

By [6, § 7.4] there exists $t \in \underline{r}$ and $\partial \in \mathrm{Der}_k B$ such that $\partial(t) = 1$. It is an easy exercise to see that this forces $\mathrm{Der}_k B = B\partial$, and $\underline{r} = Bt$. If $b \in B$ we shall write $b' = \partial(b)$.

We shall assume in all the examples we construct that $\pi : \tilde{X} \to X$ is unramified at all points. The reason for this restriction is because we can make use of the following result to simplify the calculations.

THEOREM 2 (W.C.Brown [3]) . <u>If</u> $\pi : \tilde{X} \to X$ <u>is unramified at all points then</u> $\mathcal{D}(X) \subseteq \mathcal{D}(\tilde{X})$

Thus we have , locally $\mathcal{D}(A) \subseteq \mathcal{D}(B) = B[\partial]$. First we construct examples where $H_x \simeq M_n(k)$. The easiest case is $n = 1$.

PROPOSITION 3. <u>Suppose that</u> $\# \pi^{-1}(x) > 1$. <u>Let I denote the conductor of</u> $\mathcal{O}_{X,x}$ <u>in</u> $\bar{\mathcal{O}}_{X,x}$ <u>If I is a maximal ideal of</u> $\mathcal{O}_{X,x}$, <u>then</u> $H_x \simeq k$.

<u>Proof</u>. Since $\# \pi^{-1}(x) \neq 1$, $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are not Morita equivalent , so $\mathcal{D}(B,A) \star B \neq A$. However, $I \subseteq \mathcal{D}(B,A)$ whence $I = \mathcal{D}(B,A) \star B$. But $k = A/I$ is now a faithful H_x-module. Hence $H_x \simeq k$. □

This explains [6 , Theorem 4.4] since under the hypotheses of that theorem one must have I a maximal ideal of $\mathcal{O}_{X,x}$, since $\mathcal{O}_{X,x}/I$ is a local ring contained in $\bar{\mathcal{O}}_{X,x}/I$ which is a product of fields.

It is possible for H_x to equal k without the hypothesis of Proposition 3 being satisfied. Indeed, if $\mathcal{D}(B,A) \star B$ is a maximal ideal of A then $H_x \simeq k$. This is illustrated by the following :

EXAMPLE 4 [6, § 5.7] Take $\tilde{X} = \mathbb{A}^1$, and $\mathcal{O}(\tilde{X}) = k[t]$. Define X by $\mathcal{O}(X) = k + kt^2(t-1) + t^4(t-1)k[t]$. The conductor is $t^4(t-1)k[t]$ which is not a maximal ideal of $\mathcal{O}(X)$. It is shown in [6] that $\mathcal{D}(B,A) \star B = \underline{m}$, the unique maximal ideal of $\mathcal{O}_{X,x}$ (here x is the unique singular point of X) . Again $A/\mathcal{D}(B,A) \star B$ is a faithful H_x-module so $H_x \simeq k$.

This example may be understood as follows . Let X' be the curve with $\mathcal{O}(X') = k + t^2k[t]$. We have a factorisation of Π as $\tilde{X} \xrightarrow{\psi} X' \xrightarrow{\varphi} X$ with $\Pi = \varphi\psi$ and ψ injective. Hence $\mathcal{D}(\tilde{X},X') \star \mathcal{O}(\tilde{X}) = \mathcal{O}(X')$. However, $\mathcal{O}(X) = k+t^2(t-1)\,\mathcal{O}(X')$ and $\mathcal{D}(X',X) \supseteq t^2(t-1)\,\mathcal{D}(X')$. Thus $\mathcal{D}(\tilde{X},X) \supseteq \mathcal{D}(X',X)\mathcal{D}(\tilde{X},X') \supseteq t^2(t-1)\mathcal{D}(\tilde{X},X')$. Hence $\mathcal{D}(\tilde{X},X) \star \mathcal{O}(\tilde{X}) \supseteq t^2(t-1)\,\mathcal{O}(X') = \underline{m}$. The point is that ψ is injective, and the conductor of $\mathcal{O}(X)$ in $\mathcal{O}(X')$ is a maximal ideal of $\mathcal{O}(X)$.

PROPOSITION 5. <u>Suppose that</u> $\# \Pi^{-1}(x) > 1$. <u>Suppose that the Jacobson radical of</u> $\bar{\mathcal{O}}_{X,x}$ <u>is</u> $t\,\bar{\mathcal{O}}_{X,x}$, <u>and that</u> $\mathcal{O}_{X,x} = k+kt+\ldots+kt^n+t^{n+1}\bar{\mathcal{O}}_{X,x}$. <u>Then</u> $H_x \simeq M_{n+1}(k)$.

<u>Proof</u>. Let $\underline{m}$ be the maximal ideal of $A = \mathcal{O}_{X,x}$. Then $\underline{m}B = tB$, and by Theorem 2, $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. The same argument as [6 , Lemma 5.3] shows that $\mathcal{D}(B,A) = t^{n+1}\mathcal{D}(B)$, whence $C_x = A/t^{n+1}B$ is a faithful H_x-module. Thus , the result will follow if we can show that $A/t^{n+1}B$ is a simple H_x-module, or equivalently is a simple $\mathcal{D}(A)$-module. Notice that $A/t^{n+1}B$ is generated by 1 , and that kt^n is an essential A-submodule . Thus, to show $A/t^{n+1}B$ is a simple $\mathcal{D}(A)$-module it will suffice to show that there exists $D \in \mathcal{D}(A)$ such that $D(t^n) = 1$. We proceed to show that $D := (t\partial - 1)\ldots(t\partial - n)\partial^n$ belongs to $\mathcal{D}(A)$; since $(-1)^n(n!)^{-2}D(t^n) = 1$ this will complete the proof of the Proposition.

Since $\mathcal{D}(B) = B[\partial]$ we have $D \in \mathcal{D}(B)$. The action of D on A annihilates $k + kt + \ldots + kt^{n-1}$, so it remains to show that $D\star(Bt^{n+1}) \subseteq Bt^{n+1}$. First, notice that $\partial^n \star (Bt^{n+1}) \subseteq Bt$. Secondly, notice that, for all $j \in \mathbb{N}$, $(t\partial - j)\star(Bt^j) \subseteq Bt^{j+1}$. Hence $(t\partial - n)\ldots(t\partial - 1) \star (Bt) \subseteq Bt^{n+1}$ and thus $D\star(Bt^{n+1}) \subseteq Bt^{n+1}$ and $D \in \mathcal{D}(A)$, as required. $\square$

PROPOSITION 6 . Suppose that $\# \Pi^{-1}(x) > 1$. Suppose that the Jacobson radical of $\bar{\mathcal{O}}_{X,x}$ is $t\bar{\mathcal{O}}_{X,x}$, and that $\mathcal{O}_{X,x} = k + kt + ty\bar{\mathcal{O}}_{X,x}$ where $y \in \bar{\mathcal{O}}_{X,x} \setminus t\bar{\mathcal{O}}_{X,x}$, and y is not a unit . Then $H_x \simeq \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$.

Proof. The same arguments as usual show that $\mathcal{D}(B,A) = ty\mathcal{D}(B)$ and hence $C_x = A/tyB$. Since $\dim C_x = 2$ and C_x is a faithful H_x-module, H_x embeds in $M_2(k)$. The hypothesis implies that $t \in yB$ (just use the fact that $\mathcal{O}_{X,x}$ is a ring, so contains t^2) . Note that $t\partial \in \mathrm{Der}_k A$. It is now easy to show that the images of $1,t,t\partial$ are linearly independent in $H_x = \mathcal{D}(A)/\mathcal{D}(B,A)$.

Now $\mathcal{D}(A) \subseteq \mathbb{I}(\mathcal{D}(B,A))$ the idealiser of $\mathcal{D}(B,A)$ in $\mathcal{D}(B)$. Since $\mathbb{I}(ty\mathcal{D}(B))/ty\mathcal{D}(B) \simeq \mathrm{End}_{\mathcal{D}(B)}(\mathcal{D}(B)/ty\mathcal{D}(B))$ it is straightforward (after decomposing as a sum of simple modules) to see that $\dim_k(\mathbb{I}(ty\mathcal{D}(B))/ty\mathcal{D}(B)) = \dim_k(B/tB) + 3\dim_k(B/yB)$. The next step is to explicitly describe $\mathbb{I}(ty\mathcal{D}(B))$.

Write $t = yz$. Notice that $zy' = 1 \pmod{yB}$. It follows that both y' and $zy' + 1$ are units modulo yB . Thus there exists $b \in B$ such that $2y' - b(zy'+1) \in yB$. Now one computes to check that $(t\partial - bz)\partial \in \mathbb{I}(ty\mathcal{D}(B))$. Thus $\mathbb{I}(ty\mathcal{D}(B))$ contains $B + Bt\partial + B(t\partial - bz)\partial + ty\mathcal{D}(B)$. It is straightforward to compute the dimension of this modulo $ty\mathcal{D}(B)$, and check that it is equal to $\dim_k(B/tB) + 3\dim_k(B/yB)$. It follows from the previous paragraph that this subspace is in fact equal to $\mathbb{I}(ty\mathcal{D}(B))$.

Recall that $\mathcal{D}(A) \subseteq \mathbb{I}(ty\mathcal{D}(B))$. To show that $\mathcal{D}(A) = k + kt + kt\partial + ty\mathcal{D}(B)$, it is enough to show that if $u,v \in B$ with $D = ut\partial + v(t\partial - bz)\partial$ is an element of $\mathcal{D}(A)$ then one must have $D \in k + kt + kt\partial + ty\mathcal{D}(B)$. To see this first observe that $D \star A \subseteq k + tB$, and evaluating D on t this gives $vbz \in k + tB$. However, vbz cannot be a unit since z is not (because $y \notin tB$) . Thus $vbz \in tB$ and $vb \in yB$. But b is a unit modulo yB , so $v \in yB$. Thus D is a derivation modulo $ty\mathcal{D}(B)$. But $\mathrm{Der}_k(A) = kt\partial + tyB\partial$, hence $D \in k + kt + kt\partial + ty\mathcal{D}(B)$.

Thus the images of $1,t,t\partial$ span H_x , and therefore $H_k \simeq \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$. □

PROPOSITION 7. Suppose that $\# \Pi^{-1}(x) > 1$. Suppose that the Jacobson radical of $\bar{\mathcal{O}}_{X,x}$ is $t\bar{\mathcal{O}}_{X,x}$, and that $\mathcal{O}_{X,x} = k + kt + kty + t^2\bar{\mathcal{O}}_{X,x}$ where $y \in \bar{\mathcal{O}}_{X,x} \setminus t\bar{\mathcal{O}}_{X,x}$, and y is not a unit. Then $H_x \simeq \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}$.

Proof. The argument is very similar to that in Proposition 6. One computes $\mathbb{I}(t^2B) = B + Bt\partial + B(t\partial - 1)\partial + t^2\mathcal{D}(B)$, checks that $1, t, ty, t\partial \in D(A)$ and that their images in H_x are linearly independent. And finally one shows that if $D = v(y\partial - 1)\partial + ut\partial$ belongs to $\mathcal{D}(A)$ with $u, v \in B$, then $D \in kt\partial + t^2\mathcal{D}(B)$; where H_x is spanned by $1, t, ty, t\partial$ and the result follows by considering the action of these elements on A/t^2B. □

This completes the list of examples stated at the beginning of this section. Notice in the examples where H_x is $M_2(k)$, and H_x is $\begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$, that C_x is isomorphic to $k[t]/(t^2)$ in both cases, and $C_x \simeq \mathcal{O}_{X,x}/I$ where I is the conductor of $\mathcal{O}_{X,x}$. In particular, knowing C_x and $\mathcal{O}_{X,x}/I$ does not determine H_x.

In the above examples H_x is always an indecomposable algebra, in the sense that H_x cannot be written a direct product of two non-zero subalgebras. More generally we have

PROPOSITION 8. For any X, and any $x \in X$, H_x is an indecomposable algebra.

Proof. Suppose H_x is a direct product of non-zero subalgebras. Then there exist non-zero central orthogonal idempotents $e, f \in H_x$ with $1 = e + f$. Then $C_x = H_x eC_x \oplus H_x fC_x$. However, this decomposition of C_x as a $\mathcal{D}_{X,x}$-module is also a decomposition of C_x as an $\mathcal{O}_{X,x}$-module, and hence as a C_x-module. But C_x is a local algebra, hence indecomposable. Hence either $eC_x = 0$ or $fC_x = 0$. But, either possibility contradicts the fact that C_x is a faithful H_x-module. □

§ 4. CONSTANT COEFFICIENT DIFFERENTIAL OPERATORS AND THE SPACE OF POLYNOMIAL SOLUTIONS.

Let $R = \mathbb{C}[x,y]$ be the polynomial ring in two variables, and $\mathcal{D} = \mathcal{D}(R) = C[x,y,\partial_x,\partial_y]$ the ring of differential operators on R. Let $D \in \mathcal{D}$ and set $S = \{f \in R \mid D(f) = 0\}$, the space of polynomial solutions. Observe that if $P, Q \in \mathcal{D}$ with $DP = QD$, and $f \in S$ then $P(f) \in S$ also. Define, the idealiser of $\mathcal{D}D$, $\mathbb{I}(\mathcal{D}D) = \{P \in \mathcal{D} \mid DP \in \mathcal{D}D\}$. This is a subring of $\mathcal{D}$, containing $\mathcal{D}D$ as a two sided ideal. The above observation says that S is a left $\mathbb{I}(\mathcal{D}D)$-module. Furthermore it is annihilated by $\mathcal{D}D$, so S is a left $\mathbb{I}(\mathcal{D}D)/\mathcal{D}D$-module.

Let $\sigma : \mathcal{D} \to \mathcal{D}$ be the anti-automorphism given by

$$\sigma(x) = \partial_x \ , \ \sigma(\partial_x) = x \ , \ \sigma(y) = \partial_y \ , \ \sigma(\partial_y) = y \ .$$

Setting $\sigma(D) = D^\sigma$, we have $\sigma(\mathcal{D}D) = D^\sigma \mathcal{D}$, and $\sigma(\mathrm{II}(\mathcal{D}D)) = \mathrm{II}(D^\sigma \mathcal{D})$. Thus S can be given the structure of right $\mathrm{II}(D^\sigma \mathcal{D})/D^\sigma\mathcal{D}$-module by defining $f.Q' = Q(f)$ for $Q' \in \mathrm{II}(D^\sigma \mathcal{D})$ where $Q = \sigma(Q')$.

Now consider for example, the case where $D = \partial_y{}^2 - \partial_x{}^3$ (resp. $D = \partial_y{}^2 - \partial_x{}^3 + \partial_x$). Then $D^\sigma = y^2 - x^3$ (resp. $D^\sigma = y^2 - x^3 + x$) and the space of polynomial solutions is a right $\mathrm{II}(g\mathcal{D})/g\mathcal{D}$-module where $g = y^2 - x^3$ (resp. $g = y^2 - x^3 + x$). But by [6, § 1.6], $\mathrm{II}(g\mathcal{D})/g\mathcal{D} \simeq \mathcal{D}(R/gR) = \mathcal{D}(C)$ where $C \subseteq \mathbb{A}^2$ is the curve defined by $g \in \mathbb{C}[x,y]$. Thus S is a right $\mathcal{D}(C)$-module. We will show below (for both the given examples, and more generally whenever $\Pi : \tilde{C} \to C$ is injective) that S is a simple right $\mathcal{D}(C)$-module. Thus to describe all of S we need know only one non-zero element of S and the action of $\mathcal{D}(C)$ on S. In the given examples it is clear that $1 \in S$, whence $S = 1.\mathcal{D}(C)$. So the problem of describing all polynomial solutions of the differential equation $D(f) = 0$ leads us naturally to ask for a description of $\mathcal{D}(C)$ (for example, once we know by Theorem 2.2, that $\mathcal{D}(C)$ is finitely generated, we want to know the generators) and a description of the action of $\mathcal{D}(C)$ on S.

The **procedure** we shall adopt in order to describe all of S, will be to first describe $\mathcal{D}(C)$ (through its relationship with $\mathcal{D}(\tilde{C})$ as outlined in Theorem 2.3) and thence to obtain a description of $\mathrm{II}(\mathcal{D}D)/\mathcal{D}D$ and so act on S. For example in the case $D = \partial_y{}^2 - \partial_x{}^3$, we have $\mathcal{O}(C) = \mathbb{C}[t^2,t^3]$ and as in [6, Remark 3.12], $\mathcal{D}(C) = \mathbb{C}[t^2, t^3, t\partial, t^2\partial, (t\partial - 1)\partial, t^{-1}(t\partial - 2)\partial]$ and (after the detailed considerations below), $t^2\partial$ gives rise to the element $Q = 2x\partial_y + 3y\partial_x{}^2 \in \mathrm{II}(\mathcal{D}D)$, and $t^{-1}(t\partial - 2)\partial$ gives rise to the element $P = 4x^2\partial_x + 12xy\partial_y + 9y^2\partial_x{}^2 - 2x \in \mathrm{II}(\mathcal{D}D)$. It will be shown that $S = \mathbb{C}[P].1 + Q\mathbb{C}[P].1$ and thus we obtain all elements of S by starting with $1 \in S$ and acting by Q,P as follows. The diagram indicates how solutions are obtained from previous ones by applying P and Q (we ignore scalar multiples, so although $Q(x^3 + 3y^2) = 24xy$ we just write $x^3 + 3y^2 \xrightarrow{Q} xy$).

$$\begin{array}{ccccccccccccc}
1 & \xrightarrow{P} & x & \xrightarrow{P} & x^2 & \xrightarrow{P} & x^3+3y^2 & \xrightarrow{P} & x^4+12xy^2 & \xrightarrow{P} & x^5+30x^2y^2 & \xrightarrow{P} & \cdots \\
 & & \downarrow Q & & & & \downarrow Q & & \downarrow Q & & \downarrow Q & & \\
 & & y & \longrightarrow & & & xy & \longrightarrow & x^2y & \longrightarrow & x^3y+y^3 & \longrightarrow & \cdots
\end{array}$$

One can continue to apply P and Q to obtain more solutions, and Proposition 6 below shows that one will in this way obtain a basis for $\{f \in \mathbb{C}[x,y] \mid (\partial_y^2 - \partial_x^3)(f) = 0\}$.

Two points should be observed. First, to find elements such as $P,Q \in \mathbb{I}(\mathcal{D}D)$ simply by computing inside $\mathcal{D}$ seems an impossibly difficult task. Even if one can find elements of $\mathbb{I}(\mathcal{D}D)$ one needs to know whether one has found enough elements to generate all of $\mathbb{I}(\mathcal{D}D)/\mathcal{D}D$ (hence the importance of Theorem 2.2 saying that $\mathcal{D}(C)$ is finitely generated, and hence the importance of trying to obtain a procedure to find generators of $\mathcal{D}(C)$). Secondly, to know that all polynomial solutions belong to $1.\mathcal{D}(C)$ (in the case when $\Pi: \tilde{C} \to C$ is injective) one needs to show (as we do below) that S is a simple right $\mathcal{D}(C)$-module.

It is no problem to extend the above analysis to the more general situation described in the following Proposition. First note that there is a natural anti-automorphism σ on the ring $\mathcal{D}(A^n) = \mathbb{C}[t_1,\dots,t_n,\partial_1,\dots,\partial_n]$ where $\partial_j = \partial/\partial t_j$, given by $\sigma(t_j) = \partial_j$, $\sigma(\partial_j) = t_j$ for all j.

PROPOSITION 1. <u>Let</u> $R = \mathbb{C}[t_1,\dots,t_n]$, <u>and let</u> J <u>be an ideal of</u> R <u>contained in</u> $(t_1,\dots,t_n)$. <u>Set</u> $A = R/J$, <u>and</u> $\underline{m} = (t_1,\dots,t_n)/J$. <u>Set</u> $\mathcal{D} = \mathcal{D}(R)$ <u>and let</u> $\sigma : \mathcal{D} \to \mathcal{D}$ <u>be the above anti-automorphism. Consider</u> R <u>as a left</u> $\mathcal{D}$<u>-module and define</u> $S = \{f \in R \mid D(f) = 0$ <u>for all</u> $D \in \sigma(J\mathcal{D})\}$. <u>Then</u> S <u>is a left</u> $\mathbb{I}(\sigma(J\mathcal{D}))$<u>-submodule annihilated by</u> $\sigma(J\mathcal{D})$. <u>There is an anti-automorphism</u>

$$\Phi : \mathbb{I}(\sigma(J\mathcal{D}))/\sigma(J\mathcal{D}) \longrightarrow \mathbb{I}(J\mathcal{D})/J\mathcal{D} \simeq \mathcal{D}(A)$$

<u>and thus</u> S <u>may be given the structure of a right</u> $\mathcal{D}(A)$<u>-module. As a right</u> $\mathcal{D}(A)$<u>-module,</u> S <u>is isomorphic to</u> $\mathcal{D}(A,A/\underline{m})$.

<u>Proof</u>. It is straightforward computation to see that S is an $\mathbb{I}(\sigma(J\mathcal{D}))$-submodule of R, annihilated by $\sigma(J\mathcal{D})$. The anti-isomorphism Φ is of course induced by σ, and the fact that $\mathcal{D}(A) \simeq \mathbb{I}(J\mathcal{D})/J\mathcal{D}$ is just [6, § 1.6]. Thus it remains to prove the final asertion.

Apply the left exact functor $\mathcal{D}_R(-, R/(t_1,\dots,t_n))$ to the short exact sequence of R-modules $0 \to J \to R \to A \to 0$ to obtain the exact sequence

$$0 \longrightarrow \mathcal{D}_R(A,R/(t_1,\dots,t_n)) \longrightarrow \mathcal{D}_R(R,R/(t_1,\dots,t_n)) \longrightarrow \mathcal{D}_R(J,R/(t_1,\dots,t_n)).$$

Thus $\mathcal{D}_A(A,A/\underline{m}) = \mathcal{D}_R(A,R/(t_1,\dots,t_n)) = \{Q \in \mathcal{D}/t_1\mathcal{D}+\dots+t_n\mathcal{D} \mid Q \star J = 0\}$

$$= \{Q \in \mathcal{D} \mid QJ \subseteq t_1\mathcal{D}+\dots+t_n\mathcal{D}\}/t_1\mathcal{D}+\dots+t_n\mathcal{D} .$$

Now consider $S \subseteq R \simeq \mathcal{D}/\mathcal{D}\partial_1 + \dots + \mathcal{D}\partial_n$. By definition

$$S = \{P \in \mathcal{D} \mid \sigma(J\mathcal{D})P \subseteq \mathcal{D}\partial_1 + \dots + \mathcal{D}\partial_n\}/\mathcal{D}\partial_1+\dots+\mathcal{D}\partial_n .$$

Through the anti-isomorphism Φ , S is made into a right $\mathcal{D}(A)$-module. Let $\Psi : S \to \mathcal{D}_A(A,A/\underline{m})$ be defined by

$$\Psi([P + \mathcal{D}\partial_1+\dots+\mathcal{D}\partial_n]) = [\sigma(P) + t_1\mathcal{D}+\dots+t_n\mathcal{D}] .$$

It is clear that Ψ is a vector space isomorphism. To see that Ψ is a right $\mathcal{D}(A)$-module map , let $s \in S$, and $d \in \mathcal{D}(A)$. Suppose that $s = [P + \mathcal{D}\partial_1+\dots+\mathcal{D}\partial_n]$, and $d = [\sigma(e) + J\mathcal{D}]$ for $e \in \mathbb{I}(\sigma(J\mathcal{D}))$. Then

$s.d = [P + \mathcal{D}\partial_1+\dots+\mathcal{D}\partial_n].[\sigma(e) + J\mathcal{D}] = [eP + \mathcal{D}\partial_1+\dots+\mathcal{D}\partial_n]$ and $\Psi(s.d) =$

$[\sigma(P)\sigma(e) + t_1\mathcal{D}+\dots+t_n\mathcal{D}] = [\sigma(P) + t_1\mathcal{D}+\dots+t_n\mathcal{D}].[\sigma(e) + J\mathcal{D}] = \Psi(s).d.$

Thus $S \simeq \mathcal{D}_A(A,A/\underline{m})$ as required . □

Remark. It is easier to consider S as a left $\mathcal{D}(A)^{op}$-module, and this is what we shall do in practice. That is , $\mathcal{D}(A)^{op}$ will be identified with $\mathbb{I}(\sigma(J\mathcal{D}))/\sigma(J\mathcal{D})$ and the action of this ring on S will be obtained through the restriction of the usual action of differential operators on $R = \mathbb{C}[x,y]$.

PROPOSITION 2. Let C be an irreducible affine curve, such that $\pi : \tilde{C} \to C$ is injective. Let $\underline{m}$ be a maximal ideal of $A = \mathcal{O}(C)$. Then $\mathcal{D}(A,A/\underline{m})$ is a simple right $\mathcal{D}(A)$-module.

Proof. Let $\bar{A}$ denote the integral closure of A in Fract A . By Theorem 2.3 , $\mathcal{D}(A)$ and $\mathcal{D}(\bar{A})$ are Morita equivalent. The progenerators giving the Morita equivalence are $\mathcal{D}(\bar{A},A)$ and $\mathcal{D}(A,\bar{A})$. Consider the following natural maps obtained by taking composition of differential operators :

$$\mathcal{D}_A(\bar{A},A/\underline{m}) \otimes_{\mathcal{D}(\bar{A})} \mathcal{D}_A(A,\bar{A}) \otimes_{\mathcal{D}(A)} \mathcal{D}_A(\bar{A},A) \to \mathcal{D}(A,A/\underline{m}) \otimes_{\mathcal{D}(A)} \mathcal{D}_A(\bar{A},A) \to \mathcal{D}_A(\bar{A},A/\underline{m}) .$$

Since $\mathcal{D}(A,\bar{A})\otimes_{\mathcal{D}(A)}\mathcal{D}(\bar{A},A) \to \mathcal{D}(\bar{A})$ given by composition is an isomorphism, the above map is also an isomorphism. In particular, $\mathcal{D}(A,A/\underline{m})$ corresponds to $\mathcal{D}(\bar{A},A/\underline{m})$ under the Morita equivalence. Hence to prove the result, it is enough to show that $\mathcal{D}_A(\bar{A},A/\underline{m})$ is a simple right $\mathcal{D}(\bar{A})$-module.

However, by Proposition 4 below, $\mathcal{D}_A(\bar{A},A/\underline{m}) \simeq \mathcal{D}_{\bar{A}}(\bar{A},\bar{A}/\underline{m}')$ where $\underline{m}'$ is the unique maximal ideal of $\bar{A}$ containing $\underline{m}$. By [6, § 1.3e], $\mathcal{D}(\bar{A},\bar{A}/\underline{m}') \simeq \mathcal{D}(\bar{A})/\underline{m}'\mathcal{D}(\bar{A})$, and this is a simple right $\mathcal{D}(\bar{A})$-module [6, §1.4g]. Hence the result. □

The next two results are required to complete the proof of Proposition 4.2.

LEMMA 3. Let $A = \mathcal{O}(X)$ be the co-ordinate ring of an affine irreducible variety X. Let M and N be A-modules, and $\underline{m}$ a maximal ideal of A. If $\underline{m}N = 0$, then for all n

$$\mathcal{D}_A^{\,n}(M,N) = \{\theta \in \mathrm{Hom}_k(M,N) \mid \theta(\underline{m}^{n+1}M) = 0\}.$$

Proof. Write $J = \ker(\mu : A\otimes_k A \to A)$ where μ is the multiplication map. As $A = k \oplus \underline{m}$, A is generated as a k-algebra by elements of $\underline{m}$. Hence J is generated as an ideal by $\{1\otimes a - a\otimes 1 \mid a\in \underline{m}\}$. In particular, $J \subseteq A\otimes \underline{m} + \underline{m}\otimes A$, and also $A\otimes \underline{m} \subseteq \underline{m}\otimes A + J$. Thus $J^n \subseteq A\otimes \underline{m}^n + \underline{m}\otimes A$ and $A\otimes \underline{m}^n \subseteq \underline{m}\otimes A + J^n$.

As $\underline{m}N = 0$, if $\theta\in\mathrm{Hom}_k(M,N)$ then $(\underline{m}\otimes A).\theta = 0$. Thus $J^n.\theta(M) \subseteq (A\otimes\underline{m}^n).\theta(M) = A\theta(\underline{m}^nM)$, and also $A\theta(\underline{m}^nM) \subseteq J^n.\theta(M)$. Thus $\theta(\underline{m}^{n+1}M) = 0$ if and only if $J^{n+1}.\theta = 0$, which is precisely the condition that $\theta\in\mathcal{D}_A^{\,n}(M,N)$. □

PROPOSITION 4. Let $A = \mathcal{O}(C)$ be the co-ordinate ring of an affine irreducible curve C, and set $B = \mathcal{O}(\tilde{C})$. Let $\underline{m}$ be a maximal ideal of A, and $\{\underline{m}_\lambda \mid \lambda\in\Lambda\}$ the maximal ideals of B containing $\underline{m}$. Then there is an isomorphism of right $\mathcal{D}(B)$-modules

$$\Phi : \oplus_\lambda\ \mathcal{D}_B(B,B/\underline{m}_\lambda) \to \mathcal{D}_A(B,A/\underline{m}).$$

Proof. For each λ, fix an A-module isomorphism $\varphi_\lambda : B/\underline{m}_\lambda \to A/\underline{m}$. If, for each λ, $\theta_\lambda\in\mathcal{D}_B(B,B/\underline{m}_\lambda)$ then write $\Sigma_\lambda\,\theta_\lambda$ for the element in the direct sum. Define $\Phi(\Sigma_\lambda\,\theta_\lambda) = \Sigma_\lambda\varphi_\lambda\theta_\lambda$.

First Φ is a map to $\mathcal{D}_A(B,A/\underline{m})$ because each $\theta_\lambda \in \mathcal{D}_B(B,B/\underline{m}_\lambda) \subseteq \mathcal{D}_A(B,B/\underline{m}_\lambda)$ whence $\varphi_\lambda\ \theta_\lambda \in \mathcal{D}_A(B,A/\underline{m})$. It is clear that Φ is a right $\mathcal{D}(B)$-module map. However, a word of warning is required : $\mathcal{D}(B)$ means $\mathcal{D}_B(B,B)$ not $\mathcal{D}_A(B,B)$, and one must observe that $\mathcal{D}_B(B,B) \subseteq \mathcal{D}_A(B,B)$ so $\mathcal{D}_A(B,A/\underline{m})$ really is a right $\mathcal{D}(B)$-module.

To see that Φ is injective, first observe that $\oplus_\lambda \mathcal{D}_B(B,B/\underline{m}_\lambda)$ is a direct sum of non-isomorphic simple right $\mathcal{D}(B)$-modules [6,Corollary 4.3 and § 1.4g] . Hence if $\ker \Phi \neq 0$ then some $\mathcal{D}_B(B,B/\underline{m}_\lambda)$ is contained in $\ker \Phi$. But if $\theta_\lambda \in \ker \Phi$ then $\varphi_\lambda\ \theta_\lambda = 0$, which implies $\theta_\lambda = 0$ since φ_λ is an isomorphism. Hence $\ker \Phi = 0$.

It remains to show that Φ is surjective. Choose $\theta \in \mathcal{D}_A(B,A/\underline{m})$. By Lemma 4.3 this forces $\theta(\underline{m}^n B) = 0$ for some $n > 0$. But for $r >> 0$, $(\prod_\lambda \underline{m}_\lambda)^r \subseteq \underline{m}^n B$. But B is a Dedekind domain so $(\prod_\lambda \underline{m}_\lambda)^r = \cap_\lambda \underline{m}^r{}_\lambda$ Thus $\theta(\cap_\lambda \underline{m}^r{}_\lambda) = 0$ for some $r >> 0$. Denote by $\bar{\theta}$ the map induced by θ , $\bar{\theta} : B/\cap_\lambda \underline{m}^r{}_\lambda \to A/\underline{m}$. However, $B/\cap_\lambda \underline{m}^r{}_\lambda \simeq \oplus_\lambda B/\underline{m}^r{}_\lambda$ and hence there are maps $\bar{\theta}_\lambda : B/\underline{m}^r{}_\lambda \to A/\underline{m}$ for each λ . Now define $\theta_\lambda : B \to A/\underline{m}$ by $\theta_\lambda(b) = \bar{\theta}_\lambda([b + \underline{m}^r{}_\lambda])$. Finally, consider $\varphi_\lambda{}^{-1}\theta_\lambda : B \to B/\underline{m}_\lambda$. By construction , $\varphi_\lambda{}^{-1}\ \theta_\lambda \in \mathcal{D}_B(B,B/\underline{m})$. It is clear that $\Phi(\Sigma_\lambda \varphi_\lambda{}^{-1}\theta_\lambda) = \Sigma_\lambda \theta_\lambda = \theta$. So Φ is surjective. □

This completes the proof of Proposition 4.2. The module $\mathcal{D}_A(A,A/\underline{m})$ seems to play a rather special role (when $\underline{m}$ is the maximal ideal corresponding to a singular point on the curve). For example, it plays a key role in the results in [8] . Also the following is an interesting consequence of Lemma 3.

COROLLARY 5. <u>Let</u> $A = \mathcal{O}(X)$ <u>be the co-ordinate ring of an irreducible affine variety</u> X . <u>Let</u> $\underline{m}$ <u>be a maximal ideal of</u> A . <u>Then as a right</u> A-<u>module</u> $\mathcal{D}_A(A,A/\underline{m}) \simeq E_A(A/\underline{m})$, <u>the injective hull of</u> $A/\underline{m}$.

<u>Proof</u>. Lemma 3 shows that

$$\mathcal{D}_A(A,A/\underline{m}) = \{\theta \in \mathrm{Hom}_k(A,A/\underline{m}) \mid \theta(\underline{m}^n) = 0 \quad \text{for} \quad n >> 0\} .$$

That this is now the injective envelope of $A/\underline{m} \simeq \mathrm{Hom}_A(A,A/\underline{m}) = \mathcal{D}_A^0(A,A/\underline{m})$ follows from [Bourbaki, Algèbre Homologique, § 1, Ex. 29-32]. That an earlier proof of this result could be replaced by this reference was pointed out in [8]. □

We now return to the examples at the beginning of this section. In fact we will first discuss the example where $D = \partial_y^2 - \partial_x^3$ (the other example is somewhat simpler since the corresponding curve is non-singular, and we shall comment on that in the remarks at the end of this section) .

Set $P = 4x^2\partial_x + 12xy\partial_y + 9y^2\partial_x^3 - 2x$ and $Q = 2x\partial_y + 3y\partial_x^2$.

PROPOSITION 3. Set $D = \partial_y^2 - \partial_x^3$ and $S = \{f \in \mathbb{C}[x,y] \mid D(f) = 0\}$. Then $S = C[P].1 + Q\mathbb{C}[P].1$.

Proof. Set $g = y^2 - x^3$, so $g = \sigma(D)$, and let $A = C[x,y]/g\mathbb{C}[x,y]$. By Proposition 1 , the structure of S as a left $\mathbb{I}(\mathcal{D}D)/\mathcal{D}D$-module transfers to make S a right $\mathcal{D}(A)$-module isomorphic to $\mathcal{D}(A,A/\underline{m})$ where $\underline{m} = Ax + Ay$. A careful analysis of the proof of Proposition 1 shows that $1 \in S$ corresponds to the natural algebra map $\varepsilon : A \to A/\underline{m}$ which is an element of $\mathcal{D}(A,A/\underline{m})$.

Set $\hat{P} = t^{-1}(t\partial - 2)\partial$, $\hat{Q} = t^2\partial$. We view $\hat{P},\hat{Q}$ as elements of $\mathcal{D}(A)$ with $A = \mathbb{C}[t^2,t^3] \subseteq \mathbb{C}[t]$. Since $\mathcal{D}(A) \simeq \mathbb{I}(g\mathcal{D})/g\mathcal{D}$ we can find elements $P',Q' \in \mathbb{I}(g\mathcal{D})$ which map to $\hat{P}$ and $\hat{Q}$ respectively. Such elements are

$$P' = 4x\,\partial_x^2 + 12y\partial_x\partial_y + 9x^2\partial_y^2 - 2\partial_x$$
$$Q' = 2y\partial_x + 3x^2\partial_y .$$

Notice that $P = \sigma(P')$, $Q = \sigma(Q')$.

Hence to prove the Proposition it is sufficient to show $\mathcal{D}(A,A/\underline{m}) = \varepsilon.\mathbb{C}[\hat{P}] + \varepsilon.\mathbb{C}[\hat{P}]\hat{Q}$. Recall that

$$\mathcal{D}(A,A/\underline{m}) = \cup_{n=0}^{\infty} \mathcal{D}^n{}_A(A,A/\underline{m}) = \cup_{n=0}^{\infty}\{\theta\in \mathrm{Hom}_{\mathbb{C}}(A,A/\underline{m}) \mid \theta(\underline{m}^{n+1}) = 0\} .$$

We identify $\mathcal{D}^n{}_A(A,A/\underline{m})$ with $\mathrm{Hom}_{\mathbb{C}}(A/\underline{m}^{n+1},A/\underline{m})$. Set $\mathcal{B} = \{t^j \mid 0 \leqslant j \leqslant 2n+1,\ j \neq 1\}$. This is a basis for $A/\underline{m}^{n+1}$. Set $\mathcal{B}' = \{\varepsilon\hat{P}^j \mid 0 \le j \le n\} \cup \{\varepsilon\hat{P}^j\hat{Q} \mid 2 \le j \le n+1\}$. Check that (up to a non-zero scalar multiple) $\varepsilon\hat{P}^k(t^j) = \delta_{2k,j}$ and $\varepsilon\hat{P}^k\hat{Q}(t^j) = \delta_{2k-1,j}$. Hence $\mathcal{B}' \subseteq \mathrm{Hom}_{\mathbb{C}}(A/\underline{m}^{n+1},A/\underline{m})$ is (up to non-zero scalar multiples) the dual basis to $\mathcal{B}$. In particular, it follows that

$$\mathcal{D}(A,A/\underline{m}) = \varepsilon.\mathbb{C}[\hat{P}] + \varepsilon.\mathbb{C}[\hat{P}]\hat{Q} .$$ □

Remarks (1). Proposition 6 allows one to routinely produce a basis for S ; in fact the proof essentially shows that $\{P^j(1) \mid j \geq 0\} \cup \{QP^j(1) \mid j \geq 2\}$ gives a basis for S ; this verifies the claims made at the start of this section.

(2). The elements P' and Q' of the proof are obtained as follows. We have in $A = \mathbb{C}[t^2,t^3]$ that $y = t^3$, $x = t^2$. Now $t\partial \in \mathcal{D}(A)$ satisfies $(t\partial)(y) = 3y$, $(t\partial)(x) = 2x$. The derivation on $\mathbb{C}[x,y]$ that has this effect is precisely $3y\partial_y + 2x\partial_x$. So $t\partial \in \mathcal{D}(A)$ "lifts" to $3y\partial_y + 2x\partial_x \in \mathcal{D}$. Since $t = yx^{-1}$, $t^2\partial$ "lifts" to $yx^{-1}(3y\partial_y + 2x\partial_x) = 3x^2\partial_y + 2y\partial_x$ (using the fact that in A , $y^2 = x^3$), this gives Q' . To obtain P' , re-write $P = t^{-2}(t\partial - 3)(t\partial)$, and this "lifts" to $x^{-1}(3y\partial_y + 2x\partial_x - 3)(3y\partial_y + 2x\partial_x)$. Expanding this, using the fact that $y^2 = x^3$ in A , gives P' .

(3). The other example was to describe S , the space of polynomial solutions to $D = \partial_y{}^2 - \partial_x{}^3 + \partial_x$. Here $g = \sigma(D) = y^2-x^3+x$ defines a non-singular curve $C \subseteq \mathbb{A}^2$. Thus $\mathcal{D}(C)$ is generated by $\mathcal{O}(C)$ and $\operatorname{Der} C$. It is easy to compute that $\operatorname{Der} C$ is free on δ defined by $\delta(x) = 2y$, $\delta(y) = 3x^2-1$. Lifting δ back to $\operatorname{Der} \mathbb{C}[x,y]$ we have $\delta = 2y\partial_x + (3x^2-1)\partial_y$. Applying σ , we have $P = \sigma(\delta) = 2x\partial_y + 3y\partial_x{}^2 - y$. Since $\sigma(x) = \partial_x$, $\sigma(y) = \partial_y$ the earlier analysis shows that $S = \mathbb{C}[\partial_x,\partial_y,P].1$. In fact the analogue of Proposition 6 gives $S = \mathbb{C}[P].1$. The action of P on 1 is as follows :

$$1 \xrightarrow{P} y \xrightarrow{P} 2x-y^2 \xrightarrow{P} -6xy+y^3 \xrightarrow{P} -12x^2+y^4 \xrightarrow{P} 60x^2y-20xy^3-48y+y^5 \xrightarrow{P} \cdots\cdots$$

and one continues applying P to obtain a basis for S .

§ 5. ON THE $\mathcal{D}(\mathbb{A}^2)$-MODULE $\mathcal{O}(\mathbb{A}^2)_f$.

Let $R = \mathbb{C}[x,y] = \mathcal{O}(\mathbb{A}^2)$, and let $0 \neq f \in R$ be an irreducible polynomial defining a curve $C \subsetneq \mathbb{A}^2$. By a celebrated theorem of Bernstein [1] , $R_f = \mathcal{O}(\mathbb{A}^2 \setminus C)$ is a $\mathcal{D}(\mathbb{A}^2)$-module of finite length. It is not difficult to show that R_f/R contains a unique simple $\mathcal{D}(\mathbb{A}^2)$-module (we give the proof below) . The problem we consider here is that of determining this simple submodule. We will show that if C is a non-singular curve then R_f/R is a simple $\mathcal{D}(\mathbb{A}^2)$-module. This is not difficult and is well known. It will be clear from the proof that a new idea is required to cope with the case when C is singular. The reason is that the proof relies on the fact that, if C is non-singular, then the ideal of R generated by f, $\partial f/\partial x$, $\partial f/\partial y$ equals R itself. The main result in § 5 is to show that if $\pi : \tilde{C} \to C$ is injective, then R_f/R is a simple $\mathcal{D}(\mathbb{A}^2)$- module. The details of the proof will appear elsewhere [7] and we only give a rough outline.

The reason for the interest in determining the simple submodule of R_f/R is as follows. Let X be a non-singular variety and $Y \subset X$ a closed irreducible subvariety (possibly singular) of codimension 1 in X, defined by $0 \neq f \in \mathcal{O}(X)$. Then $\mathcal{O}(X \setminus Y)/\mathcal{O}(X) = \mathcal{O}(X)_f/\mathcal{O}(X)$ has a unique simple $\mathcal{D}(X)$-submodule, which we denote by $\mathcal{L}(Y,X)$. Under the equivalence of categories between regular holonomic $\mathcal{D}_X$-modules, and the category of perverse sheaves on X, $\mathcal{L}(Y,X)$ (which is regular holonomic) corresponds to $IC.(Y)$ the intersection homology complex associated to $Y \subset X$.

The main result in this section, namely Theorem 4, can be proved in a quite different (and less algebraic way) through using the Riemann-Hilbert correspondence. I would like to thank J.-L.Brylinski for showing me how to do this.

PROPOSITION 1. The $\mathcal{D}(\mathbb{A}^2)$-module $M = \mathcal{O}(\mathbb{A}^2)_f/\mathcal{O}(\mathbb{A}^2)$ has a unique simple submodule, for any $0 \neq f \in \mathcal{O}(\mathbb{A}^2)$.

Proof. Observe that if $N_1, N_2 \subseteq M$ are non-zero $\mathcal{O}(\mathbb{A}^2)$-submodules then $N_1 \cap N_2 \neq 0$. It follows that the same is true of any two non-zero $\mathcal{D}(\mathbb{A}^2)$-submodules. Because M is of finite length as a $\mathcal{D}(\mathbb{A}^2)$-module it contains some simple submodule S, say. By the first observation, S must be contained in every non-zero $\mathcal{D}(\mathbb{A}^2)$-submodule of M. Hence the conclusion. □

We will next show that when C, the curve defined by an irreducible $f \in \mathcal{O}(\mathbb{A}^2)$, is non-singular, the module $\mathcal{O}(\mathbb{A}^2)_f/\mathcal{O}(\mathbb{A}^2)$ is simple (this is certainly well known, but we cannot find a proof to refer the reader to). To do this, first observe that $f^{-1}R/R \subseteq R$ is an $\mathbb{I}(\mathcal{D}f)$-submodule, is annihilated by $\mathcal{D}f$, and is therefore an $\mathbb{I}(\mathcal{D}f)/\mathcal{D}f$-module. However, there is an isomorphism of k-algebras $\mathbb{I}(\mathcal{D}f)/\mathcal{D}f \simeq \mathbb{I}(f\mathcal{D})/f\mathcal{D}$; this isomorphism is obtained from $\psi : \mathbb{I}(\mathcal{D}f) \to \mathbb{I}(f\mathcal{D})$ given by $\psi(D) = D'$ where $D \in \mathcal{D}$ is the unique element satisfying $fD = D'f$ for $D \in \mathbb{I}(\mathcal{D}f)$. Thus, as $\mathbb{I}(f\mathcal{D})/f\mathcal{D} \simeq \mathcal{D}(C)$ by [6, § 1.6], it follows that $f^{-1}R/R$ is a left $\mathcal{D}(C)$-module. The point is

PROPOSITION 2. As a left $\mathcal{D}(C)$-module $f^{-1}R/R$ is isomorphic to $\mathcal{O}(C) = R/fR$ with its natural $\mathcal{D}(C)$-module structure.

Proof. Easy. □

THEOREM 3. Let $0 \neq f \in \mathcal{O}(\mathbb{A}^2)$ be an irreducible polynomial defining a curve C. If C is non-singular then $\mathcal{O}(\mathbb{A}^2)_f / \mathcal{O}(\mathbb{A}^2)$ is a simple $\mathcal{D}(A^2)$-module.

Proof. First we show that $M = \mathcal{O}(A^2)_f / \mathcal{O}(A^2)$ is generated by f^{-1}. Clearly $\mathcal{D}(A^2).f^{-1}$ contains $\partial_x(f^{-1}) = -f_x f^{-2}$, $\partial_y(f^{-1}) = -f_y f^{-2}$ and $f^{-1} = ff^{-2}$. Since C is non-singular, $1 \in \mathcal{O}(A^2)f_x + \mathcal{O}(A^2)f_y + \mathcal{O}(A^2)f$. Thus we obtain $f^{-2} \in \mathcal{D}(A^2).f^{-1}$. An induction argument applying ∂_x and ∂_y to f^{-n} for each $n > 0$ completes the proof of the fact that $\mathcal{D}(A^2).f^{-1} = M$.

Now to see that M is simple, we need only show that every non-zero submodule of M contains f^{-1}. Pick $0 \neq m \in M$, and consider $\mathcal{D}(A^2).m$. Clearly this contains an element of the form af^{-1} with $a \in \mathcal{O}(A^2) \setminus \mathcal{O}(A^2)_f$. Thus $0 \neq af^{-1} \in f^{-1}\mathcal{O}(A^2)/\mathcal{O}(A^2)$. Consider $f^{-1}\mathcal{O}(A^2)/\mathcal{O}(A^2)$ as a left $\mathcal{D}(C)$-module. As such it is isomorphic to $\mathcal{O}(C)$. However, $\mathcal{O}(C)$ is a simple $\mathcal{D}(C)$-module because C is non-singular. Therefore $f^{-1} \in \mathcal{D}(A^2).af^{-1}$. □

Remark. (1) The above proof gives a very explicit argument as to why f^{-1} generates $\mathcal{O}(A^2)_f / \mathcal{O}(A^2)$. Later we shall show that $\mathcal{O}(\mathbb{A}^2)_f / \mathcal{O}(A^2)$ is a simple $\mathcal{D}(\mathbb{A}^2)$-module whenever $\pi : \tilde{C} \to C$ is injective. Hence in that case also f^{-1} generates $\mathcal{O}(A^2)_f / \mathcal{O}(A^2)$. However, our proof will not explain in such an explicit manner, why $f^{-n} \in \mathcal{D}(A^2).f^{-1}$. Hence it is an interesting question (interesting for this author, anyway) to find in some explicit cases (for example, $f = y^2 - x^3$) operators D_n such $D_n.f^{-1} = f^{-n}$ in $\mathcal{O}(A^2)_f / \mathcal{O}(A^2)$.

(2) It is clear that all the above arguments work in greater generality. That is, if X is a non-singular variety and $0 \neq f \in \mathcal{O}(X)$ an irreducible polynomial defining a hypersurface $Y \subset X$, then similar considerations (to the above) apply to $\mathcal{O}(X)_f / \mathcal{O}(X)$ as a $\mathcal{D}(X)$-module.

THEOREM 4. Let $0 \neq f \in \mathcal{O}(\mathbb{A}^2)$ be an irreducible polynomial defining a curve C. Suppose that $\pi : \tilde{C} \to C$ is injective. Then $\mathcal{O}(\mathbb{A}^2)_f / \mathcal{O}(\mathbb{A}^2)$ is a simple $\mathcal{D}(\mathbb{A}^2)$-module.

Sketch of Proof. The goal is to show that each $f^{-n}R/R$, $R = \mathcal{O}(\mathbb{A}^2)$, is a simple left $\mathbb{I}(\mathcal{D}f^n)$-module, where $\mathcal{D} = \mathcal{D}(\mathbb{A}^2)$. It will then follow at once that R_f/R is a simple left $\mathcal{D}$-module.

It is clear that for $n \in \mathbb{N}$, $f^{-n}R/R$ is a left $\mathbb{I}(\mathcal{D}f^n)/\mathcal{D}f^n$-module. However, $\mathbb{I}(\mathcal{D}f^n)/\mathcal{D}f^n \simeq \mathcal{D}(R/f^nR)$, the ring of differential operators on R/f^nR, and it is easy to see that as a left $\mathcal{D}(R/f^nR)$-module, $f^{-n}R/R$ is isomorphic to R/f^nR. Hence the aim is to show that R/f^nR is a simple $\mathcal{D}(D/f^nR)$-module for all $n \in \mathbb{N}$. The case $n = 1$ is precisely Theorem 2.3 above. For $n > 1$ we must extend the results in [6]. This is done in [7], and here we just sketch the main steps of the argument.

There is an inclusion of algebras

$$R/f^nR \subseteq R/fR \otimes_k k[z]/(z^n) = \mathcal{O}(C) \otimes_k k[z]/(z^n) \subseteq \mathcal{O}(\tilde{C}) \otimes_k k[z]/(z^n) \subseteq \mathrm{Fract}(R/f^nR),$$

such that R/f^nR is of finite codimension in $\mathcal{O}(\tilde{C}) \otimes_k k[z]/(z^n)$, and the induced map on the spectra is bijective. One observes that

$$\mathcal{D}(\mathcal{O}(\tilde{C}) \otimes_k k[z]/(z^n)) \simeq \mathcal{D}(\tilde{C}) \otimes_k \mathcal{D}(k[z]/(z^n)) \simeq \mathcal{D}(\tilde{C}) \otimes_k M_n(k) ,$$

and this latter algebra is Morita equivalent to $\mathcal{D}(\tilde{C})$. One therefore can apply the same ideas as in [6, §§2,3] to show that, if

$$(\dagger) \qquad \mathcal{D}(\mathcal{O}(\tilde{C}) \otimes_k k[z]/(z^n), R/f^nR) * (\mathcal{O}(\tilde{C}) \otimes_k k[z]/(z^n)) = R/f^nR$$

then $\mathcal{D}(R/f^nR)$ is Morita equivalent to $\mathcal{D}(\mathcal{O}(\tilde{C}) \otimes k[z]/(z^n))$. Because of the bijectivity of the map on the spectra, $(\dagger)$ can be established by imitating the proof of [6, Theorem 3.4]. Then, from the Morita equivalence it follows that $\mathcal{D}(R/f^nR)$ is a simple ring, and hence R/f^nR is a simple $\mathcal{D}(R/f^nR)$-module. □

Theorem 4 has been obtained independently by van Doorn and van den Essen [8].

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