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An equivalence of categories for graded modules over monomial algebras and path algebras of quivers

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A R T I C L E I N F O

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A B S T R A C T

Let A be a finitely generated connected graded k-algebra defined by a finite number of monomial relations, or, more generally, the path algebra of a finite quiver modulo a finite number of relations of the form "path = 0". Then there is a finite directed graph, Q, the Ufnarovskii graph of A, for which there is an equivalence of categories $\mathsf{QGr}\,A \equiv \mathsf{QGr}(kQ)$. Here $\mathsf{QGr}\,A$ is the quotient category $\mathsf{Gr}\,A/\mathsf{Fdim}$ of graded A-modules modulo the subcategory consisting of those that are the sum of their finite dimensional submodules. The proof makes use of an algebra homomorphism $A \to kQ$ that may be of independent interest.

1. Introduction

1.1. Throughout k is a field.

Let A be an $\mathbb{N}$-graded k-algebra.

The category of $\mathbb{Z}$-graded *right* A-modules with degree-preserving homomorphisms is denoted by $\mathsf{Gr}\,A$ and $\mathsf{Fdim}\,A$ is its full subcategory consisting of modules that are the sum of their finite dimensional submodules. Since $\mathsf{Fdim}\,A$ is a Serre subcategory of $\mathsf{Gr}\,A$ (it is, in fact, a localizing subcategory) we may form the quotient category

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$$\mathsf{QGr}\, A := \frac{\mathsf{Gr}\, A}{\mathsf{Fdim}\, A}.$$

We are interested in the structure of $\mathsf{QGr}\, A$ for monomial algebras.

1.2. A connected graded monomial algebra is a free algebra modulo an ideal generated by words in the letters generating the free algebra. More explicitly, if $w_1, \dots, w_r$ are words in the letters $x_1, \dots, x_g$, then

$$A = \frac{k\langle x_1, \dots, x_g\rangle}{(w_1, \dots, w_r)} \tag{1.1}$$

is a finitely presented monomial algebra.

Our main result applies to a more general class of monomial algebras, namely those of the form kQ'/I where Q' is a finite quiver (Section 2.1) and I an ideal generated by a finite set of paths in Q'. Such algebras can be described without mentioning quivers: let K be a finite product of copies of k, $T_K V$ the tensor algebra of a K-bimodule V that has a finite k-basis $a_1, \dots, a_g$, and

$$A = \frac{T_K V}{(p_1, \dots, p_r)} \tag{1.2}$$

where each p_j is a word in the a_i's.

1.3. *The main result*

Theorem 1.1. *Let A be a monomial algebra of the form* (1.2)*. There is a quiver Q and an equivalence of categories*

$$\mathsf{QGr}\, A \equiv \mathsf{QGr}\, kQ\,.$$

The structure and properties of $\mathsf{QGr}\, kQ$ are described in [5].

The proof of Theorem 1.1 uses result of Artin and Zhang, Proposition 2.1 below, in an essential way.

When A is of the form (1.1) we can take Q to be its Ufnarovskii graph (Section 3) and there is then a homomorphism $f : A \to kQ$ such that the functor $-\bigotimes_A kQ$ induces the equivalence in Theorem 1.1. This is proved in Section 4.1; see Theorem 4.2 for a precise statement.

In Section 4.2, Theorem 1.1 is proved for algebras of the form (1.2): if A is of the form (1.2) its subalgebra generated by k and A_1 is of the form (1.1) and has finite codimension in A so, by Artin and Zhang's result and Theorem 1.1 for algebras of the form (1.1), Theorem 1.1 holds for algebras of the form (1.2).

1.4. *Quadratic monomial algebras*

If A is monomial algebra of the form (1.1) with $\deg w_i = 2$ for all i we call A a *quadratic* monomial algebra. The proof of Theorem 1.1 for quadratic monomial algebras is much simpler than the general case. We give that proof in Section 6.1.

Let A be an arbitrary finitely presented connected graded monomial algebra. By Backelin and Fröberg [2], the Veronese subalgebra $A^{(n)} \subset A$ is quadratic for $n \gg 0$; by Verevkin [9], $\mathsf{QGr}\, A \equiv \mathsf{QGr}\, A^{(n)}$, so Theorem 1.1 holds for A if it holds for $A^{(n)}$. However, if Theorem 1.1 is proved for A by first proving it for $A^{(n)}$ the quiver Q is the Ufnarovskii graph for $A^{(n)}$ which is more complicated than that for A (see Section 6.3 for an example illustrating this).

That is why we prove Theorem 1.1 directly in Section 4.1, i.e., without passing to a Veronese subalgebra.

2. Preliminaries

2.1. Notation

The letter Q will always denote a directed graph, or quiver, with a finite number of vertices and arrows—loops and multiple arrows between vertices are allowed.

We write kQ for the path algebra of Q. The finite paths in Q, including the trivial paths at each vertex, form a basis for kQ and multiplication is given by concatenation of paths. If a is an arrow that ends where the arrow b begins we write

$$ab := \text{the path "}a\text{ followed by }b\text{"}.$$

We set $ab = 0$ if b does not begin where a ends. Likewise, if a path p ends where a path q begins, pq denotes the path *first traverse p then q*.

We make kQ an $\mathbb{N}$-graded algebra by declaring that a path is homogeneous of degree equal to its length.

2.2. Throughout, modules are *right* modules.

Proposition 2.1. *(See [1, Prop. 2.5].) Let $\phi : A \to B$ be a homomorphism of graded k-algebras. If* $\ker\phi$ *and* $\operatorname{coker}\phi$ *belong to* $\mathsf{Fdim}\, A$*, then $-\bigotimes_A B$ induces an equivalence of categories* $\mathsf{QGr}\, A \to \mathsf{QGr}\, B$.

Lemma 2.2. *Let A and B be $\mathbb{N}$-graded k-algebras generated by $A_0 + A_1$ and $B_0 + B_1$ respectively. Let $\phi : A \to B$ be a homomorphism of graded k-algebras. If $B_0\phi(A_m) \subset \phi(A_m)$ and $B_1\phi(A_m) \subset \phi(A_{m+1})$ for some $m \in \mathbb{N}$, then* $\operatorname{coker}\phi$ *belongs to* $\mathsf{Fdim}\, A$.

Proof. We can replace A by its image in B so we will do that; i.e., without loss of generality, A is a graded subalgebra of B and ϕ is the inclusion map.

If $n \geqslant 2$ and $B_{n-1}A_m \subset A_{m+n-1}$, then

$$B_nA_m = B_1B_{n-1}A_m \subset B_1A_{m+n-1} = B_1A_mA_{n-1} \subset A_{m+1}A_{n-1} = A_{m+n}.$$

It follows that $B_nA_m \subset A_{m+n}$ for all $n \geqslant 0$. Thus B/A is annihilated on the right by A_m and therefore belongs to $\mathsf{Fdim}\, A$. $\square$

3. The Ufnarovskii graph of a connected graded monomial algebra

Throughout this paper G is a fixed finite set of *letters* or *generators*, $\langle G\rangle$ is the free monoid generated by G, and $k\langle G\rangle$ is the free k-algebra generated by G. Elements of $\langle G\rangle$ are called words. Throughout, F denotes a fixed *finite* set of words and

$$A := \frac{k\langle G\rangle}{(F)} \tag{3.1}$$

is the quotient by the ideal (F) generated by F. Such A is called a monomial algebra.

There is no loss of generality in assuming that $G \cap F = \emptyset$. We will make that assumption.

We make A a graded algebra by placing G in degree one. Thus $A_1 = kG$.

3.1. Words

The words in F are said to be forbidden. A word is illegal if it belongs to (F) and legal otherwise. The set of legal words is denoted by L, and $L_r := L \cap G^r$ is the set of legal words of length r. The image of L_r in A is a basis for A_r; see, for example, [3, Lem. 2.2].

Throughout we use the notation

$$\begin{aligned} \ell + 1 &:= \text{the longest length of a forbidden word} \\ &= \max\{\ell+1 \mid F \cap G^{\ell+1} \neq \emptyset\}, \quad \text{and} \\ L_{\leqslant r} &:= \{\text{legal words of length} \leqslant r\}. \end{aligned}$$

3.2. Notation

The letters s, t, u, v, w, will always denote words.

If u and w are words we write

$$u \lhd w$$

if $w = uv$ for some word v.

The symbols x, y, and x_i, will always denote elements of G. The notation $x_i \lhd w$ therefore means that x_i is the first letter of w.

3.3. The Ufnarovskii graph

The Ufnarovskii graph of A is the directed graph Q, or $Q(A)$ if we need to specify A, defined as follows (see [3, Sect. 12.2], [7,8]).

The set of vertices of Q is

$$Q_0 = L_\ell.$$

The set of arrows of Q is in bijection with the set $L_{\ell+1}$ as follows,

$$Q_1 = \{a_w \mid w \in L_{\ell+1}\}.$$

If $w \in L_{\ell+1}$, then there are unique $s, t \in Q_0$ and unique $x, y \in G$ such that $w = sy = xt \in L$ and we declare that the arrow a_w corresponding to w goes from s to t.

Given $s, t \in Q_0$, there is at most one arrow from s to t.

Suppose $n > 0$. If $x_1 \dots x_{n+\ell}$ is a legal word of length $n + \ell$ there is a length-n path

$$x_1 \dots x_\ell \longrightarrow x_2 \dots x_{\ell+1} \longrightarrow \cdots \longrightarrow x_{n+1} \dots x_{n+\ell} \tag{3.2}$$

in Q. This provides a bijection between legal words of length $n + \ell$ and paths of length n (see the proof of [7, Thm. 3] and the remark at [3, p. 157]).

3.4. Labeling arrows and paths

We write a_w for the arrow corresponding to $w \in L_{\ell+1}$. The path in (3.2) is therefore $a_{x_1\ldots x_{\ell+1}} a_{x_2\ldots x_{\ell+2}} \cdots a_{x_n\ldots x_{n+\ell}}$.

Suppose there is an arrow $u \to v$. Then $uy = xv$ for unique x and y in G, and we attach the label x to the arrow $u \to v$. We denote this by $u \xrightarrow{x} v$. The following facts will be used often:

- The label attached to the arrow a_w is the first letter of w.
- The existence of an arrow $u \xrightarrow{x} v$ implies that $x \lhd u$ and $u \lhd xv$.

We extend the labeling to paths: the label attached to a concatenation of arrows is the concatenation of the labels attached to the arrows in the path—for example, the label attached to the path in (3.2) is $x_1 \ldots x_n$. In general, there will be different paths with the same label: for example, the labels on the Ufnarovskii graph for $A = k\langle x, y\rangle/(y^3)$ are

$$\begin{tikzcd} x^2 \arrow[loop left, "x"] \arrow[d, "x"'] & y^2 \arrow[d, "y"] \\ xy \arrow[ur, "x"] \arrow[r, bend left, "x"] & yx. \arrow[ul, "y"'] \arrow[l, bend left, "y"] \end{tikzcd} \tag{3.3}$$

The Ufnarovskii graph for $k\langle x, y, z\rangle/(z^2, zy)$ appears in Section 5.

The following observation is surely known to the experts.

Lemma 3.1. *Suppose there is a path with the label $x_1 \ldots x_r$, say*

$$v_0 \xrightarrow{x_1} v_1 \xrightarrow{x_2} \cdots \xrightarrow{x_r} v_r. \tag{3.4}$$

Let $v_r = x_{r+1} \ldots x_{r+\ell}$.

(1) *$v_{i-1} = x_i \ldots x_{i+\ell-1}$ for all $i = 1, \ldots, r+1$.*
(2) *$x_1 \ldots x_r v_r$ is a legal word.*
(3) *$x_1 \ldots x_r \notin (F)$.*

Proof. The hypothesis implies $v_{i-1} \lhd x_i v_i$ and $x_i \lhd v_{i-1}$ for all $i = 1, \ldots, r$. An induction argument, or simply noticing the pattern in the equalities

$$\begin{aligned} v_r &= x_{r+1} \ldots x_{r+\ell}, \\ v_{r-1} &= x_r \ldots x_{r+\ell-1}, \\ v_{r-2} &= x_{r-1} \ldots x_{r+\ell-2}, \quad \text{etc.} \end{aligned}$$

proves (1).

(2) To prove $x_1 \ldots x_r v_r$ is legal it suffices to show its subwords of length $\ell + 1$ are legal. Such a subword is of the form $x_i \ldots x_{i+\ell-1} x_{i+\ell}$ for some i in the range $1 \leqslant i \leqslant r$; this subword is equal to $v_{i-1} x_{i+\ell} = x_i v_i$ and is legal because there is an arrow $v_{i-1} \to v_i$.

(3) Since a subword of a legal word is legal, (3) follows from (2). $\square$

The contrapositive of part (3) of Lemma 3.1 is useful so we record it separately.

Lemma 3.2. *If $x_1 \dots x_r$ is an illegal word, then there are no paths labeled $x_1 \dots x_r$.*

The converse of Lemma 3.2 is false. For example, x is a legal word when $A = k\langle x, y\rangle/(xy, x^2)$ but the Ufnarovskii graph of A is

$$Q = \begin{tikzcd} y \arrow[loop left, "a_{yy}"] \arrow[r, "a_{yx}"] & x \end{tikzcd}$$

with labels

$$\begin{tikzcd} y \arrow[loop left, "y"] \arrow[r, "y"] & x. \end{tikzcd} \tag{3.5}$$

3.5. The homomorphism $k\langle G\rangle/(F) \to kQ$

Let $f : k\langle G\rangle \to kQ$ be the unique algebra homomorphism such that for all $x \in G$,

$$f(x) = \text{the sum of all arrows labeled } x$$

or 0 if there are no arrows labeled x.

Hence, if $x_1 \dots x_r \in G^r$,

$$f(x_1 \dots x_r) = \text{the sum of all paths labeled } x_1 \dots x_r \tag{3.6}$$

or 0 if there are no such paths. More formally,

$$f(x) = 0 \quad \text{if } xL_\ell \cap L_{\ell+1} = \emptyset, \quad \text{and}$$
$$f(x) = \sum_{\substack{w \in Q_1 \\ x \lhd w}} a_w \quad \text{if } xL_\ell \cap L_{\ell+1} \neq \emptyset.$$

Since $f(G) \subset Q_1$, f is a homomorphism of graded k-algebras.

Proposition 3.3. *The homomorphism $f : k\langle G\rangle \to kQ$ induces a homomorphism of graded algebras from A to kQ.*

Proof. Lemma 3.2 and (3.6) imply $f(w) = 0$ for all $w \in F$. $\square$

Lemma 3.4. *Let $x_1 \dots x_r \in G^r$. There is a path labeled $x_1 \dots x_r$ if and only if $x_1 \dots x_r L_\ell \cap L \neq \emptyset$.*

Proof. ($\Rightarrow$) Suppose there is a path

$$v_0 \xrightarrow{x_1} v_1 \xrightarrow{x_2} \cdots \xrightarrow{x_r} v_r.$$

Write $v_r = x_{r+1} \dots x_{r+\ell}$. Since $x_i v_i$ is legal for all $i = 1, \dots, r$ and $x_i v_i = x_i x_{i+1} \dots x_{i+\ell-1}$ all subwords of $x_1 \dots x_r v_r$ of length $\ell + 1$ are legal. It follows that $x_1 \dots x_r v_r$ is legal.

($\Leftarrow$) Suppose $x_1 \dots x_r L_\ell \cap L \neq \emptyset$. Let $v_r = x_{r+1} \dots x_{r+\ell}$ be a vertex such that $x_1 \dots x_r v_r$ is legal. For $i = 1, \dots, r$, define

$$v_{i-1} := x_i \dots x_{i+\ell-1}.$$

This is a legal word, of length ℓ, because it is a subword of the legal word $x_1 \dots x_r v_r$. Since $v_{i-1} \lhd x_i v_i$ there is an arrow $v_{i-1} \xrightarrow{x_i} v_i$. Concatenating these arrows produces a path labeled $x_1 \dots x_r$. $\square$

Lemma 3.5. *Let $x_1 \dots x_r$ be a legal word of length $r \geqslant \ell$. There is a path labeled $x_1 \dots x_r$ if and only if there is a path labeled $x_{r-\ell+1} \dots x_r$.*

Proof. The lemma is true for $r = \ell$ so suppose $r > \ell$.

($\Rightarrow$) This is obvious.

($\Leftarrow$) Suppose there is a path

$$v_{r-\ell} \xrightarrow{x_{r-\ell+1}} v_{r-\ell+1} \longrightarrow \cdots \longrightarrow v_{r-1} \xrightarrow{x_r} v_r.$$

Write $v_r = x_{r+1} \dots x_{r+\ell}$.

By Lemma 3.4, $x_1 \dots x_r$ is legal if $x_1 \dots x_r v_r$ is. The word $x_1 \dots x_r v_r$ is legal if all its subwords of length $\ell + 1$ are legal. The proof of Lemma 3.4 showed that $x_{r-\ell+1} \dots x_r v_r$ is legal. All subwords of $x_{r-\ell+1} \dots x_r x_{r+1} \dots x_{r+\ell}$ are therefore legal so it only remains to show that $x_i \dots x_{i+\ell}$ is legal for all $i \leqslant r - \ell$. If $i \leqslant r - \ell$, then $x_i \dots x_{i+\ell}$ is a subword of $x_1 \dots x_r$ and therefore legal. $\square$

3.6. *The kernel of f*

The homomorphism f need not be injective: for example, by looking at the labels on the quiver (3.5) above one sees that $f(x) = 0$ when $A = k\langle x, y\rangle/(xy, x^2)$.

Lemma 3.6. *Let $w_1, \dots, w_n$ be pairwise distinct legal words. If $f(w_i) \neq 0$ for all i, then $\{f(w_1), \dots, f(w_n)\}$ is linearly independent.*

Proof. Since f preserves degree we can assume that $w_1, \dots, w_n$ have the same length, say r. By definition, $f(w_i)$ is the sum of the paths labeled w_i; hence if $i \neq j$ no path that appears in $f(w_i)$ appears in $f(w_j)$. But the paths of length r are linearly independent elements of kQ so $\{f(w_1), \dots, f(w_n)\}$ is linearly independent.

Theorem 3.7. *The kernel of the homomorphism $f : k\langle G\rangle \to kQ$ is equal to $(F) + I$ where I is the left ideal generated by the set*

$$S := \{x_1 \dots x_s \in G^s \mid s \leqslant \ell \text{ and there is no path labeled } x_1 \dots x_s\}.$$

Proof. By Proposition 3.3, $\ker f$ contains the ideal (F). Since $f(x_1 \dots x_r)$ is the sum of all the paths labeled $x_1 \dots x_r$, $S \subset \ker f$. Hence $(F) + I \subset \ker f$.

Since (F) is spanned by words, Lemma 3.6 implies $\ker f$ is spanned by (F) and various legal words. Suppose $x_1 \dots x_r$ is a legal word such that $f(x_1 \dots x_r) = 0$. This implies there is no path labeled $x_1 \dots x_r$ so, if $r \leqslant \ell$, $x_1 \dots x_r$ is in S and therefore in I. On the other hand, if $r \geqslant \ell + 1$, Lemma 3.5 implies $x_{r-\ell+1} \dots x_r$ is in S, whence $x_1 \dots x_r \in I$. $\square$

Information about the cokernel of f is given in Proposition 4.1.

4. The proof of Theorem 1.1

4.1. The proof of Theorem 1.1 when A is as in (1.1)

Let A be as in (1.1) and adopt the notation in (3.1). We will prove Theorem 1.1 by applying Proposition 2.1 to the induced homomorphism $\bar{f}: A \to kQ$. Before doing that we must check that the hypotheses of Proposition 2.1 hold: we must show that the kernel and cokernel of $\bar{f}$ belong to $\mathsf{Fdim}\, A$.

Proposition 4.1. *Let $\bar{f}: A \to kQ$ be the homomorphism induced by f. Then* $\ker \bar{f}$ *and* $\operatorname{coker} \bar{f}$ *belong to* $\mathsf{Fdim}\, A$.

Proof. Let I and S be as in Theorem 3.7 and write $\bar{I}$ and $\bar{S}$ for their images in A. Thus, $\bar{I} = \ker \bar{f}$ and $\ker \bar{f}$ is generated as a left ideal by $\bar{S}$.

Given the description of $\ker f$ in Theorem 3.7, it suffices to show that $\bar{I} A_\ell = 0$.

Let $x_1 \dots x_s \in S$. By Lemma 3.4, $x_1 \dots x_r L_\ell \cap L = \emptyset$; in other words, $x_1 \dots x_r L_\ell \subset (F)$. Taking the image of this equality in A we conclude that $\bar{S} A_\ell = 0$. It follows that $\bar{I} A_\ell = 0$. Thus $\ker f$ belongs to $\mathsf{Fdim}\, A$.

By Lemma 2.2, to show $\operatorname{coker} \bar{f}$ belongs to $\mathsf{Fdim}\, A$ it suffices to show that

$$(kQ_0)\bar{f}(A_\ell) \subset \bar{f}(A_\ell) \quad \text{and} \quad (kQ_1)\bar{f}(A_\ell) \subset \bar{f}(A_{\ell+1}).$$

To do this it suffices to show that $Q_0 f(L_\ell) \subset f(L_\ell)$ and $Q_1 f(L_\ell) \subset f(L_{\ell+1})$.

Let $x_1 \dots x_\ell \in L_\ell$. By Lemma 3.1(1), every path labeled $x_1 \dots x_\ell$ begins at the vertex $v_0 = x_1 \dots x_\ell$.

Let e be a trivial path and p a path labeled $x_1 \dots x_\ell$; since p begins at v_0, $ep = p$ if e is the trivial path at v_0, and $ep = 0$ if e is some other trivial path. Hence $ef(x_1 \dots x_\ell)$ is either 0 or $f(x_1 \dots x_\ell)$. It follows that $Q_0 f(x_1 \dots x_\ell) = \{f(x_1 \dots x_\ell)\}$ and $Q_0 f(L_\ell) = f(L_\ell)$.

Let a be an arrow and p a path labeled $x_1 \dots x_\ell$. If a does not end at v_0, then $ap = 0$ because p begins at v_0; thus, if a does not end at v_0, then $af(x_1 \dots x_\ell) = 0$.

We now assume a ends at v_0; i.e., $v_{-1} \xrightarrow{a} v_0$ and the arrow a is labeled by the first letter of v_{-1}, say x_0. The path ap is therefore labeled $x_0 x_1 \dots x_\ell$. Since $v_0 \lhd x_0 v_1$, a is the only arrow labeled x_0 that ends at v_0. Therefore

$$\begin{aligned} af(x_1 \dots x_\ell) &= f(x_0) f(x_1 \dots x_\ell) \\ &= f(x_0 x_1 \dots x_\ell). \end{aligned}$$

In particular, $af(x_1 \dots x_\ell) \in f(L_{\ell+1})$.

This completes the proof that $Q_1 f(L_\ell) \subset f(L_{\ell+1})$ and, as explained before, this implies $\operatorname{coker} \bar{f}$ belongs to $\mathsf{Fdim}\, A$. $\square$

Theorem 4.2. *Let A be a connected graded monomial algebra as in* (1.1) *and/or* (3.1)*. Let Q be its Ufnarovskii graph and view kQ as a left A-module through the homomorphism $\bar{f}: A \to kQ$. Then $- \bigotimes_A kQ$ induces an equivalence of categories* $\mathsf{QGr}\, A \equiv \mathsf{QGr}\, kQ$.

Proof. This follows from Propositions 2.1 and 4.1. $\square$

4.2. The proof of Theorem 1.1 when A is as in (1.2)

Let Q' be a finite quiver and $A = kQ'/I$ the quotient of its path algebra by an ideal generated by a finite number of paths. (Thus A is a more general kind of monomial algebra.) The subalgebra

$$A' = k \oplus A_1 \oplus A_2 \oplus \cdots$$

is of finite codimension in A so $A/A' \in \mathsf{Fdim}\, A'$. Proposition 2.1 therefore implies that $-\bigotimes_{A'} A$ induces an equivalence of categories

$$\mathsf{QGr}\, A' \equiv \mathsf{QGr}\, A. \tag{4.1}$$

Since A' is a monomial algebra of the form (1.1), Theorem 4.2 gives an equivalence

$$\mathsf{QGr}\, A' \equiv \mathsf{QGr}\, kQ \tag{4.2}$$

where Q is the Ufnarovskii graph of A'. By (4.1) and (4.2),

$$\mathsf{QGr}\, A \equiv \mathsf{QGr}\, kQ\,.$$

This completes the proof of Theorem 1.1 for kQ'/I.

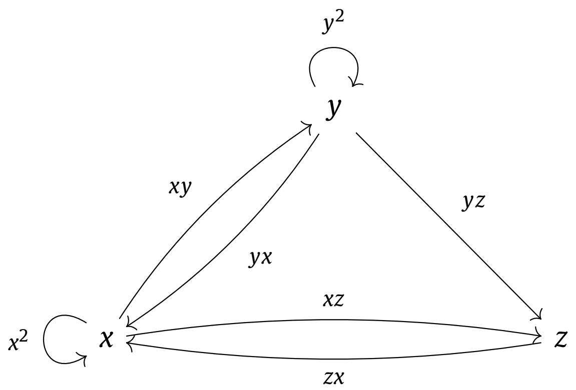
5. An example

Let $A = k\langle x, y, z\rangle/(z^2, zy)$. Since $\ell = 1$, $Q_0 = \{x, y, z\}$. The arrows for $Q(A)$ correspond to the legal words of length two, namely

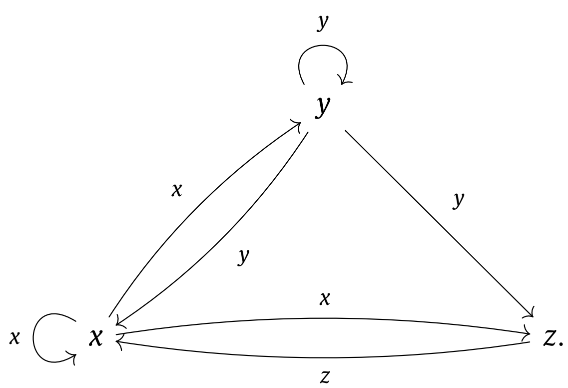
$$\{x^2, xy, xz, y^2, yx, yz, z^2, zx, zy\} - \{z^2, zy\}.$$

The Ufnarovskii graph of A is therefore

$Q(A) =$



(the arrows are denoted by w rather than a_w) with labels



Thus, the homomorphism f is

$$f(x) = a_{x^2} + a_{xy} + a_{xz},$$
$$f(y) = a_{y^2} + a_{yx} + a_{yz},$$
$$f(z) = a_{zx}.$$

6. Connected graded quadratic monomial algebras

Section 6.1 contains a short proof of Theorem 4.2 for connected graded monomial algebras with quadratic relations. Section 6.2 shows that Theorem 4.2 for an arbitrary finitely presented connected graded monomial algebra A can be deduced from the quadratic case.

6.1. Let A be a quadratic monomial algebra and Q its Ufnarovskii graph.

The defining relations for A have length 2 so $\ell = 1$. The set of vertices for Q is therefore in bijection with G. There is an arrow a_{xy} from vertex x to vertex y if and only if $xy \notin F$ and that arrow is labeled x if it exists. It follows that the map $f : k\langle G\rangle \to kQ$ defined in Section 3 can be defined as follows:

$$f(x) = \text{the sum of all arrows that start at } x.$$

Thus, if $r \geqslant 2$, then

$$f(x_1 \dots x_r) = \begin{cases} pf(x_r) & \text{where } p \text{ is the unique path labeled} \\ & x_1 \dots x_{r-1} \text{ that ends at vertex } x_r; \\ 0 & \text{if there is no such } p. \end{cases}$$

In particular, if $xy \in F$, there is no arrow from x to y so $f(xy) = 0$. Thus $f(F) = 0$ and there is an induced map $\bar{f} : A \to kQ$.

The lemmas in Section 3 are either trivial or unnecessary in the quadratic case. The proof that $\ker \bar{f}$ belongs to $\mathsf{Fdim}\, A$ is also much simpler.

6.2. Let n be a positive integer. The nth Veronese subalgebra of a $\mathbb{Z}$-graded algebra B is

$$B^{(n)} := \bigoplus_{i \in \mathbb{Z}} B_{in}.$$

Theorem 6.1 *(Backelin–Fröberg). (See [2, Prop. 3].) If A is a connected graded k-algebra with defining relations of degree $\leqslant d + 1$, then $A^{(n)}$ is a quadratic algebra for all $n \geqslant d$.*

Theorem 6.2 *(Verevkin). (See [9, Thm. 4.4].) Let A be a connected graded algebra generated by A_1. Then $\mathsf{QGr}\, A \equiv \mathsf{QGr}\, A^{(n)}$ for all positive integers n.*

Proposition 6.3. *If Theorem* 1.1 *holds for connected graded quadratic monomial algebras it holds for all connected graded monomial algebras.*

Proof. Let A be a monomial algebra and give ℓ, F and G the meanings they have in Section 3.

By Theorem 6.1, $A^{(\ell)}$ is a quadratic algebra. Because A is a monomial algebra so is $A^{(\ell)}$. By Theorem 6.2, $\mathsf{QGr}\, A \equiv \mathsf{QGr}\, A^{(\ell)}$. Hence if Theorem 1.1 holds for $A^{(\ell)}$, then $\mathsf{QGr}\, A \equiv \mathsf{QGr}\, kQ'$ where Q' is the Ufnarovskii graph for $A^{(\ell)}$. $\square$

6.3. The Ufnarovskii graph for $A^{(\ell)}$ is more complicated than that for A. For example, the Ufnarovskii graph for $A = k\langle x, y\rangle/(y^3)$ is

$$\begin{tikzcd}
Q := x^2 \arrow[loop left, "x^3"] \arrow[d, "x^2 y"'] & y^2 \arrow[d, "y^2 x"] \\
xy \arrow[ur, "xy^2"] \arrow[r, bend left, "xyx"] & yx \arrow[ul, "yx^2"'] \arrow[l, bend left, "yxy"]
\end{tikzcd} \tag{6.1}$$

where the arrows are denoted by w rather than a_w. The homomorphism $\bar{f} : A \to kQ$ is given by

$$\bar{f}(x) = a_{x^3} + a_{x^2 y} + a_{xyx} + a_{xy^2},$$
$$\bar{f}(y) = a_{yx^2} + a_{yxy} + a_{y^2 x}.$$

The 2-Veronese subalgebra of A is generated by $s = x^2$, $t = xy$, $u = yx$, and $v = y^2$. We have

$$A^{(2)} \cong \frac{k\langle s, t, u, v\rangle}{(vu, tv, v^2)}$$

so its Ufnarovskii graph is

$$\begin{tikzcd}
 & v \arrow[dl, "vt"'] \arrow[d, bend left, "vs"] & \\
Q'' = t \arrow[loop left, "t^2"] \arrow[r, bend left, "ts"] \arrow[rr, bend right=50, "tu"'] & s \arrow[l, bend left, "st"] \arrow[r, bend left, "su"] \arrow[u, bend left, "sv"] \arrow[loop below, "s^2"] & u \arrow[loop right, "u^2"] \arrow[l, bend left, "us"] \arrow[ul, "uv"'] \arrow[ll, bend left=60, "ut"]
\end{tikzcd}$$

The homomorphism $f : k\langle s, t, u, v\rangle/(vu, tv, v^2) \to kQ''$ is given by

$$\bar{f}(s) = a_{s^2} + a_{st} + a_{su} + a_{sv},$$
$$\bar{f}(t) = a_{t^2} + a_{ts} + a_{tu},$$
$$\bar{f}(u) = a_{u^2} + a_{us} + a_{ut} + a_{uv},$$
$$\bar{f}(v) = a_{vs} + a_{vt}.$$

7. A remark

The results in [5] and [6] show that many different Q give rise to the equivalent categories $\mathsf{QGr}\, kQ$. Thus, given a finitely presented connected graded monomial algebra A, the Ufnarovskii graph is not the only Q for which $\mathsf{QGr}\, A$ is equivalent to $\mathsf{QGr}\, kQ$.

Consider, in particular,

$$A = \frac{k\langle x, y\rangle}{(y^3)}.$$

The Ufnarovskii graphs for A and $A^{(2)}$ appear in Section 6.3. Since $A^{(\ell)}$ is quadratic for all $\ell \geqslant 2$, $\mathsf{QGr}\, kQ(A) \equiv \mathsf{QGr}\, kQ(A^{(\ell)})$ for all $\ell \geqslant 2$.

Furthermore, by [4], $\mathsf{QGr}\, A$ is also equivalent to $\mathsf{QGr}\, kQ'$ where

$$Q' = \begin{tikzcd} 0 \arrow[loop left] \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[r, bend left] & 2 \arrow[ll, bend left] \end{tikzcd} \tag{7.1}$$

There is a direct proof of the equivalence $\mathsf{QGr}\, kQ(A) \equiv \mathsf{QGr}\, kQ'$.

Theorem 7.1. *(See [6].) Let L and R be $\mathbb{N}$-valued matrices such that LR and RL make sense. Let Q^{LR} be the quiver with incidence matrix LR and Q^{RL} the quiver with incidence matrix RL. There is an equivalence of categories*

$$\mathsf{QGr}\, kQ^{LR} \equiv \mathsf{QGr}\, kQ^{RL}.$$

The equivalence $\mathsf{QGr}\, kQ(A) \equiv \mathsf{QGr}\, kQ'$ follows from Theorem 7.1 because $Q(A) = Q^{LR}$ and $Q' = Q^{RL}$ where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

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