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Journal of Algebra 354 (2012) 95–109

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# A non-commutative homogeneous coordinate ring for the degree six del Pezzo surface ☆

S. Paul Smith

*Department of Mathematics, Box 354350, Univ. Washington, Seattle, WA 98195, United States*

ARTICLE INFO

*Article history:*
Received 27 April 2011
Available online 16 January 2012
Communicated by Michel Van den Bergh

*MSC:*
14A22
16S38
16W50

*Keywords:*
Homogeneous coordinate ring
Noncommutative
del Pezzo surface
Graded algebra

ABSTRACT

Let $R$ be the free $\mathbb{C}$-algebra on $x$ and $y$ modulo the relations $x^5 = yxy$ and $y^2 = xyx$ endowed with the $\mathbb{Z}$-grading $\deg x = 1$ and $\deg y = 2$. The ring $R$ appears, in somewhat hidden guise, in a paper on quiver gauge theories. Let $\mathbb{B}_3$ denote the blow up of $\mathbb{CP}^2$ at three non-colinear points. The main result in this paper is that the category of quasi-coherent $\mathcal{O}_{\mathbb{B}_3}$-modules is equivalent to the quotient of the category of $\mathbb{Z}$-graded $R$-modules modulo the full subcategory of modules that are the sum of their finite dimensional submodules. This reduces almost all representation-theoretic questions about $R$ to algebraic geometric questions about the del Pezzo surface $\mathbb{B}_3$. For example, the generic simple $R$-module has dimension six. Furthermore, the main result combined with results of Artin, Tate, Van den Bergh, and Stephenson implies that $R$ is a noetherian domain of global dimension three.

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## 1. Introduction

We will work over the field of complex numbers.

**1.1.** The surface obtained by blowing up $\mathbb{P}^2$ at three non-colinear points is, up to isomorphism, independent of the points. It is called the del Pezzo surface of degree six and we will denote it by $\mathbb{B}_3$.

**1.2.** Let $R$ be the free $\mathbb{C}$-algebra on $x$ and $y$ modulo the relations

$$x^5 = yxy \quad \text{and} \quad y^2 = xyx. \tag{1.1}$$

☆ The author was supported by the National Science Foundation, Award No. 0602347.
*E-mail address:* smith@math.washington.edu.

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doi:10.1016/j.jalgebra.2011.11.030

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Give $R$ a $\mathbb{Z}$-grading by declaring that

$$\deg x = 1 \quad \text{and} \quad \deg y = 2.$$

The ring $R$ arises, in somewhat hidden guise, in a paper about string theory [5] (see Section 1.5). The present paper concerns only the mathematical properties of $R$ and its relation to the degree 6 del Pezzo surface.

**1.3.** The main result in this paper establishes the following surprising relationship between $R$ and the degree six del Pezzo surface.

**Theorem 1.1.** *Let $R$ be the non-commutative algebra $\mathbb{C}[x, y]$ defined by the relations* (1.1)*. Let* $\mathsf{Gr}\, R$ *be the category of $\mathbb{Z}$-graded left $R$-modules. There is an equivalence of categories*

$$\mathsf{Qcoh}\,\mathbb{B}_3 \equiv \frac{\mathsf{Gr}\,R}{\mathsf{Fdim}\,R}$$

*where the left-hand side is the category of quasi-coherent $\mathcal{O}_{\mathbb{B}_3}$-modules and the right-hand side is the quotient category modulo the full subcategory* $\mathsf{Fdim}\,R$ *consisting of those modules that are the sum of their finite dimensional submodules.*

Theorem 1.1 is a consequence of the following result.

**Theorem 1.2.** *Let $R$ be the non-commutative algebra $\mathbb{C}[x, y]$ defined by the relations* (1.1)*. Let $\mathcal{L} = \mathcal{O}(-E)$ be the invertible $\mathcal{O}_{\mathbb{B}_3}$-module corresponding to a $(-1)$-curve $E$ and $\sigma$ an order* 6 *automorphism of $\mathbb{B}_3$ that cyclically permutes the six $(-1)$-curves on $\mathbb{B}_3$. Then $R$ is isomorphic to the twisted homogeneous coordinate ring*

$$B(\mathbb{B}_3, \mathcal{L}, \sigma) := \bigoplus_{n \geqslant 0} H^0(\mathbb{B}_3, \mathcal{L}_n)$$

*where*

$$\mathcal{L}_n := \mathcal{L} \otimes (\sigma^*)\mathcal{L} \otimes \cdots \otimes (\sigma^*)^{n-1}\mathcal{L}.$$

In the terminology of Artin, Tate, and Van den Bergh [1] and Artin and Van den Bergh [3], $B(\mathbb{B}_3, \mathcal{L}, \sigma)$ is a *twisted homogeneous coordinate ring* of $\mathbb{B}_3$. Results of Artin, Tate, and Van den Bergh, and Stephenson [10] now imply that $R$ is a 3-dimensional Artin–Schelter regular algebra and therefore has the following properties.

**Corollary 1.3.** *Let $R$ be the non-commutative algebra $\mathbb{C}[x, y]$ defined by the relations* (1.1)*. Then*

(1) *$R$ is a left and right noetherian domain*;
(2) *$R$ has global homological dimension* 3;
(3) *$R$ is Auslander–Gorenstein and Cohen–Macaulay in the non-commutative sense*;
(4) *the Hilbert series of $R$ is the same as that of the weighted polynomial ring on three variables of weights* 1, 2, *and* 3;
(5) *$R$ is a finitely generated module over its center* [9, *Corollary* 2.3];
(6) *$R^{(6)} := \bigoplus_{n=0}^{\infty} R_{6n}$ is isomorphic to $\bigoplus_{n=0}^{\infty} H^0(\mathbb{B}_3, \mathcal{O}(-nK))$ where $K = K_{\mathbb{B}_3}$ is the canonical divisor on $\mathbb{B}_3$*;
(7) $\operatorname{Spec} R^{(6)}$ *is the canonical cone over $\mathbb{B}_3$, i.e., the cone obtained by collapsing the zero section of the total space of the canonical bundle over $\mathbb{B}_3$.*

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This close connection between $R$ and $\mathbb{B}_3$ means that almost all aspects of the representation theory of $R$ can be expressed in terms of the geometry of $\mathbb{B}_3$. We plan to address this question in another paper.

**1.4.** The justification for calling $R$ a non-commutative homogeneous coordinate ring for $\mathbb{B}_3$ is the similarity between the equivalence of categories in Theorem 1.1 and following theorem of Serre [7]:

if $X \subset \mathbb{P}^n$ is the scheme-theoretic zero locus of a graded ideal $I$ in the polynomial ring $S = \mathbb{C}[x_0, \dots, x_n]$ with its standard grading, and $A = S/I$, then there is an equivalence of categories

$$\mathsf{Qcoh}\, X \equiv \frac{\mathsf{Gr}\, A}{\mathsf{Fdim}\, A} \tag{1.2}$$

where the right-hand side is the quotient category of $\mathsf{Gr}\, A$, the category of graded $A$-modules, by the full subcategory $\mathsf{Fdim}\, A$ consisting of modules whose non-zero finitely generated submodules have support only at the origin.

**1.5. Motivation** The results in this paper are a prerequisite for some results in [8] where three superpotential algebras appearing in the string theory literature are investigated by relating them to twisted homogeneous coordinate rings. In [5], Beasley and Plesser study a superpotential algebra they dub the $dP_3I$ path algebra. In [8], we will show that the $dP_3I$ path algebra is isomorphic to $R \rtimes \mu_6$, the skew group ring for the 6th roots of unity acting on $R$ by $\xi \cdot r = \xi^n r$ for $r \in R_n$; the isomorphism is established in [8]. An intimate understanding of $R$ therefore leads to a detailed understanding of the $dP_3I$ path algebra. The $dP_3$ in the notation $dP_3I$ refers to the de Pezzo surface obtained by blowing up 3 non-colinear points in $\mathbb{P}^2$. The I in $dP_3I$ is to distinguish this algebra from two other path algebras with relations that Beasley and Plesser associate to the degree-six del Pezzo surface.

## 2. $R = \mathbb{C}[x, y]$ with $x^5 = yxy$ and $y^2 = xyx$ is an iterated Ore extension

The following result is a straightforward calculation. The main point of it is to show that $R$ has the same Hilbert series as the weighted polynomial ring on three variables of weights 1, 2, and 3.

**Proposition 2.1.** *(See Stephenson [10,11].) The ring $R := \mathbb{C}[x, y]$ with defining relations*

$$x^5 = yxy \quad \text{and} \quad y^2 = xyx$$

*is an iterated Ore extension of the polynomial ring $\mathbb{C}[w]$. Explicitly, if $\zeta$ is a fixed primitive 6th root of unity, $R$ has the following properties.*

(1) *$R = \mathbb{C}[w][z; \sigma][x; \tau, \delta]$ where $\sigma \in \operatorname{Aut}\mathbb{C}[z]$, $\tau \in \operatorname{Aut}\mathbb{C}[w][z; \sigma]$, and $\delta$ is a $\tau$-derivation defined as follows*

$$\begin{aligned} \sigma(w) &= \zeta w, & & \\ \tau(w) &= -\zeta^2 w, & \tau(z) &= \zeta z, \\ \delta(w) &= z, & \delta(z) &= -w^2. \end{aligned}$$

(2) *A set of defining relations of $R = \mathbb{C}[z, w, x]$ is given by*

$$zw = \zeta wz,$$
$$xw = -\zeta^2 wx + z,$$
$$xz = \zeta zx - w^2.$$

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(3) *$R$ has basis* $\{w^i z^j x^k \mid i, j, k \geqslant 0\}$.
(4) *$R$ is a noetherian domain.*
(5) *The Hilbert series of $R$ is* $(1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1}$.

**Proof.** Define the elements

$$\begin{aligned} w &:= y - x^2, \\ z &:= xw + \zeta^2 wx \\ &= xy + \zeta^2 yx - \zeta x^3 \end{aligned}$$

of $R$. Since $y$ belongs to the subalgebra of $R$ generated by $x$ and $w$, $\mathbb{C}[x, y] = \mathbb{C}[x, w] = \mathbb{C}[x, w, z]$. It is easy to check that

$$zw = \zeta wz, \qquad xw = z - \zeta^2 wx, \qquad xz = \zeta zx - w^2. \tag{2.1}$$

Let $R'$ be the free algebra $\mathbb{C}\langle w, x, z\rangle$ modulo the relations in (2.1). We will show $R'$ is isomorphic to $R$. We already know there is a homomorphism $R' \to R$ and we will now exhibit a homomorphism $R \to R'$ by showing there are elements $x$ and $Y$ in $R'$ that satisfy the defining relations for $R$. Define the element $Y := w + x^2$ in $R'$. A straightforward computation in $R'$ gives

$$xwx - x^2 w = w^2 + wx^2$$

so

$$Y^2 = w^2 + x^2 w + wx^2 + x^4 = xwx + x^4 = xYx.$$

The next calculation uses the identity $1 - \xi + \xi^2 = 0$ repeatedly. Deep breath...

$$\begin{aligned} YxY &= (w + x^2)xw + wx^3 + x^5 \\ &= (w + x^2)(z - \zeta^2 wx) + [wx^3 + x^5] \\ &= x^2 z - \zeta^2 x^2 wx + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= x(\zeta zx - w^2) - \zeta^2 x(z - \zeta^2 wx)x + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= (\zeta - \zeta^2)xzx - xw^2 - \zeta xwx^2 + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= (\zeta - \zeta^2)(\zeta zx - w^2)x - (z - \zeta^2 wx)w - \zeta(z - \zeta^2 wx)x^2 + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= (\zeta^2 - \zeta^3)zx^2 - (\zeta - \zeta^2)w^2 x - zw + \zeta^2 wxw - \zeta zx^2 - wx^3 + [wz - \zeta^2 w^2 x + wx^3 + x^5] \\ &= (\zeta^2 - \zeta^3 - \zeta)zx^2 + \zeta^2 wxw + [(1 - \zeta)wz - \zeta w^2 x + x^5] \\ &= \zeta^2 wxw + [(1 - \zeta)wz - \zeta w^2 x + x^5] \\ &= \zeta^2 w(z - \zeta^2 wx) + [-\zeta^2 wz - \zeta w^2 x + x^5] \\ &= x^5. \end{aligned}$$

Since $YxY = x^5$, $R$ is isomorphic to $R'$. Hence $R$ is an iterated Ore extension as claimed. The other parts of the proposition follow easily. □

It is an immediate consequence of the relations that $x^6 = y^3$. Hence $x^6$ is in the center of $R$.

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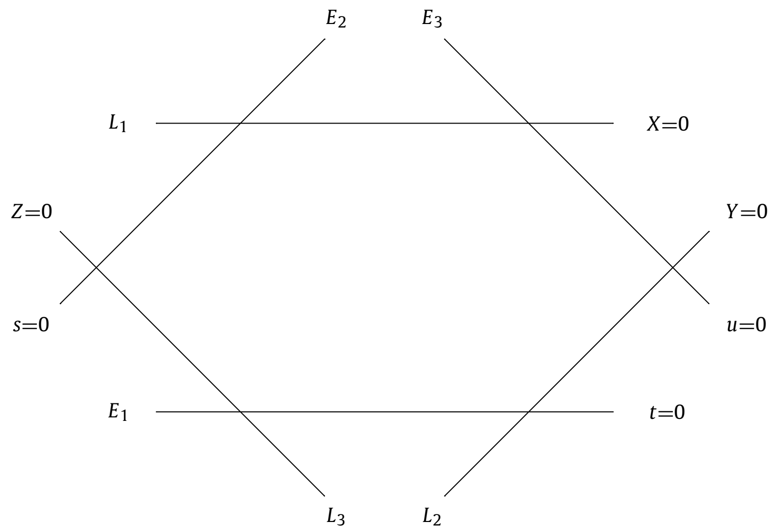
## 3. The del Pezzo surface $\mathbb{B}_3$

Let $\mathbb{B}_3$ be the surface obtained by blowing up the complex projective plane $\mathbb{P}^2$ at three non-colinear points. We will write

$$\pi : \mathbb{B}_3 \to \mathbb{P}^2$$

for the morphism that contracts the exceptional curves $E_1$, $E_2$, and $E_3$. The $(-1)$-curves on $\mathbb{B}_3$ lie in the following configuration




(3.1)

where $L_1$, $L_2$, and $L_3$ are the strict transforms of the lines in $\mathbb{P}^2$ spanned by the points that are blown up. (The labeling of the equations for the $(-1)$-curves is explained in Section 3.3.)

The union of the $(-1)$-curves is an anti-canonical divisor so we write

$$-K := L_1 + L_2 + L_3 + E_1 + E_2 + E_3$$

($K$ for canonical). This is, of course, an ample divisor.

**3.1. The Picard group of $\mathbb{B}_3$** The morphism $\pi : \mathbb{B}_3 \to \mathbb{P}^2$ induces an injective group homomorphism $\pi^* : \operatorname{Pic}\mathbb{P}^2 \to \operatorname{Pic}\mathbb{B}_3$. We write $H = \pi^* L$ where $L$ is a line in $\mathbb{P}^2$. Hence

$$\operatorname{Pic}\mathbb{B}_3 = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3.$$

We identify $\operatorname{Pic}\mathbb{B}_3$ with $\mathbb{Z}^4$ by using the ordered basis

$$H,\ -E_2,\ -E_1,\ -E_3.$$

Thus

$$H = (1,0,0,0), \qquad E_1 = (0,0,-1,0), \qquad E_2 = (0,-1,0,0), \qquad E_3 = (0,0,0,-1).$$

In this basis the anti-canonical divisor is

$$-K = (3,1,1,1).$$

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The Picard group may be presented more symmetrically as

$$\operatorname{Pic}\mathbb{B}_3 = \frac{\bigoplus_{i=1}^{3}(\mathbb{Z}L_i \oplus \mathbb{Z}E_i)}{(E_i + L_j = E_j + L_i \mid 1 \leqslant i, j \leqslant 3)}.$$

It follows that

$$H = L_1 + E_2 + E_3 = L_2 + E_1 + E_3 = L_3 + E_1 + E_2$$

and

$$L_1 = (1,1,0,1), \qquad L_2 = (1,0,1,1), \qquad L_3 = (1,1,1,0).$$

**3.2. Cox's homogeneous coordinate ring** By definition, Cox's homogeneous coordinate ring [6] for a complete smooth toric variety $X$ is

$$S := \bigoplus_{[\mathcal{L}]\in \operatorname{Pic} X} H^0(X, \mathcal{L}).$$

From now on, $S$ denotes Cox's homogeneous coordinate ring for $\mathbb{B}_3$.

Let $X, Y, Z, s, t, u$ be coordinate functions on $\mathbb{C}^6$. One can present $\mathbb{B}_3$ as a toric variety by defining it as the orbit space

$$\mathbb{B}_3 := \frac{\mathbb{C}^6 - W}{(\mathbb{C}^\times)^4}$$

where the irrelevant locus, $W$, is the union of nine codimension two subspaces, namely

$$\begin{array}{lll} X = t = 0, & X = Y = 0, & s = t = 0, \\ Y = s = 0, & Y = Z = 0, & u = t = 0, \\ Z = u = 0, & Z = X = 0, & s = u = 0 \end{array} \tag{3.2}$$

and $(\mathbb{C}^\times)^4$ acts with weights

$$\begin{array}{ccccc} X & 1 & 1 & 0 & 1 \\ Y & 1 & 0 & 1 & 1 \\ Z & 1 & 1 & 1 & 0 \\ s & 0 & -1 & 0 & 0 \\ t & 0 & 0 & -1 & 0 \\ u & 0 & 0 & 0 & -1\,. \end{array}$$

Therefore $S$ is the $\mathbb{Z}^4$-graded polynomial ring

$$S = \mathbb{C}[X, Y, Z, s, t, u]$$

with the degrees of the generators given by their weights under the $(\mathbb{C}^\times)^4$ action, e.g., $\deg X = (1,1,0,1)$, $\deg u = (0,0,0,-1)$, etc. It follows from Cox's results [6, Section 3] that

$$\mathsf{Qcoh}\,\mathbb{B}_3 \equiv \frac{\mathsf{Gr}(S, \mathbb{Z}^4)}{\mathsf{T}}$$

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where $\mathsf{Gr}(S,\mathbb{Z}^4)$ is the category of $\mathbb{Z}^4$-graded $S$-modules and $\mathsf{T}$ is the full subcategory consisting of the modules that are sums of finitely generated $S$-modules supported on $W$.

**3.3.** The labeling of the $(-1)$-curves in the diagram (3.1) is explained by the existence of the morphisms

$$\begin{array}{ccccc} & & \mathbb{B}_3 & & \\ & \overset{\pi}{\swarrow} & & \searrow & \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \end{array} \qquad\qquad \begin{array}{ccc} & (X,Y,Z,s,t,u) & \\ \overset{\pi}{\swarrow} & & \searrow \\ (Xsu, Ytu, Zst) & & (YZt, XZs, XYu) \end{array}$$

that collapse the $(-1)$-curves: for example, $\pi$ contracts the three divisors $t=0$, $s=0$, and $u=0$, i.e., $E_1$, $E_2$, and $E_3$.

**3.4. An order six automorphism $\sigma$ of $\mathbb{B}_3$** The cyclic permutation of the six $(-1)$-curves on $\mathbb{B}_3$ extends to a global automorphism of $\mathbb{B}_3$ of order six. We now make this explicit.

The category of graded rings consists of pairs $(A,\Gamma)$ consisting of an abelian group $\Gamma$ and a $\Gamma$-graded ring $A$. A morphism $(f,\theta):(A,\Gamma)\to(B,\Upsilon)$ consists of a ring homomorphism $f:A\to B$ and a group homomorphism $\theta:\Gamma\to\Upsilon$ such that $f(A_i)\subset B_{\theta(i)}$ for all $i\in\Gamma$.

Let $\tau:S\to S$ be the automorphism induced by the cyclic permutation

$$X \longrightarrow u \longrightarrow Y \underset{\tau}{\longrightarrow} t \longrightarrow Z \longrightarrow s \quad (s \to X) \tag{3.3}$$

and let $\theta:\mathbb{Z}^4\to\mathbb{Z}^4$ be left multiplication by the matrix

$$\theta=\begin{pmatrix} 2 & -1 & -1 & -1\\ 1 & -1 & -1 & 0\\ 1 & 0 & -1 & -1\\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

Then $(\tau,\theta):(S,\mathbb{Z}^4)\to(S,\mathbb{Z}^4)$ is an automorphism in the category of graded rings. The irrelevant locus (3.2) is stable under the action of $\tau$, so $\tau$ induces an automorphism $\sigma$ of $\mathbb{B}_3$. It follows from the definition of $\tau$ in (3.3) that $\sigma$ cyclically permutes the six $-1$-curves.

Since $(\tau,\theta)^6=\mathrm{id}_{(S,\mathbb{Z}^4)}$ the order of $\sigma$ divides six. But the action of $\sigma$ on the set of $(-1)$-curves has order six, so $\sigma$ has order six as an automorphism of $\mathbb{B}_3$.

**3.5.** Fix a primitive cube root of unity $\omega$. The left action of $\theta$ on $\mathbb{Z}^4=\operatorname{Pic}\mathbb{B}^3$ has eigenvectors

$$v_1=\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\qquad v_2=\begin{pmatrix}3\\1\\1\\1\end{pmatrix},\qquad v_3=\begin{pmatrix}0\\1\\\omega\\\omega^2\end{pmatrix},\qquad v_4=\begin{pmatrix}0\\1\\\omega^2\\\omega\end{pmatrix},$$

with corresponding eigenvalues $-1$, $+1$, $\omega^2$, $\omega$.

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## 4. A twisted hcr for $\mathbb{B}_3$

In this section we will prove the main theorem: $R$ is isomorphic to a twisted homogeneous coordinate ring $B = B(\mathbb{B}_3, \mathcal{L}, \sigma)$ for $\mathbb{B}_3$. The degree-$n$ homogeneous component of $B$ is the global sections of an invertible $\mathcal{O}_{\mathbb{B}_3}$-module $\mathcal{L}_n$; i.e., $B_n = H^0(\mathbb{B}_3, \mathcal{L}_n)$. After defining $\mathcal{L}_n$ in Section 4.1 we prove some vanishing results for its cohomology that will be used later to prove that $B$ is generated as a $\mathbb{C}$-algebra by $B_1$ and $B_2$. Since $R$ is generated as a $\mathbb{C}$-algebra by $x \in R_1$ and $y \in R_2$ this will allow us to prove that the homomorphism $\Phi : R \to B$ defined in Proposition 4.5 is surjective. We also compute $\dim H^0(\mathbb{B}_3, \mathcal{L}_n) = \dim B_n$ and observe that this is the same as $\dim R_n$ which allows us to conclude that $\Phi$ is an isomorphism.

We write $K$ for the canonical divisor on $\mathbb{B}_3$.

**4.1. A sequence of line bundles on $\mathbb{B}_3$** We will blur the distinction between a divisor $D$ and the class of the line bundle $\mathcal{O}(D)$ in $\operatorname{Pic}\mathbb{B}_3$.

We define a sequence of divisors: $D_0$ is zero; $D_1$ is the line $L_1$; for $n \geqslant 1$

$$D_n := \left(1 + \theta + \cdots + \theta^{n-1}\right)(D_1).$$

We will write $\mathcal{L}_n := \mathcal{O}(D_n)$. Therefore

$$\mathcal{L}_n = \mathcal{L}_1 \otimes \sigma^*\mathcal{L}_1 \otimes \cdots \otimes \left(\sigma^*\right)^{n-1}\mathcal{L}_1.$$

For example,

$$\begin{aligned}
\mathcal{O}(D_1) &= \mathcal{L}_1 = \mathcal{O}(1,1,0,1) = \mathcal{O}(L_1),\\
\mathcal{O}(D_2) &= \mathcal{L}_2 = \mathcal{O}(1,1,0,0) = \mathcal{O}(L_1 + E_3),\\
\mathcal{O}(D_3) &= \mathcal{L}_3 = \mathcal{O}(2,1,1,1) = \mathcal{O}(L_1 + E_3 + L_2),\\
\mathcal{O}(D_4) &= \mathcal{L}_4 = \mathcal{O}(2,1,0,1) = \mathcal{O}(L_1 + E_3 + L_2 + E_1),\\
\mathcal{O}(D_5) &= \mathcal{L}_5 = \mathcal{O}(3,2,1,1) = \mathcal{O}(L_1 + E_3 + L_2 + E_1 + L_3),\\
\mathcal{O}(D_6) &= \mathcal{L}_6 = \mathcal{O}(3,1,1,1) = \mathcal{O}(L_1 + E_3 + L_2 + E_1 + L_3 + E_2)\\
&= \mathcal{O}(-K).
\end{aligned}$$

**Lemma 4.1.** *Suppose $m \geqslant 0$ and $0 \leqslant r \leqslant 5$. Then*

$$D_{6m+r} = D_r - mK.$$

**Proof.** Since $\theta^6 = 1$,

$$\begin{aligned}
\sum_{i=0}^{6m+r-1} \theta^i &= \left(1 + \theta + \cdots + \theta^5\right)\sum_{j=0}^{m-1} \theta^{6j} + \theta^{6m}\left(1 + \theta + \cdots + \theta^{r-1}\right)\\
&= \left(1 + \theta + \cdots + \theta^{r-1}\right) + m\left(1 + \theta + \cdots + \theta^5\right)
\end{aligned}$$

where the sum $1 + \theta + \cdots + \theta^{r-1}$ is empty and therefore equal to zero when $r = 0$. Therefore $D_{6m+r} = D_r + mD_6 = D_r - mK$, as claimed. □

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**4.2. Vanishing results** For a divisor $D$ on a smooth surface $X$, we write

$$h^i(D) := \dim H^i\big(X, \mathcal{O}_X(D)\big).$$

We need to know that $h^1(D) = h^2(D) = 0$ for various divisors $D$ on $\mathbb{B}_3$.

If $D - K$ is ample, then the Kodaira Vanishing Theorem implies that $h^0(K - D) = h^1(K - D) = 0$ and Serre duality then gives $h^2(D) = h^1(D) = 0$.

The notational conventions in Section 3.1 identify $\operatorname{Pic}\mathbb{B}_3$ with $\mathbb{Z}^4$ via

$$aH - cE_1 - bE_2 - dE_3 \equiv (a, b, c, d).$$

The intersection form on $\mathbb{B}_3$ is given by

$$H^2 = 1, \qquad E_i \cdot E_j = -\delta_{ij}, \qquad H \cdot E_i = 0,$$

so the induced intersection form on $\mathbb{Z}^4$ is

$$(a, b, c, d) \cdot \big(a', b', c', d'\big) = aa' - bb' - cc' - dd'.$$

**Lemma 4.2.** *Let $D = (a, b, c, d) \in \operatorname{Pic}\mathbb{B}_3 \equiv \mathbb{Z}^4$. Suppose that*

$$(a + 3)^2 > (b + 1)^2 + (c + 1)^2 + (d + 1)^2 \tag{4.1}$$

*and*

$$b, c, d > -1, \quad \text{and} \quad a + 1 > b + c, b + d, c + d. \tag{4.2}$$

*Then $D - K$ is ample, whence $h^1(D) = h^2(D) = 0$.*

**Proof.** The effective cone is generated by $L_1$, $L_2$, $L_3$, $E_1$, $E_2$, and $E_3$ so, by the Nakai–Moishezon criterion, $D - K$ is ample if and only if $(D - K)^2 > 0$ and $(D - K) \cdot L_i > 0$ and $(D - K) \cdot E_i > 0$ for all $i$. Now $D - K = (a + 3, b + 1, c + 1, d + 1)$, so $(D - K)^2 > 0$ if and only if (4.1) holds and $(D - K) \cdot D' > 0$ for all effective $D'$ if and only if (4.2) holds.

Hence the hypothesis that (4.1) and (4.2) hold implies that $D - K$ is ample. The Kodaira Vanishing Theorem now implies that $h^0(K - D) = h^1(K - D) = 0$. Serre duality now implies that $h^2(D) = h^1(D) = 0$. □

**Lemma 4.3.** *For all $n \geqslant 0$, $h^1(D_n) = h^2(D_n) = 0$.*

**Proof.** The value of $D_n$ for $0 \leqslant n \leqslant 6$ is given explicitly in Section 4.1. We also note that $D_7 = D_1 + D_6 = (4, 2, 1, 2)$. It is routine to check that conditions (4.1) and (4.2) hold for $D = D_n$ when $n = 0, 2, 3, 4, 5, 6, 7$. Hence $h^1(D_n) = h^2(D_n) = 0$ when $n = 0, 2, 3, 4, 5, 6, 7$.

We now consider $D_1$ which is the $(-1)$-curve $X = 0$. (Since $(D_1 - K) \cdot D_1 = 0$, $D_1 - K$ is not ample so we can't use Kodaira Vanishing as we did for the other small values of $n$.) It follows from the exact sequence $0 \to \mathcal{O}_{\mathbb{B}_3} \to \mathcal{O}_{\mathbb{B}_3}(D_1) \to \mathcal{O}_{D_1}(D_1) \to 0$ that $H^p(\mathbb{B}_3, \mathcal{O}_{\mathbb{B}_3}(D_1)) \cong H^p(\mathbb{B}_3, \mathcal{O}_{D_1}(D_1))$ for $p = 1, 2$. However, $D_1 \cong \mathbb{P}^1$, $\mathcal{O}_{D_1}(D_1)$ is the normal sheaf for $D_1 \subset \mathbb{B}_3$, and, since $D_1$ can be contracted to a smooth point on the degree 7 del Pezzo surface, $\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{D_1}(-1)$. Therefore $H^p(\mathbb{B}_3, \mathcal{O}_{D_1}(D_1)) \cong H^p(\mathbb{P}^1, \mathcal{O}(-1))$ which is zero for $p = 1, 2$. It follows that $h^1(D_1) = h^2(D_1) = 0$.

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Thus $h^1(D_n) = h^2(D_n) = 0$ when $0 \leqslant n \leqslant 7$. We have also shown that $D_n - K$ is ample when $2 \leqslant n \leqslant 7$. We now argue by induction. Suppose $n \geqslant 8$ and $D_{n-6} - K$ is ample. Now $D_n - K = D_{n-6} - K - K$. Since a sum of ample divisors is ample, $D_n - K$ is ample. It follows that $h^1(D_n) = h^2(D_n) = 0$. □

**4.3. The twisted homogeneous coordinate ring $B(\mathbb{B}_3, \mathcal{L}, \sigma)$** We assume the reader is somewhat familiar with the notion of twisted homogeneous coordinate rings. Standard references for that material are [1–4].

The notion of a $\sigma$-ample line bundle [3] plays a key role in the study of twisted homogeneous coordinate rings. Because $\mathcal{L}_6$ is the anti-canonical bundle and therefore ample, $\mathcal{L}_1$ is $\sigma$-ample. This allows us to use the results of Artin and Van den Bergh in [3] to conclude that the twisted homogeneous coordinate ring

$$B = B(\mathbb{B}_3, \mathcal{L}, \sigma) = \bigoplus_{n=0}^{\infty} B_n = \bigoplus_{n=0}^{\infty} H^0(\mathbb{B}_3, \mathcal{L}_n) \tag{4.3}$$

is such that

$$\mathsf{Qcoh}\,\mathbb{B}_3 \equiv \frac{\mathsf{Gr}\,B}{\mathsf{Fdim}\,B} \tag{4.4}$$

where $\mathsf{Fdim}\,B$ is the full subcategory of $\mathsf{Gr}\,B$ consisting of those graded modules that are the sum of their finite dimensional submodules. Artin and Van den Bergh [3] show that the equivalence (4.4) implies that $B$ has a host of good properties.

**4.4.** We will now compute the dimensions $h^0(D_n)$ of the homogeneous $B_n$ of $B$. We will show that $B$ has the same Hilbert series as the non-commutative ring $R$, i.e., the same Hilbert series as the weighted polynomial ring with weights 1, 2, and 3. The Hilbert series of $R$ was computed in Proposition 2.1.

As usual we write $\chi(D) = h^0(D) - h^1(D) + h^2(D)$. The Riemann–Roch formula is

$$\chi\big(\mathcal{O}(D)\big) = \chi(\mathcal{O}) + \frac{1}{2} D \cdot (D - K).$$

We have $\chi(\mathcal{O}_{\mathbb{B}_3}) = 1$ and $K^2 = 6$.

**Lemma 4.4.** *Suppose* $0 \leqslant r \leqslant 5$. *Then*

$$h^0(D_{6m+r}) = \begin{cases} (m+1)(3m+r) & \text{if } r \neq 0, \\ 3m^2 + 3m + 1 & \text{if } r = 0 \end{cases}$$

*and*

$$\sum_{n=0}^{\infty} h^0(D_n) t^n = \frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

*In particular, $B$ and $R$ have the same Hilbert series.*

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**Proof.** Computations for $1 \leqslant r \leqslant 5$ give $D_r^2 = r - 2$ and $D_r \cdot K = -r$. Hence

$$\begin{aligned}\chi(D_{6m+r}) &= 1 + \frac{1}{2}(D_r - mK) \cdot \bigl(D_r - (m+1)K\bigr) \\ &= 1 + \frac{1}{2}\bigl(D_r^2 - (2m+1)D_r \cdot K + 6m(m+1)^2\bigr) \\ &= (3m+r)(m+1)\end{aligned}$$

for $m \geqslant 0$ and $1 \leqslant r \leqslant 5$. When $r = 0$, $D_r = 0$ so

$$\chi(D_{6m}) = 3m^2 + 3m + 1.$$

By Lemma 4.3, $\chi(D_n) = h^0(D_n)$ for all $n \geqslant 0$ so it follows from the formula for $\chi(D_n)$ that

$$h^0(D_{n+6}) - h^0(D_n) = n + 6 \tag{4.5}$$

for all $n \geqslant 0$.

To complete the proof of the lemma, it suffices to show that $h^0(D_n)$ is the coefficient of $t^n$ in the Taylor series expansion

$$f(t) := \frac{1}{(1-t)(1-t^2)(1-t^3)} = \sum_{n=0}^{\infty} a_n t^n.$$

Because

$$\bigl(1 - t^6\bigr) f(t) = \bigl(1 - t + t^2\bigr)(1-t)^{-2} = 1 + \sum_{n=1}^{\infty} n t^n,$$

it follows that

$$\begin{aligned}\bigl(1 - t^6\bigr) f(t) &= a_0 + a_1 t + \cdots + a_5 t^5 + \sum_{n=0}^{\infty} (a_{n+6} - a_n) t^{n+6} \\ &= 1 + t + 2t + \cdots + 5t^5 + \sum_{n=6}^{\infty} n t^n \\ &= 1 + t + 2t + \cdots + 5t^5 + \sum_{n=0}^{\infty} (n+6) t^{n+6}.\end{aligned}$$

In particular, if $0 \leqslant r \leqslant 5$, $a_r = h^0(D_r)$. We now complete the proof by induction. Suppose we have proved that $a_i = h^0(D_i)$ for $i \leqslant n + 5$. By comparing the expressions in the Taylor series we see that

$$a_{n+6} = a_n + (n+6) = h^0(D_n) + n + 6 = h^0(D_{n+6})$$

where the last equality is given by (4.5). $\square$

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**4.4.1. Remark** It wasn't necessary to compute $\chi(D_n)$ in the previous proof. The proof only used the fact that $\chi(D_{n+6}) - \chi(D_n) = n+6$ which can be proved directly as follows

$$\begin{aligned}
\chi(D_{n+6}) - \chi(D_n) &= \frac{1}{2}D_{n+6}\cdot(D_{n+6}-K) - \frac{1}{2}D_n\cdot(D_n-K) \\
&= \frac{1}{2}(D_{n+6}-D_n)\cdot(D_{n+6}+D_n-K) \\
&= -K\cdot\big(D_r-(m+1)K\big) \\
&= 6(m+1) - K\cdot D_r \\
&= n+6.
\end{aligned}$$

**4.5. The isomorphism $R \to B(\mathbb{B}_3, \mathcal{L}, \sigma)$** By definition, $B_n = H^0(\mathbb{B}_3, \mathcal{L}_n)$. Since Cox's homogeneous coordinate ring, $S = \mathbb{C}[X, Y, Z, s, t, u]$, is the direct sum of $H^0(\mathbb{B}_3, \mathcal{L})$ as $[\mathcal{L}]$ ranges over $\operatorname{Pic}\mathbb{B}_3$, each $B_n$ is a subspace of $S$. In particular, $B$ itself is a subspace of $S$, but

*the multiplication in $B$ is not that in $S$.*

The ring $B$ has the following basis elements in the following degrees:

| deg $= n$ | $\mathcal{L}_n$ | basis for $B_n$ | | | | |
|---|---|---|---|---|---|---|
| 1 | $\mathcal{O}(1,1,0,1)$ | $X$ | | | | |
| 2 | $\mathcal{O}(1,1,0,0)$ | $Xu$ | $Zt$ | | | |
| 3 | $\mathcal{O}(2,1,1,1)$ | $XYu$ | $YZt$ | $XZs$ | | |
| 4 | $\mathcal{O}(2,1,0,1)$ | $XYtu$ | $YZt^2$ | $XZst$ | $X^2su$ | |
| 5 | $\mathcal{O}(3,2,1,1)$ | $XYZtu$ | $YZ^2t^2$ | $XZ^2st$ | $X^2Zsu$ | $X^2Yu^2$ |
| 6 | $\mathcal{O}(3,1,1,1)$ | $XYZstu$ | $YZ^2st^2$ | $XZ^2s^2t$ | $X^2Zs^2u$ | $X^2Ysu^2$ |
| | | | | | $XY^2tu^2$ | $Y^2Zt^2u$. |

The multiplication in $B$ is Zhang's twisted multiplication [12] with respect to the automorphism $\tau$ defined in (3.3): the product in $B$ of $a \in B_m$ and $b \in B_n$ is

$$a *_B b := a\tau^m(b). \tag{4.6}$$

To make it clear whether a product is being calculated in $B$ or $S$ we will write $x$ for $X$ considered as an element of $B$ and $y$ for $Zt$ considered as an element of $B$. Therefore, for example,

$$\begin{aligned}
x^5 &= X\tau(X)\tau^2(X)\tau^3(X)\tau^4(X)\tau^5(X) \\
&= XuYtZ \\
&= (Zt)Y(uX) \\
&= Zt\tau^2(X)\tau^3(Zt) \\
&= yxy
\end{aligned}$$

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and

$$y^2 = Zt\tau^2(Zt) = Zt(sX) = X(sZ)t = X\tau(zt)\tau^3(X) = xyx.$$

The following proposition is an immediate consequence of these two calculations.

**Proposition 4.5.** *Let $R$ be the free algebra $\mathbb{C}\langle x, y\rangle$ modulo the relations $x^5 = yxy$ and $y^2 = xyx$. Then there is a $\mathbb{C}$-algebra homomorphism*

$$\Phi : R = \mathbb{C}[x, y] \to B(\mathbb{B}_3, \mathcal{L}, \sigma), \quad x \mapsto X,\ y \mapsto Zt.$$

**Lemma 4.6.** *The homomorphism in Proposition* 4.5 *is an isomorphism in degrees* $\leqslant 6$.[1]

**Proof.** By Proposition 2.1, $R$ has Hilbert series $(1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1}$, so the dimension of $R_n$ in degrees $1, 2, 3, 4, 5, 6$ is $1, 2, 3, 4, 5, 7$.

The $n$th row in the following table gives a basis for $B_n$, $1 \leqslant n \leqslant 6$. One proceeds down each column by multiplying on the right by $x$. There wasn't enough room on a single line for $B_6$ so we put the last two entries for $B_6$ on a new line.

$$\begin{array}{lllll}
x = X, & & & & \\
x^2 = Xu, & y = Zt, & & & \\
x^3 = XYu, & yx = YZt, & xy = XZs, & & \\
x^4 = XYtu, & yx^2 = YZt^2, & y^2 = XZst, & x^2y = X^2su, & \\
x^5 = XYZtu, & yx^3 = YZ^2t^2, & y^2x = XZ^2st, & xy^2 = X^2Zsu, & x^3y = X^2Yu^2, \\
x^6 = XYZstu, & yx^4 = YZ^2st^2, & y^2x^2 = XZ^2s^2t, & xy^2x = X^2Zs^2u, & x^2y^2 = X^2Ysu^2, \\
 & & & yx^2y = Y^2Zt^2u, & x^4y = XY^2tu^2.
\end{array}$$

These calculations involving $x$ and $y$ are made by using the formula (4.6) in the same way it was used to show that $x^5 = yxy$. □

**Lemma 4.7.** *$\mathcal{L}_2$ is generated by its global sections.*

**Proof.** A line bundle on a variety is generated by its global sections if and only if for each point on the variety there is a section of the bundle that does not vanish at that point. In this case, $H^0(\mathbb{B}_3, \mathcal{L}_2)$ is spanned by $Xu$ and $Zt$. One can see from the diagram (3.1) that the zero locus of $Xu$ does not meet the zero locus of $Zt$, so the common zero locus of $Xu$ and $Zt$ is empty. □

**Proposition 4.8.** *As a $\mathbb{C}$-algebra, $B$ is generated by $B_1$ and $B_2$.*

**Proof.** It follows from the explicit calculations in Lemma 4.6 that the subalgebra of $B$ generated by $B_1$ and $B_2$ contains $B_m$ for all $m \leqslant 6$. It therefore suffices to prove that the twisted multiplication map $B_2 \otimes B_n \to B_{n+2}$ is surjective for all $n \geqslant 5$.

By definition, $B_2 = H^0(\mathcal{L}_2)$ and this has dimension two so, by Lemma 4.7, there is an exact sequence $0 \to \mathcal{N} \to B_2 \otimes \mathcal{O}_{\mathbb{B}_3} \to \mathcal{L}_2 \to 0$ for some line bundle $\mathcal{N}$. In fact, $\mathcal{N} \cong \mathcal{L}_2^{-1}$.

By definition, $\mathcal{L}_{n+2} = \mathcal{L}_2 \otimes \mathcal{M}$ where $\mathcal{M} \cong \mathcal{O}(D_{n+2} - D_2)$, and the twisted multiplication map $B_2 \otimes B_n \to B_{n+2}$ is the ordinary multiplication map

$$B_2 \otimes H^0(\mathcal{M}) = H^0(\mathcal{L}_2) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L}_2 \otimes \mathcal{M}).$$

[1] We will eventually prove that $\Phi$ is an isomorphism in all degrees but the low degree cases need to be handled separately.

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Applying $-\otimes\mathcal{M}$ to the exact sequence $0\to\mathcal{L}_2^{-1}\to B_2\otimes\mathcal{O}_{\mathbb{B}_3}\to\mathcal{L}_2\to 0$ and taking cohomology gives an exact sequence

$$0\to H^0\big(\mathcal{L}_2^{-1}\otimes\mathcal{M}\big)\to B_2\otimes H^0(\mathcal{M})\to H^0(\mathcal{L}_2\otimes\mathcal{M})\to H^1\big(\mathcal{L}_2^{-1}\otimes\mathcal{M}\big).$$

Hence, if $h^1(\mathcal{L}_2^{-1}\otimes\mathcal{M})=0$, then $B_2B_n=B_{n+2}$.

We will now show that $h^1(\mathcal{L}_2^{-1}\otimes\mathcal{M})=0$. Since $\mathcal{L}_2^{-1}\otimes\mathcal{M}\cong\mathcal{O}(-D_2+D_{n+2}-D_2)$ and $n+2\geqslant 7$,

$$\mathcal{L}_2^{-1}\otimes\mathcal{M}\cong\mathcal{O}(-D_{6m+r}-2D_2-K)$$

for suitable integers $m$ and $r$ such that $6m+r\geqslant 7$ and $0\leqslant r\leqslant 5$.

By Lemma 4.1, $D_{6m+r}-2D_2-K=D_r-2D_2-(m+1)K$. By Lemma 4.2, to show that $h^1(D_r-2D_2-(m+1)K)=0$ it suffices to show that conditions (4.1) and (4.2) hold for the divisors $D$ in the following table:

| | | $D:=D_r-2D_2-(m+1)K\in\operatorname{Pic}\mathbb{B}_3$ |
|---|---|---|
| $r=0$ | $m\geqslant 2$ | $(3m+1,m-1,m+1,m+1)\in\mathbb{Z}^4$ |
| $r=1$ | $m\geqslant 1$ | $(3m+2,m,m+1,m+2)$ |
| $r=2$ | $m\geqslant 1$ | $(3m+2,m,m+1,m+1)$ |
| $r=3$ | $m\geqslant 1$ | $(3m+3,m,m+2,m+2)$ |
| $r=4$ | $m\geqslant 1$ | $(3m+3,m,m+1,m+2)$ |
| $r=5$ | $m\geqslant 1$ | $(3m+4,m+1,m+2,m+2)$. |

This is a routine task. □

**Theorem 4.9.** *Let $R$ be the free algebra $\mathbb{C}\langle x,y\rangle$ modulo the relations $x^5=yxy$ and $y^2=xyx$. The $\mathbb{C}$-algebra homomorphism*

$$\Phi:R=\mathbb{C}[x,y]\to B(\mathbb{B}_3,\mathcal{L},\sigma),\quad x\mapsto X,\ y\mapsto Zt,$$

*is an isomorphism of graded algebras.*

**Proof.** By Lemma 4.6, $B_1$ and $B_2$ are in the image of $\Phi$. By Proposition 4.8, $B$ is generated by $B_1$ and $B_2$. Hence $\Phi$ is surjective. But $\Phi(R_n)\subset B_n$, and $R$ and $B$ have the same Hilbert series, so $\Phi$ is also surjective. □

Consider $R^{(3)}\supset\mathbb{C}[x^3,xy,yx]$. Since $\dim R_6=7=(\dim R_3)^2-2$ there is a 2-dimensional space of quadratic relations among the elements $x^3$, $xy$, and $yx$. Hence $R^{(3)}$ is not a 3-dimensional Artin–Schelter regular algebra. The relations in the degree two component of $R^{(3)}$ are generated by

$$\big(x^3\big)^2=(xy)^2=(yx)^2.$$

## Acknowledgments

The author is grateful to the following people: Amer Iqbal for bringing the paper [5] to his attention; Paul Hacking and Sándor Kovács for passing on some standard results about del Pezzo surfaces and algebraic geometry; Darin Stephenson for telling the author that $R$ is an iterated Ore extension and for useful remarks about his papers [10] and [11]; Mark Blunk for pointing out an error in an earlier version of this paper.

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