MATH 1A

Review Sheet Solutions

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Last Revised:

September 29, 2013

Limits of Sequences

Problem 1: *Write a rigorous definition of what the following means:* $\lim_{n\to\infty} a_n = c$ *(i.e. write the definition for a sequence to converge) three times.*

Solution: For every $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n > N$ implies that $|a_n - c| < \varepsilon$. ■

Problem 2: *Let* $a_n = \dfrac{n}{10n+7}$. *Does* $\lim_{n\to\infty} a_n$ *exist? If so, prove it. If not, justify.*

Solution: Yes, the limit does exist. We claim that $\lim a_n = \frac{1}{10}$.

Scratch Work: We want $|a_n - \dfrac{1}{10}| < \varepsilon$. Well,

$$\begin{aligned} \left|a_n - \frac{1}{10}\right| &= \left|\frac{n}{10n+7} - \frac{1}{10}\right| \\ &= \left|\frac{10n - (10n+7)}{10(10n+7)}\right| \\ &= \left|\frac{-7}{10(10n+1)}\right| \\ &= \frac{7}{10(10n+7)} \end{aligned}$$

So, if we want

$$\left|a_n - \frac{1}{10}\right| < \varepsilon$$

we want

$$\frac{7}{10(10n+7)} < \varepsilon$$

Solving this inequality in the standard way shows it is equivalent to

$$n > \frac{7}{100}\left(\frac{1}{\varepsilon} - 10\right)$$

So, we would want to take $N = \left\lceil \frac{7}{100}\left(\frac{1}{\varepsilon} - 10\right)\right\rceil$, right? The only problem with this is that technically we want $N \in \mathbb{N}$ (where, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$). Unfortunately, for some ε this expression gives us a negative number. In particular, it's the subtraction of 10 that causes N sometimes to be negative. Since, if one N works, any larger one works, we can just toss the -10 to get that a valid choice is $N = \left\lceil \frac{7}{100\varepsilon}\right\rceil$. ♣

Let $\varepsilon > 0$ be given. Let $N = \left\lceil \frac{7}{100\varepsilon}\right\rceil$. Then, by the scratch work we see that if $n > N$ then

$$\left|a_n - \frac{1}{10}\right| < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, the conclusion follows. ■

Problem 3: *Let* $a_n = \frac{1}{n^2 + 2n + 1}$. *Does* $\lim_{n\to\infty} a_n$ *exist? If so, prove it. If not, justify.*

Solution: Yes, $\lim a_n$ exists. In fact, we claim that $\lim a_n = 0$.

Scratch Work: We first note that $a_n = \frac{1}{(n+1)^2}$. So, then we want $|a_n - 0| < \varepsilon$. But,

$$\begin{aligned} |a_n - 0| &= \left|\frac{1}{(n+1)^2} - 0\right| \\ &= \left|\frac{1}{(n+1)^2}\right| \\ &= \frac{1}{(n+1)^2} \end{aligned}$$

So, we see that if we want $|a_n - 0| < \varepsilon$ we want

$$\frac{1}{(n+1)^2} < \varepsilon$$

Solving this for n shows that we want

$$n > \sqrt{\frac{1}{\varepsilon}} - 1$$

So, it would seem that we should take $N = \left\lceil \sqrt{\frac{1}{\varepsilon}} - 1 \right\rceil$. But, we see that we run into the same problem. For some values of $\varepsilon > 0$ we'd have that $N = 0$, which is not a natural number. So, we employ the same trick as in problem 1--we forget that -1. So, we see that $N = \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil$ is good. ♣

Let $\varepsilon > 0$ be given. Let $N = \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil$. Now, if $n > N$, then by our scratch work we see that

$$|a_n - 0| < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, the conclusion follows. ■

Problem 4: *State and prove the Squeeze Theorem for sequences. Use it to prove that*

$$\lim_{n\to\infty} \frac{n^2 + n + 48769504 + \sin(n)}{n^4 + \pi n^2 - 2} = 0$$

Solution: The squeeze theorem states the following

Theorem (Squeeze Theorem): *Let a_n, b_n, and c_n be real sequences. Suppose that*

$$a_n \leqslant b_n \leqslant c_n$$

for all $n \geqslant 1$. If $\lim a_n = \lim c_n = c$, then $\lim b_n = c$.

Proof: Let $\varepsilon > 0$ be given. Since $\lim a_n = c$, we may choose $N_1 \in \mathbb{N}$ such that $n > N_1$ implies that $|a_n - c| < \varepsilon$. Similarly, since $\lim c_n = c$, we may choose $N_2 \in \mathbb{N}$ such that $n > N$ implies that $|c_n - c| < \varepsilon$. Let $N = \max\{N_1, N_2\}$. Suppose that $n > N$. Note that since $n > N_1$ we have that

$$|a_n - c| < \varepsilon \quad \Longleftrightarrow \quad -\varepsilon < a_n - c < \varepsilon$$

Similarly, since $n > N_2$, we have that

$$|c_n - c| < \varepsilon \quad \Longleftrightarrow \quad -\varepsilon < c_n - c < \varepsilon$$

We claim that

$$|b_n - c| < \varepsilon \quad \Longleftrightarrow \quad -\varepsilon < b_n - c < \varepsilon$$

But, note that since $a_n \leqslant b_n \leqslant c_n$ we have that

$$-\varepsilon < a_n - c \leqslant b_n - c$$

and

$$b_n - c \leqslant c_n - c < \varepsilon$$

Thus,

$$-\varepsilon < a_n - c \leqslant b_n - c \leqslant c_n - c < \varepsilon$$

and so

$$-\varepsilon < a_n - c < \varepsilon$$

as desired. Since $\varepsilon > 0$ was arbitrary, the conclusion follows. ∎

Now, let's use the squeeze theorem to prove the desired limit is zero. Let us note that for all $n \geqslant 1$ we have that

$$n^2 + n + 4876904 + \sin(n) \leqslant n^2 + n^2 + 4876904n^2 + n^2 = 4876907n^2$$

and

$$n^4 - \pi n^2 - 2 \geqslant n^4$$

Thus,

$$\frac{n^2 + n + 48769504 + \sin(n)}{n^4 + \pi n^2 - 2} \leqslant \frac{4876907n^2}{n^4} = \frac{4876907}{n^2}$$

Now, note that for all $n \geqslant 1$ we have that

$$n^2 + n + 4876904 + \sin(n) \geqslant n^2 - 1$$

and

$$n^4 - \pi n^2 - 2 \leqslant n^4$$

Thus,

$$\frac{n^2 - 1}{n^4} \frac{n^2 + n + 48769504 + \sin(n)}{n^4 + \pi n^2 - 2}$$

Thus, putting the two inequalities we've derived together gives

$$\frac{n^2 - 1}{n^4} \frac{n^2 + n + 48769504 + \sin(n)}{n^4 + \pi n^2 - 2} \leqslant \frac{4876907}{n^2}$$

Since the two outward bounding sequences converge to 0 (exercise left to the reader) the Squeeze Theorem implies that

$$\lim_{n \to \infty} \frac{n^2 + n + 48769504 + \sin(n)}{n^4 + \pi n^2 - 2} = 0$$

as desired. ■

Limits of Functions

Problem 5: *Write a rigorous definition of what the following means:* $\lim_{x \to a} f(x) = L$ *(i.e. write the definition of a limit converging).*

Solution: Let $\varepsilon > 0$ be given. There exists a $\delta > 0$, such that if $0 < |x-a| < \delta$ then $|f(x)-L| < \varepsilon$.■

Problem 6:

a) *Use the definition of the limit to show that* $\lim_{x \to 0} 1 = 1$.

b) *Generalize the above to show that* $\lim_{x \to 0} c = c$ *where* c *is any real constant.*

c) *Use the definition of the limit to show that* $\lim_{x \to a} 1 = 1$ *where* a *is any real number.*

d) *Generalize part c) to show that* $\lim_{x \to a} c = c$ *where* a *and* c *are any (possibly unrelated) real constants.*

Solution:

a) Let $\varepsilon > 0$ be given. Let $\delta = 1$, note then that if $0 < |x-0| < \delta$, we have that $|1-1| = 0 < \varepsilon$. Note that since $\varepsilon > 0$ was arbitrary, the conclusion follows.

Remark 1: Note that we could have taken ANY δ we wanted here (why?). ♦

b) Let $\varepsilon > 0$ be given. Let $\delta = 1$. Then, if $0 < |x-0| < \delta$ we have that $|c-c| = 0 < \varepsilon$. Note that since $\varepsilon > 0$ was arbitrary, the conclusion follows.

Remark 2: Same remark as the last remark. ♦

c) Let $\varepsilon > 0$ be given. Let $\delta = 1$. Note that if $0 < |x-a| < \delta$, then $|1-1| = 0 < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the conclusion follows.

Remark 3: Same remark as the last remark about the remark before that remark. ♦

d) Let $\varepsilon > 0$ be given. Let $\delta = 1$. Note if $0 < |x-a| < \delta$, then $|c-c| = 0 < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the conclusion follows.

Remark 4: I'm not even going to attempt this one... (remarkception) ♦

Problem 7: *Let* $f(x) = \pi x - 1$. *Does* $\lim_{x \to \pi} f(x)$ *exist? If so, prove it. If not, justify.*

Solution: Yes, the limit does exist. In fact, we claim that $\lim_{x \to \pi} f(x) = \pi^2 - 1$.

Scratch Work: Let us first mess around with $|f(x) - (\pi^2 - 1)|$. Well:

$$\begin{aligned} |f(x) - (\pi^2 - 1)| &= |\pi x - 1 - (\pi^2 - 1)| \\ &= |\pi x - \pi^2| \\ &= \pi |x - \pi| \end{aligned}$$

So, if we want $|f(x) - (\pi^2 - 1)| < \varepsilon$, we really want $\pi|x - \pi| < \varepsilon$, and so we want $|x - \pi| < \dfrac{\varepsilon}{\pi}$. Thus, it behooves us to take $\delta = \dfrac{\varepsilon}{\pi}$. ♣

Let $\varepsilon > 0$ be given. Let $\delta = \dfrac{\varepsilon}{\pi}$. Note then that if $0 < |x - \pi| < \delta$, then $|f(x) - (\pi^2 - 1)| < \varepsilon$ by our scratch work. Since $\varepsilon > 0$ was arbitrary, the conclusion follows. ■

Problem 8: *Let* $f(x) = x^2 + 2x + 7$. *Does* $\lim_{x \to 1} f(x)$ *exist? If so, prove it. If not, justify.*

Solution: Yes, the limit does exist. In fact, we claim that $\lim_{x \to 1} f(x) = 10$.

Scratch Work: Let's screw around with $|f(x) - 10|$. Well:

$$\begin{aligned} |f(x) - 10| &= |x^2 + 2x + 7 - 10| \\ &= |x^2 + 2x - 3| \\ &= |(x - 1)(x + 3)| \\ &= |x - 1||x + 3| \end{aligned}$$

Now, as always, we have made appear the term we wanted (namely $|x - 1|$) but it's accompanied by a scary function of x. We might be scared, as always, that as we choose x to make $|x - 3|$ tiny, that $|x + 3|$ might get back, actually stopping $|x - 3||x + 3|$ from getting small. So, let's try and bound $|x + 3|$ by a constant, so this fear disappears. Taking a hint from Dr. Coward, what if we restrict ourselves to only looking at $|x - 1| < \frac{1}{1000}$? This manifests itself in our language by always choosing $\delta \leqslant \frac{1}{1000}$. Well, if $|x - 1| < \frac{1}{1000}$, then

$$-\frac{1}{1000} < x - 1 < \frac{1}{1000}$$

so

$$4-\frac{1}{1000}<x+3<4+\frac{1}{1000}$$

Thus, we easily see that $|x+3|<5$. Thus, we see that

$$|f(x)-10|=|x-1||x+3|\leqslant 5|x-1|$$

So, if we want to make $|f(x)-10|<\varepsilon$, it really suffices to make $5|x-1|<\varepsilon$, or $|x-1|<\frac{\varepsilon}{5}$. This last condition manifests itself by always considering $\delta\leqslant\frac{\varepsilon}{5}$. Thus, we see the two conditions we've imposed are $\delta\leqslant\frac{1}{1000}$ and $\delta\leqslant\frac{\varepsilon}{5}$. Since we want both conditions to hold for δ, we merely take $\delta=\min\{\frac{1}{1000},\frac{\varepsilon}{5}\}$ for then δ will certainly satisfy both conditions. ♣

So, let $\varepsilon>0$ be given. Let $\delta=\min\{\frac{1}{1000},\frac{\varepsilon}{5}\}$. If $0|<|x-1|<\delta$, then the scratch work shows that

$$|f(x)-1|<\varepsilon$$

Since $\varepsilon>0$ was given, the conclusion follows. ■

Problem 9: *State and prove the Squeeze Theorem for functions (include a drawn picture of why it makes sense IN ADDITION to the proof). Use it to prove the following equality:*

$$\lim_{x\to 0}x^4\sin\left(\frac{1}{x^2}\right)$$

(see discussion notes for a similar problem/hint).

Solution:

Theorem (Squeeze Theorem): *Let $f(x), g(x)$ and $h(x)$ be real valued functions. Suppose that*

$$f(x)\leqslant g(x)\leqslant h(x)$$

for all x near a. If $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L$, then $\lim_{x\to a}g(x)=L$.

Proof: Suppose that $f(x)\leqslant g(x)\leqslant h(x)$ whenever $|x-a|<r_0$ (this is true for some r_0 since we assumed the inequality was true near a). Let $\varepsilon>0$ be given. Since $\lim_{x\to a}f(x)=L$, there exists $\delta_1>0$ such that if $0<|x-a|<\delta_1$ then

$$|f(x)-L|<\varepsilon \quad\Longleftrightarrow\quad -\varepsilon<f(x)-L<\varepsilon$$

Similarly, since $\lim_{x \to a} h(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then

$$|h(x) - L| < \varepsilon \quad \Longleftrightarrow \quad -\varepsilon < |h(x) - L| < \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2, r_0\}$. Suppose that $0 < |x - a| < \delta$. We want to show that

$$|g(x) - L| < \varepsilon \quad \Longleftrightarrow -\varepsilon < g(x) - L < \varepsilon$$

Note that since $|x - a| < r_0$ we have that $f(x) \leqslant g(x)$. But, we also have that $0 < |x - a| < \delta_1$, and so we see that

$$-\varepsilon < f(x) - L \leqslant g(x) - L$$

Also, since $|x - a| < r_0$ we have that $g(x) \leqslant h(x)$. But, we also have that $0 < |x - a| < \delta_2$, and so we have that

$$g(x) - L \leqslant h(x) - L < \varepsilon$$

Putting these inequalities together gives that

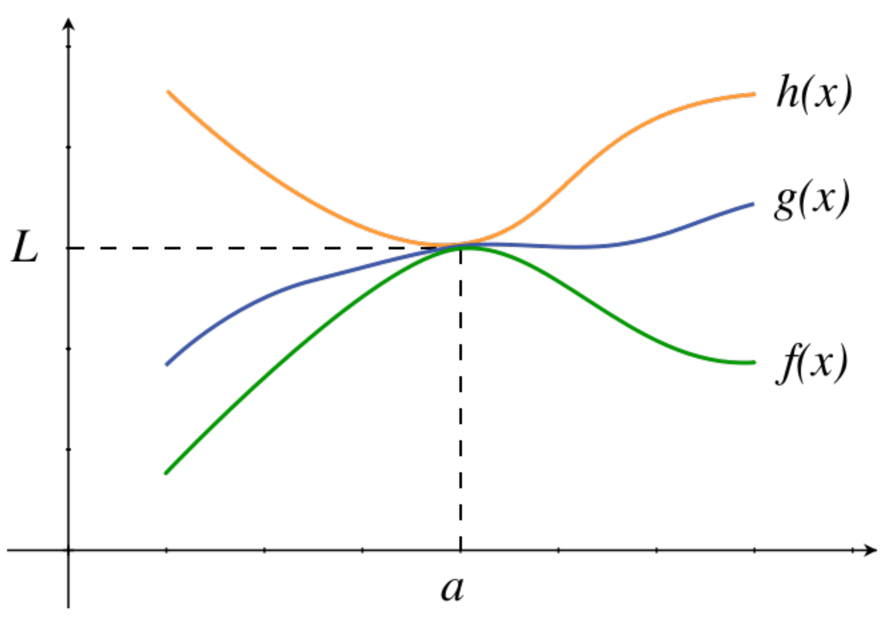
$$-\varepsilon < f(x) - L \leqslant g(x) - L \leqslant h(x) - L < \varepsilon$$

In particular, we see that

$$-\varepsilon < g(x) - L < \varepsilon$$

as desired. Since $\varepsilon > 0$ was arbitrary, the conclusion follows. ■

The picture for the proof the Squeeze Theorem would be something like the following (taken from Wikipedia):



We see that since $h(x)$ and $f(x)$ go to L as x goes to a, and $g(x)$ is stuck in between the two, that $g(x)$ is "squeezed"/forced to also approach L as $x \to a$.

Now, let's use the Squeeze Theorem to show that $\lim_{x \to 0} x^4 \sin\left(\frac{1}{x^2}\right)$. As usual, we want to get rid of the nastiest term, which, in this case, is $\sin\left(\frac{1}{x^2}\right)$. Using the standard estimation

$$-1 \leqslant \sin\left(\frac{1}{x^2}\right) \leqslant 1$$

gives

$$-x^4 \leqslant x^4 \sin\left(\tfrac{1}{x^2}\right) \leqslant x^4$$

Since the bounding functions have limit 0 at 0 (I leave this to you), we may conclude by the squeeze theorem that

$$\lim_{x \to 0} x^4 \sin\left(\frac{1}{x^2}\right) = 0$$

as desired. ■

Continuity and the Intermediate Value Theorem

Problem 10: *Write the definition of what it means for a function $f(x)$ to be continuous at $x = a$ three times.*

Solution: The function f is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$. ■

Problem 11: *Interpret the statement of Problem 6, d) as a statement about continuity.*

Solution: Problem 6), part d) is really just saying that every constant function $f(x) = c$ is continuous at every point of $\mathbb{R}$. ■

Problem 12: *Show that the function $f(x) = x$ is continuous at every point a of $\mathbb{R}$.*

Solution: We need to show that for every $a \in \mathbb{R}$ we have that $\lim_{x \to a} f(x) = f(a)$. Or, substitution $f(x) = x$, we must show that for every $a \in \mathbb{R}$ we have that $\lim_{x \to a} x = a$.

To do this, we let $\varepsilon > 0$. Let $\delta = \varepsilon$. Then, if $0 < |x - a| < \delta$ then certainly $|x - a| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the conclusion follows. ■

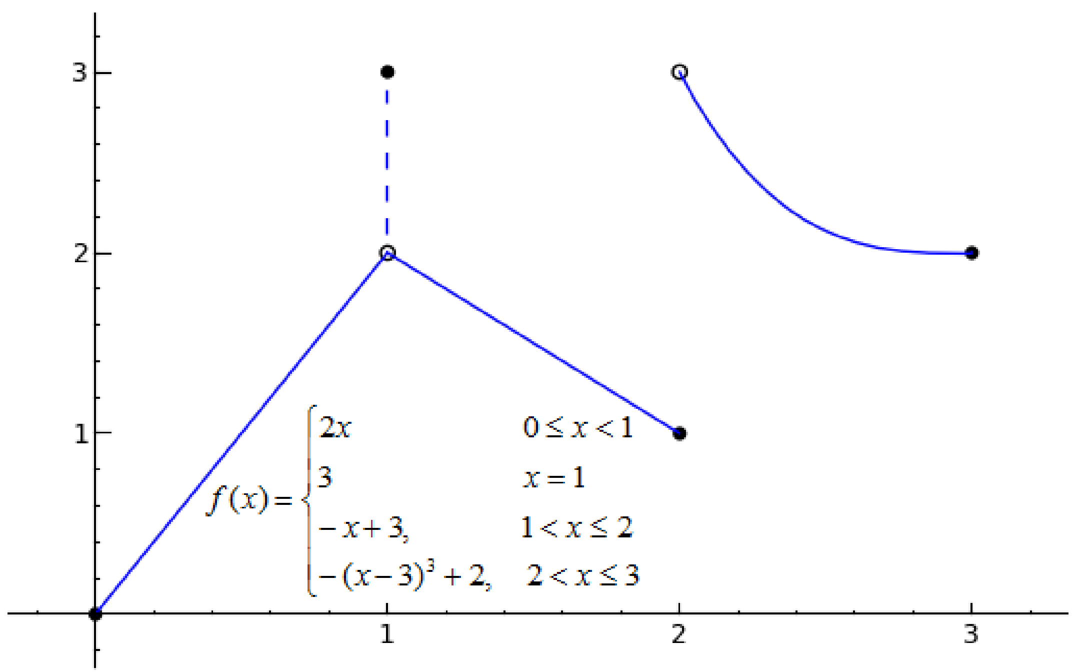
Problem 13: *Use various limit laws (be sure to state clearly what they are!) as well as the conclusions of Problem 12 and Problem 11 to justify why any polynomial is continuous (hint: think about a polynomial as being built out of combinations of the function $f(x) = x$ and constant functions).*

Solution: Every polynomial is of the form $\sum_{j=0}^{n} a_j x^j$ for some n and some constants $a_1, \dots, a_n \in \mathbb{R}$. Note that since we have proven $g(x) = x$ is continuous for all $a \in \mathbb{R}$, we have by the multiplication limit law that $g(x)^m = x^m$ is continuous for all $a \in \mathbb{R}$ (why?). Also, for any $c \in \mathbb{R}$ we have that cx^m is also continuous, being the product of continuous functions (using the product rule for limits again). From this, we see that each term $a_j x^j$ is continuous. Finally, the sum $\sum_{j=0}^{n} a_j x^j$ is continuous, being the sum of continuous functions (we are using the sum law for limits here--how?). From this we may conclude that every polynomial is indeed continuous. ■

Problem 14: *Draw three pictures. The first should depict a function continuous at $x = 1$. The second should depict a function with a jump discontinuity at $x = 1$. The third should depict a function which is discontinuous at $x = 1$, but which isn't a jump discontinuity. Discuss briefly how these last two don't satisfy what you wrote (three times!) in Problem 10, in particular, what is different between the two types of discontinuity.*

Solution: Pretty much any function will work for drawing a function continuous at $x = 1$ (e.g. a parabola). The key is that the limit of $f(x)$ as $x \to a$ is actually where the value is at that point.

A picture of a jump discontinuity at 1 would be something like the following (image credit: http://www.sagemath.org/calctut/pix-calctut/continuity04.png)



You see that the limit of $f(x)$ as $x \to 1$ exists, it's equal to 2, but that $f(2) = 3 \neq 2$.

The same image as above shows a discontinuity at $x = 2$, which is not a jump discontinuity. Namely, in this case, $\lim_{x\to 2} f(x)$ doesn't even exist! Indeed, $\lim_{x\to 2^+} f(x) = 3$ and $\lim_{x\to 2^-} f(x) = 1$.

The big difference between jump discontinuity, and the other type of discontinuity is whether or not the limit of the function at the point even exists. ∎

Problem 15:

a) *State the Intermediate Value Theorem. Draw a picture with a brief explanation to support the theorem.*

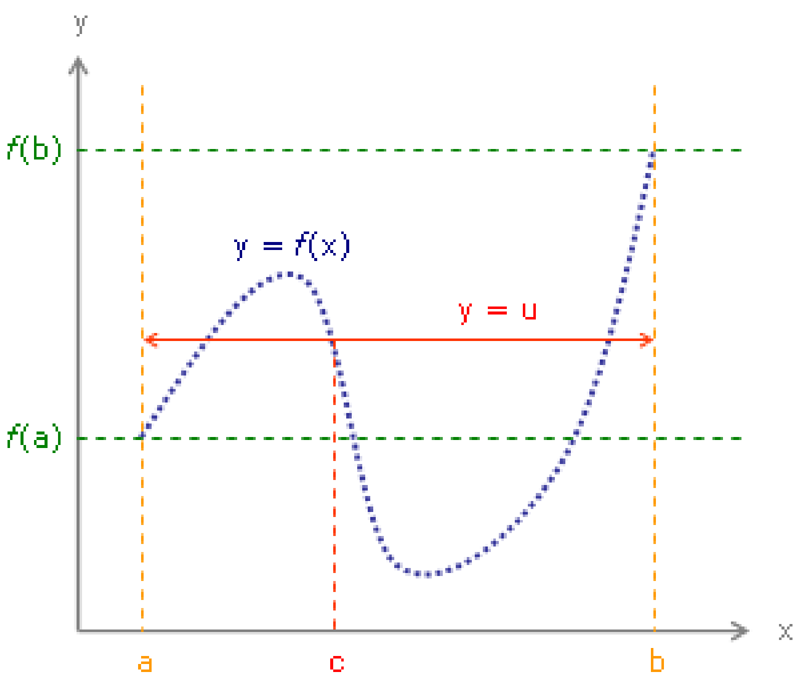
b) *Use the Intermediate Value Theorem to show that $f(x) = e^x - x - 2$ has a root in $[1,2]$.*

c) *Use the intermediate value theorem to show that $g(x) = x^5 - x + 1$ has a root on $[-1,1]$.*

Solution:

a) **Theorem (Intermediate Value Theorem):** *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) \leqslant u \leqslant f(b)$ or $f(b) \leqslant u \leqslant f(a)$, then there exists some $c \in [a,b]$ such that $f(c) = u$.*

A picture intuitively justifying the Intermediate Value Theorem would be something like the following (image credit: http://figures.boundless.com/50cf94e8e4b07bfa7a41efab/full/intermediatevaluetheore



You see that u is a value with $f(a) \leqslant u \leqslant f(b)$. What it should mean that there exists some value $c \in [a,b]$ such that $f(c) = u$, is that the graph of f should intersect the line $y = u$ somewhere on $[a,b]$. But, note that the graph of f must connect the points $(a, f(a))$ and $(b, f(b))$. There is clearly no way to connect these points without lifting your hand (i.e. continuously), and also not intersecting the line $y = u$. Thus, the graph of f must intersect the line at some point as desired.

b) Note that $f(x)$ is continuous on $[1,2]$. We merely note that $f(1) = e - 3 < 0$, and $f(2) = e^2 - 4 > 0$. Thus, by the Intermediate Value Theorem, since $f(1) < 0 < f(2)$ there must exist some $x_0 \in [1,2]$ such that $f(x_0) = 0$ as desired.

c) We note that $g(x)$ is continuous on $[-1,1]$. Note then that $g(-1) = -1$, and $g(1) = 1$. Thus, by the Intermediate Value Theorem there exists some $x_0 \in [-1,1]$ such that $g(x_0) = 0$ as desired.

One-sided Limits

Problem 16: *Write a rigorous definition of what the following means:* $\lim_{x\to a^-} f(x) = L$*. Do the same for* $\lim_{x\to a^+} f(x) = L$*.*

Solution: The definition of $\lim_{x\to a^+} f(x) = L$ is the following: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $a < x < a + \delta$ implies that $|f(x) - L| < \varepsilon$.

The definition of $\lim_{x\to a^-} f(x) = L$ is the following: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $a - \delta < x < a$ implies that $|f(x) - L| < \varepsilon$. ■

Problem 17: *Draw a picture of a function* $f(x)$ *such that* $\lim_{x\to 1^-} f(x) = 2$ *and* $\lim_{x\to 1^+} f(x) = -1$*. Find an actual such* $f(x)$*.*

Solution: A function whose graph looks like that of the graph of

$$f(x) = \begin{cases} 2 & \text{if} \quad x < 1 \\ \pi^e & \text{if} \quad x = 1 \\ -1 & \text{if} \quad x > 1 \end{cases}$$

works (did it matter what I declared $f(1)$ to be?). ■

Problem 18: *Draw a picture of a function* $f(x)$ *such* $\lim_{x\to 3^-} f(x)$ *exists, but* $\lim_{x\to 3^+} f(x)$ *does not exist.*

Solution: One such picture would be the graph of

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 3 \\ 100 & \text{if} \quad x = 3 \\ \dfrac{1}{x-3} & \text{if} \quad x > 3 \end{cases}$$

(does it matter what I put as $f(3)$?). ■

Problem 19: *State and prove the theorem relating the three notions of left-sided limit, right-sided limit, and limits (e.g. which two are equivalent to the third?).*

Solution: The statement is the following:

Theorem: *Let $f(x)$ be a real valued function. Then, $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$.*

Proof: Suppose first that $\lim_{x\to a} f(x) = L$. To see that $\lim_{x\to a^+} f(x) = L$, let $\varepsilon > 0$ be arbitrary. By assumption that $\lim_{x\to a} f(x) = L$, there exists some $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$. But, if $a < x < a + \delta$, then $0 < |x - a| < \delta$, and so we see that if $a < x < a + \delta$ we have that $|f(x) - L| < \varepsilon$. Since ε was arbitrary, it follows that $\lim_{x\to a^+} f(x) = L$.

To see that $\lim_{x\to a^-} f(x) = L$. Let $\varepsilon > 0$ be arbitrary. By assumption that $\lim_{x\to a} f(x) = L$, there exists some $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$. But, note that if $a-\delta < x < a$, then $0 < |x - a| < \delta$. Thus, if $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{x\to a^-} f(x) = L$ as desired.

Now, suppose that $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$. Let's show that $\lim_{x\to a} f(x) = L$. Let $\varepsilon > 0$ be given. By assumption that $\lim_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ such that if $a - \delta_1 < x < a$ then $|f(x)-L| < \varepsilon$. Similarly, since $\lim_{x\to a^+} f(x) = L$, there exists some $\delta_2 > 0$ such that if $a < x < a+\delta_2$, then $|f(x) - L| < \varepsilon$. Now, let $\delta = \min\{\delta_1, \delta_2\}$. Now, suppose that $0 < |x - a| < \delta$. If $x > a$, we see that $a < x < a + \delta \leqslant a + \delta_2$, in which case $|f(x) - L| < \varepsilon$ by definition of δ_1. If $x < a$, then $a > x > a - \delta \geqslant a - \delta_1$, and so $|f(x) - L| < \varepsilon$ by definition of δ_1. Regardless, we see that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the conclusion follows. ■

Limit Computations

Problem 20: *Compute*

$$\lim_{t\to -1} \frac{t^2+2t+1}{t+1}$$

Solution: We obviously can't just plug in -1, since the top and bottom are both zero at that point. But, noting that $t^2+2t+1=(t+1)^2$ we can see that for $t \neq -1$ we have that

$$\frac{t^2+2t+1}{t+1} = t+1$$

Thus, since limits are insensitive to the actual action of a function at the point x is approaching, we have that

$$\lim_{t\to -1} \frac{t^2+2t+1}{t+1} = \lim_{t\to -1}(t+1) = -1+1 = 0$$

■

Problem 21: *Let* $f(x) = \dfrac{3}{(x+1)^3}$*. State the definition of* $f'(x)$ *(in terms of a limit) and compute it.*

Solution: By definition, $f'(x)$ is equal to the following:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{3}{(x+1+h)^3} - \dfrac{3}{(x+1)^3}}{h}$$

But,

$$\begin{aligned}
\frac{\dfrac{3}{(x+1+h)^3} - \dfrac{3}{(x+1)^3}}{h} &= \frac{\dfrac{3(x+1)^3 - 3(x+1+h)^3}{(x+1+h)^3(x+1)^3}}{h} \\
&= \frac{3(x+1)^3 - 3(x+1+h)^3}{h(x+1+h)^3(x+1)^3} \\
&\overset{\star}{=} \frac{3(x+1-(x+1+h))((x+1)^2+(x+1)(x+1+h)+(x+1+h)^2)}{h(x+1+h)^3(x+1)^3} \\
&= \frac{-3h((x+1)^2+(x+1)(x+1+h)+(x+1+h)^2)}{h(x+1+h)^3(x+1)^3} \\
&= \frac{-3((x+1)^2+(x+1)(x+1+h)+(x+1+h)^2)}{(x+1+h)^3(x+1)^3}
\end{aligned}$$

where the starred step was achieved via the formula $x^3-y^3=(x-y)(x^2+xy+y^2)$. Now, note that in this last expression we can plug in $h=0$ to get

$$\frac{-3((x+1)^2+(x+1)^2+(x+1)^2)}{(x+1)^6} = \frac{-6(x+1)^2}{(x+1)^6} = \frac{-6}{(x+1)^4}$$

Thus, we see that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{(x+1+h)^3} - \frac{3}{(x+1)^3}}{h} \\ &= \lim_{h\to 0} \frac{-3((x+1)^2+(x+1)(x+1+h)+(x+1+h)^2)}{(x+1+h)^3(x+1)^3} \\ &= \frac{-6}{(x+1)^4} \end{aligned}$$

Note that this matches up with the formula we know for derivatives of polynomials (remember, you can't actually use this! It's just a checking technique).

Remark 5: The one tricky step of the above calculations is the usage of the factorization $x^3-y^3 = (x-y)(x^2+xy+y^2)$. Of course, you could have just went on a violent FOILing rampage. You would have gotten the right answer, but in the numerator you would have gotten a bunch of terms that happen to factor to $-6(x+1)^2$. Whatever makes you happy, and whatever works. ♦

Problem 22: *Compute*

$$\lim_{w\to 2} \frac{\sqrt{w+2}-2}{w-2}$$

Solution: As usual, we rationalize:

$$\begin{aligned} \frac{\sqrt{w+2}-2}{w-2} &= \frac{\sqrt{w+2}-2}{w-2}\,\frac{\sqrt{w+2}+2}{\sqrt{w+2}+2} \\ &= \frac{w+2-4}{(w-2)(\sqrt{w+2}+2)} \\ &= \frac{w-2}{(w-2)(\sqrt{w+2}+2)} \\ &= \frac{1}{\sqrt{w+2}+2} \end{aligned}$$

Now, note that we can just plug 2 into this last expression to get $\frac{1}{4}$. Thus, we see that

$$\lim_{w\to 0} \frac{\sqrt{w+2}-2}{w-2} = \lim_{w\to 0} \frac{1}{\sqrt{w+2}+2} = \frac{1}{4}$$

So, we're done.

Remark 6: The astute amongst you will realize that this is merely a limit describing $f'(2)$ for $f(w) = \sqrt{w+2}$. Indeed, note that $f'(w) = \frac{1}{2}(w+2)^{\frac{-1}{2}}$, so that $f'(2) = \frac{1}{2}4^{\frac{-1}{2}} = \frac{1}{4}$. ♦