Math 1A

Review Sheet

Author:

Alex Youcis

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Limits of Sequences

Problem 1: *Write a rigorous definition of what the following means:* $\lim\limits_{n \to \infty} a_n = c$ *(i.e. write the definition for a sequence to converge) three times.*

Problem 2: *Let* $a_n = \dfrac{n}{10n + 7}$. *Does* $\lim\limits_{n \to \infty} a_n$ *exist? If so, prove it. If not, justify.*

Problem 3: *Let* $a_n = \dfrac{1}{n^2 + 2n + 1}$. *Does* $\lim\limits_{n \to \infty} a_n$ *exist? If so, prove it. If not, justify.*

Problem 4: *State and prove the Squeeze Theorem for sequences. Use it to prove that*

$$\lim_{n \to \infty} \frac{n^2 + n + 48769504 + \sin(n)}{n^4 + \pi n^2 - 2} = 0$$

Limits of Functions

Problem 5: *Write a rigorous definition of what the following means: $\lim_{x \to a} f(x) = L$ (i.e. write the definition of a limit converging).*

Problem 6:

a) Use the definition of the limit to show that $\lim_{x \to 0} 1 = 1$.

b) Generalize the above to show that $\lim_{x \to 0} c = 1$ where c is any real constant.

c) Use the definition of the limit to show that $\lim_{x \to a} 1 = 1$ where a is any real number.

d) Generalize part c) to show that $\lim_{x \to a} c = c$ where a and c are any (possibly unrelated) real constants.

Problem 7: *Let $f(x) = \pi x - 1$. Does $\lim_{x \to \pi} f(x)$ exist? If so, prove it. If not, justify.*

Problem 8: *Let $f(x) = x^2 + 2x + 7$. Does $\lim_{x \to 1} f(x)$ exist? If so, prove it. If not, justify.*

Problem 9: *State and prove the Squeeze Theorem for functions (include a drawn picture of why it makes sense IN ADDITION to the proof). Use it to prove the following equality:*

$$\lim_{x \to 0} x^4 \sin\left(\frac{1}{x^2}\right)$$

(see discussion notes for a similar problem/hint).

Continuity and the Intermediate Value Theorem

Problem 10: *Write the definition of what it means for a function $f(x)$ to be continuous at $x = a$ three times.*

Problem 11: *Interpret the statement of Problem 6, d) as a statement about continuity.*

Problem 12: *Show that the function $f(x) = x$ is continuous at every point a of $\mathbb{R}$.*

Problem 13: *Use various limit laws (be sure to state clearly what they are!) as well as the conclusions of Problem 12 and Problem 11 to justify why any polynomial is continuous (hint: think about a polynomial as being built out of combinations of the function $f(x) = x$ and constant functions).*

Problem 14: *Draw three pictures. The first should depict a function continuous at $x = 1$. The second should depict a function with a jump discontinuity at $x = 1$. The third should depict a function which is discontinuous at $x = 1$, but which isn't a jump discontinuity. Discuss briefly how these last two don't satisfy what you wrote (three times!) in Problem 10, in particular, what is different between the two types of discontinuity.*

Problem 15:

a) *State the Intermediate Value Theorem. Draw a picture with a brief explanation to support the theorem.*

b) *Use the Intermediate Value Theorem to show that $f(x) = e^x - x - 2$ has a root in $[1, 2]$.*

c) *Use the intermediate value theorem to show that $g(x) = x^5 - x + 1$ has a solution on $[-1, 1]$.*

One-sided Limits

Problem 16: *Write a rigorous definition of what the following means: $\lim_{x \to a^-} f(x) = L$. Do the same for $\lim_{x \to a^+} f(x) = L$.*

Problem 17: *Draw a picture of a function $f(x)$ such that $\lim_{x \to 1^-} f(x) = 2$ and $\lim_{x \to 1^+} f(x) = -1$. Find an actual such $f(x)$.*

Problem 18: *Draw a picture of a function $f(x)$ such $\lim_{x \to 3^-} f(x)$ exists, but $\lim_{x \to 3^+} f(x)$ does not exist.*

Problem 19: *State and prove the theorem relating the three notions of left-sided limit, right-sided limit, and limits (e.g. which two are equivalent to the third?).*

Limit Computations

Problem 20: *Compute*

$$\lim_{t \to -1} \frac{t^2 + 2t + 1}{t + 1}$$

Problem 21: *Let* $f(x) = \dfrac{3}{(x + 1)^3}$. *State the definition of* $f'(x)$ *(in terms of a limit) and compute it.*

Problem 22: *Compute*

$$\lim_{w \to 2} \frac{\sqrt{w + 2} - 2}{w - 2}$$