SUMMATION FORMULAS

(*Last edited November 25, 2013 at 3:52pm.*)

Suppose we are asked to find the integral $\int_a^b x^4\,dx$ using the limit definition. We start to compute

$$\begin{aligned}\int_a^b x^4\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(a+i\frac{b-a}{n}\right)^4\frac{b-a}{n}\\ &= \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left(a^4+4a^3\frac{b-a}{n}i+6a^2\left(\frac{b-a}{n}\right)^2 i^2+4a\left(\frac{b-a}{n}\right)^3 i^3+\left(\frac{b-a}{n}\right)^4 i^4\right)\\ &= \lim_{n\to\infty}\frac{b-a}{n}\left(a^4\sum_{i=1}^{n}1+4a^3\frac{b-a}{n}\sum_{i=1}^{n}i+6a^2\left(\frac{b-a}{n}\right)^2\sum_{i=1}^{n}i^2+4a\left(\frac{b-a}{n}\right)^3\sum_{i=1}^{n}i^3+\left(\frac{b-a}{n}\right)^4\sum_{i=1}^{n}i^4\right)\ ;\end{aligned}$$

and we know $\sum_{i=1}^n 1 = n$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$, and $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$, but you may not know what the formula for $\sum_{i=1}^n i^4$ is. The purpose of this document is to explain how to find $\sum_{i=1}^n i^k$ for any n and k. To this end, define

$$S_{n,k} := \sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + (n-1)^k + n^k$$

and suppose that you want to find $S_{n,k}$ in terms of $S_{n,0}, S_{n,1}, \ldots, S_{n,k-1}$. We have

$$\begin{array}{rcccccccccc} S_{n,k+1} &=& \binom{k+1}{0}1^{k+1} &+& \binom{k+1}{0}2^{k+1} &+& \cdots &+& \binom{k+1}{0}(n-1)^{k+1} &+& \binom{k+1}{0}n^{k+1}\\ \binom{k+1}{1}S_{n,k} &=& \binom{k+1}{1}1^{k} &+& \binom{k+1}{1}2^{k} &+& \cdots &+& \binom{k+1}{1}(n-1)^{k} &+& \binom{k+1}{1}n^{k}\\ \vdots && \vdots && \vdots && \cdots && \vdots && \vdots\\ \binom{k+1}{k}S_{n,1} &=& \binom{k+1}{k}1^{1} &+& \binom{k+1}{k}2^{1} &+& \cdots &+& \binom{k+1}{k}(n-1)^{1} &+& \binom{k+1}{k}n^{1}\\ \binom{k+1}{0}S_{n,0} &=& \binom{k+1}{k+1}1^{0} &+& \binom{k+1}{k+1}2^{0} &+& \cdots &+& \binom{k+1}{k+1}(n-1)^{0} &+& \binom{k+1}{k+1}n^{0} \end{array}$$

and

$$\binom{k+1}{0}j^{k+1}+\binom{k+1}{1}j^{k}+\cdots+\binom{k+1}{k}j^{1}+\binom{k+1}{k+1}j^{0}=(j+1)^{k+1}$$

by the Binomial Theorem, so adding the columns above gives

$$\begin{array}{rcccccccccc} \sum_{i=0}^{k+1}\binom{k+1}{i}S_{n,k+1-i} &=& (1+1)^{k+1} &+& (2+1)^{k+1} &+& \cdots &+& (n-1+1)^{k+1} &+& (n+1)^{k+1}\\ &=& 2^{k+1} &+& 3^{k+1} &+& \cdots &+& n^{k+1} &+& (n+1)^{k+1}\\ &=& S_{n+1,k+1}-1 &&&&&&&& \end{array}$$

and we have $S_{n+1,k+1} = S_{n,k+1} + (n+1)^{k+1}$ so

$$\sum_{i=0}^{k+1} \binom{k+1}{i} S_{n,k+1-i} = S_{n,k+1} + (n+1)^{k+1} - 1$$

and

$$\sum_{i=1}^{k+1} \binom{k+1}{i} S_{n,k+1-i} = (n+1)^{k+1} - 1$$

thus

$$(k+1)S_{n,k} = (n+1)^{k+1} - 1 - \sum_{i=2}^{k+1} \binom{k+1}{i} S_{n,k+1-i}$$

and

$$\boxed{S_{n,k} = \frac{1}{k+1}\left((n+1)^{k+1} - 1 - \sum_{i=2}^{k+1} \binom{k+1}{i} S_{n,k+1-i}\right)}\,.$$

Let me verify the formula for the case $k = 3$:

$$\begin{aligned} S_{n,3} &= \frac{1}{4}\left((n+1)^4 - 1 - \left(6 \cdot \frac{n(n+1)(2n+1)}{6} + 4 \cdot \frac{n(n+1)}{2} + 1 \cdot n\right)\right) \\ &= \frac{1}{4}\big((n+1)^4 - 1 - (n(n+1)(2n+1) + 2n(n+1) + n)\big) \\ &= \frac{n+1}{4}\big((n+1)^3 - (n(2n+1) + 2n + 1)\big) \\ &= \frac{n+1}{4}\left(n^3 + n^2\right) \\ &= \left(\frac{n(n+1)}{2}\right)^2 . \end{aligned}$$