Here's a "practice midterm" I wrote. Note that this isn't really a practice midterm, since it's much longer than the midterm could ever be. To give you a sense of the amount of time you should expect to have on the midterm, for each problem I'm specifying my guess as to approximately how much time you'd have for such a problem on the exam. If you're a bit over time on some problems that's fine as long as you're under time on others. If you end up way over time on one type of problem you may want to continue studying that area, even if you know how to do the problems.

There are "solutions" at the end; these are not all well-written like you should write them on the exam, but should help you along if you get stuck and give you a way to check your work. However, most of the solutions to the newer material are adequately written, to give you an idea of how to write on the exam.

Also, some problems on this midterm are more challenging or otherwise less likely to appear on an exam - I've marked such problems by an asterisk (*). If I were you, I'd do all the non-asterisked problems first and make sure you understand them well, then if you have time go back and try the asterisked ones. By the way, if you get stuck anywhere or have other problems you'd like the answers to, try plugging them into Wolfram Alpha and clicking on "step-by-step solution" (you may need an account for this, but they're easy to make).

Finally, to give you a good sense of where you're at in terms of time and how capable you are at writing things up cleanly, try writing each question as you solve it as you would on an exam (especially the definitions/theorems). Then read through your solutions the next day and see if they still make sense (and verify your definitions/theorems) - if not, there's something that could be written up better. Finally, if you have questions about anything send me an email.

Practice Midterm

Problem 1: (12 mins)

1. Define the limit of a sequence. (In other words, write "$\lim_{n\to\infty} a_n = L$ means ....")
2. Prove that $\lim_{n\to\infty} \frac{n}{n-1} = 1$.
3. Prove that $\lim_{n\to\infty} \frac{1}{n^2+\sin n} = 0$.

Problem 2: (8 mins)

1. Define the limit of a function at ∞. (In other words, write "$\lim_{x\to\infty} f(x) = L$ means ....")
2. Prove that $\lim_{x\to\infty} x\sqrt{\frac{1}{x^3+1}} = 0$. (Hint: if that +1 weren't there, this would be much easier...)

Problem 3: (3 mins, +3 more for the asterisked part)

1. Define the limit of a function at a.
2. Define the right- and left-handed limits of a function at a.

3.* How would you change your answer to (1) to get a definition of continuity? Why is that equivalent to the definition of continuity you learned in class?

Problem 4: (8 mins each)

Without using limit laws:

1. Prove that $\lim_{x\to 3^-} \sqrt{3-x} = 0$.
2. Prove that $\lim_{x\to 4} x^2 - x + 1 = 13$.
3. Prove that $\lim_{x\to \frac{1}{2}} \frac{1-x}{1+x} = \frac{1}{3}$.
4. Prove that $\lim_{x\to 1} x^3 - 3x + 4 = 2$. (If you need help factoring, remember that $x - a$ must be a factor, so divide it out... or just give it to Wolfram Alpha.)

Problem 5*: (10 mins)

1.* Write down a definition that makes sense for $\lim_{x\to a} f(x) = \infty$. (Hint: you will not have an ϵ here.)

2.* Write down a definition for $\lim_{x\to -\infty} f(x) = -\infty$. (Hint: you will not have an ϵ or a δ here.)

3.* Briefly explain in words what your definitions are doing.

4.* Use your definition to prove that $\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$.

Problem 6: (3 mins + 8 mins + 12 mins)

1. State the various limit laws. Be explicit about when they apply (e.g., does $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ if $\lim_{x\to a} f(x)$ or $\lim_{x\to a} g(x)$ do not exist?).
2. Prove that $\lim_{x\to a} -6f(x) = -6 \lim_{x\to a} f(x)$.
3. Prove via the definition of limit that if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

Problem 7: (12 mins)

1. State the Squeeze Theorem.
2. Prove the Squeeze Theorem.
3. Prove that $\lim_{x\to 0} x^2 \sin(e^{1/x}) = 0$. (Name any theorems you use!)
4. Why could you NOT solve this problem with the product limit law, e.g. $\lim_{x\to 0} x^2 \sin(e^{1/x}) \neq \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin(e^{1/x})$?

Problem 8: (1 min + 8 mins + 15 mins)

1. State the definition of continuity.
2. Where is $f(x) = \lfloor x \rfloor$ continuous? Prove it. (That is, prove it is continuous everywhere you say it is, and prove that it isn't continuous everywhere you say it isn't; to do that, find the left- and right-handed limits.)

3.* Prove via the definition of limit that $\lim_{x \to 3} \lfloor x \rfloor$ doesn't exist. Conclude that $\lfloor x \rfloor$ isn't continuous at $x = 3$. (In other words, for any potential limit L find an ϵ, say $\epsilon = 0.3$ (anything $\leq \frac{1}{2}$ will work) such that there is no $\delta > 0$ such that whenever $0 < |x - 3| < \delta$, we have $|\lfloor x \rfloor - L| < \epsilon$. Note that this is essentially backwards from a usual limit problem. Also, to be completely formal you will need to use the Triangle Inequality!)

Problem 9: (3 mins, most of it for (1))

1. State the Intermediate Value Theorem. (Formally, not via a picture)
2. Illustrate your response to (1), and illustrate why the Intermediate Value Theorem does not hold when f is not continuous (or defined) on the relevant interval.
3. State the Triangle Inequality.

Problem 10: (3 mins + 3 mins + 7 mins)

1. Show that $f(x) = 2x^4 + 7x - 4$ has a root between $x = -1$ and $x = 2$.
2. Show that $f(x) = \sin(\pi x)$ intersects $g(x) = x^2$ somewhere between $x = \frac{1}{2}$ and $x = 1$. (Hint: subtract them and use the Intermediate Value Theorem.)

3.* Show that $f(x) = e^x - \frac{1}{x^2}$ has a zero between $x = -1$ and $x = 1$. (Careful! If you didn't notice that f isn't defined at 0, you did this problem wrong. Remember that for the Intermediate Value Theorem to apply, f must both exist and be continuous in the entire closed interval).

Problem 11: (5 mins each, more or less)

Compute the following limits via limit laws, or show that they don't exist (usually by computing left- and right-handed limits).

1. $\lim_{x \to 1} \frac{x^2 - 1}{|x - 1|}$
2. $\lim_{x \to 0^+} x\sqrt{1 + \sqrt{1 + \sqrt{1 + \frac{1}{x^8}}}}$ What if the 8 were replaced by a 7 or 9? What if it were a 2-sided limit (a bit tricky)?
3. $\lim_{x \to \infty} \frac{3000x^2 - 2x + 7}{x^3 - x + 1}$
4. $\lim_{x \to -\infty} x - \frac{x}{1 + \frac{1}{x}}$

5. $\lim_{t\to 1} \frac{3-\sqrt{t+8}}{t-1}$

6. $\lim_{x\to 3} \lceil x \rceil$

Problem 12: (2 mins + 7 mins + 12 mins + 5 mins)

For this problem, you may use any limit laws you wish without explanation.

1. Write down the (limit) definition of derivative.

2. Prove that $\frac{d}{dx}\left(x^2\right) = 2x$ by the above definition.

3. Prove that $\frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(x-1)^2}$ by the above definition.

4. Prove that if $f(x)$ and $g(x)$ are differentiable, then $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$.

Problem 13: (10 mins)

1. Let $f(x) = \begin{cases} x^2 - 2x + 2 & x < 0 \\ e^x + 1 & 0 \le x < 1 \\ \cos(\pi x) & x \ge 1 \end{cases}$ Where is f continuous? Prove it. (Like above, prove it is continuous everywhere you say it is, and prove that it isn't continuous everywhere you say it isn't.)

Problem 14: (7 mins)

1. Sketch the graph of a single function $f(x)$ with all the following properties:
(a) f has domain $\mathbb{R} - \{1\}$ (b) $\lim_{x\to 1} f(x) = 2$ (c) $\lim_{x\to 3^+} f(x) = 6$ (d) $\lim_{x\to 3^-} f(x) = -1$
(e) $f(3) = 0$ (f) $\lim_{x\to -4} f(x) = 5$ (g) f is discontinuous at $x = -4$ (h) $\lim_{x\to 7} f(x) = \infty$
(i) $f(7) = 2$ (j) $\lim_{x\to\infty} f(x) = 1$ (k) $\lim_{x\to -\infty} f(x) = -\infty$

Problem 15: (5 mins)

1. Find the slope of the function $f(x) = \sqrt[4]{x}$ at $x = 0$.

2. Why doesn't that conflict with the definition of function (i.e., the vertical line test)?

Solutions

Problem 1: (12 mins)

1. $\lim_{n\to\infty} a_n = L$ means: for any $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|a_n - L| < \epsilon$. (N is usually assumed to be an integer).

2. We have $|a_n - L| = \left|\frac{n}{n-1} - 1\right| = \left|\frac{1}{n-1}\right| = \frac{1}{n-1}$. We want this $< \epsilon$, so set $\frac{1}{n-1} = \epsilon$ to get $N \geq 1 + \frac{1}{\epsilon}$. Then $N = \lceil 1 + \frac{1}{\epsilon} \rceil$ will work (prove it!).

3. Use the fact that $\left|\frac{1}{n^2+\sin n} - 0\right| \leq \left|\frac{1}{n^2-1}\right|$, since $-1 \leq \sin n \leq 1$. Then $|a_n - L| = \left|\frac{1}{n^2+\sin n} - 0\right| \leq \frac{1}{n^2-1}$. We want $|a_n - L| < \epsilon$, so set $\frac{1}{n^2-1} = \epsilon$ to get $N = \lceil \sqrt{1 + \frac{1}{\epsilon}} \rceil$.

Problem 2: (8 mins)

1. $\lim_{x\to\infty} f(x) = L$ means: for any $\epsilon > 0$, there exists an N such that if $x \geq N$, then $|f(x) - L| < \epsilon$. (Note the similarity to the definition of limit of a sequence.)

2. We have $|f(x) - L| = \left|x\sqrt{\frac{1}{x^3+1}} - 0\right| \leq \left|x\sqrt{\frac{1}{x^3}}\right| = \sqrt{\frac{1}{x}}$. We want this to be $< \epsilon$, so set $\sqrt{\frac{1}{x}}$ equal to ϵ to get $N = \frac{1}{\epsilon^2}$ (that is, for $x > N = \frac{1}{\epsilon^2}$ we have $|f(x) - L| = ... = \sqrt{\frac{1}{x}} < \epsilon$).

Problem 3: (3 mins, +3 more for the asterisked part)

1. $\lim_{x\to a} f(x) = L$ means: for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.
2.
   - $\lim_{x\to a^+} f(x) = L$ means: for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $a < x < a + \delta$, then $|f(x) - L| < \epsilon$. (You may also say $0 < x - a < \delta$.)
   - $\lim_{x\to a^-} f(x) = L$ means: for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $a - \delta < x < a$, then $|f(x) - L| < \epsilon$. (You may also say $-\delta < x - a < 0$.)

3.* Replace $0 < |x - a| < \delta$ with $0 \leq |x - a| < \delta$ (same thing as just $|x - a| < \delta$). Now $f(a)$ itself is also required to be within any ϵ of L, which can only happen when $f(a) = L = \lim_{x\to a} f(x)$, the definition of continuity.

Problem 4: (8 mins each)

1. Assume $3 - \delta < x < 3$, or $-\delta < x - 3 < 0$. Negate this inequality to get $0 < 3 - x < \delta$, and relate to $|f(x) - L|$:
   $|f(x) - L| = |\sqrt{3 - x} - 0| = \sqrt{3 - x} < \sqrt{\delta}$. You want $|f(x) - L| < \epsilon$, so pick $\delta = \epsilon^2$ (prove it works!).

2. Assume $0 < |x - 4| < \delta$. Write out $|f(x) - L| = |x^2 - x + 1 - 13| = |x - 4| \cdot |x + 3|$. Assuming $\delta \leq 1$, you know $|x + 3| \leq 8$. Then $\delta = \min(\frac{\epsilon}{8}, 1)$ should work.

3. Assume $0 < |x - \frac{1}{2}| < \delta$. Write out $|f(x) - L| = \left|\frac{1-x}{1+x} - \frac{1}{3}\right| = \left|\frac{2-4x}{3(1+x)}\right| = |-4| \cdot \frac{|x-\frac{1}{2}|}{|3(1+x)|} < 4\delta \cdot \frac{1}{|3(1+x)|}$. Assuming $\delta \leq 1$, you have $-\frac{1}{2} \leq x \leq \frac{3}{2}$, so $\frac{1}{|3(1+x)|} \leq \frac{2}{3}$ (the function is maximized for smallest x). Then $|f(x) - L| < 4\delta \cdot \frac{2}{3} = \frac{8}{3}\delta$, so picking $\delta = \min(\frac{3}{8}\epsilon, 1)$ should work (prove it!).

4. Assume $0 < |x - 1| < \delta$. Factor to get $|f(x) - L| = |x^2 - 3x + 4 - 2| = |x - 1|^2 \cdot |x + 2|$. Assuming $\delta \leq 1$, this is $< 4\delta^2$ and you want it $< \epsilon$, so pick $\delta = \frac{\sqrt{\epsilon}}{2}$ (how'd I get that? Set $4\delta^2$ equal to ϵ and solve for δ). Thus $\delta = \min(\frac{\sqrt{\epsilon}}{2}, 1)$ works. Remember, you still need to prove it.

Problem 5*: (8 mins)

1.* $\lim_{x \to a} f(x) = \infty$ means: for any M, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) \geq M$. (You can replace $f(x) \geq M$ with $f(x) > M$, if you prefer.)

2.* $\lim_{x \to -\infty} f(x) = -\infty$ means: for any M, there exists an N such that if $x \leq N$, then $f(x) \leq M$. (Either or both inequalities can be $<$ or $\leq$ signs with no change in meaning.)

Note how modular all these definitions of limits are. Just replace $0 < |x - a| < \delta$ with $x > N$ or $x < N$, and replace $|f(x) - L| < \epsilon$ with $f(x) \geq M$ or $f(x) \leq M$ depending on what you want to define.

3.* (1) says that $f(x)$ gets as large as we want close enough to a. (2) says that $f(x)$ gets as small (negative) as we want (and stays that small) for small (negative) enough x.

4.* If $0 < |x - 3| < \delta$, then $\frac{1}{(x-3)^2} \geq \frac{1}{\delta^2}$. We want $f(x) = \frac{1}{(x-3)^2} \geq M$, so set $\frac{1}{\delta^2} = M$ to get $\delta = \frac{1}{\sqrt{M}}$. Whenever $0 < |x - 3| < \delta = \frac{1}{\sqrt{M}}$, we have $f(x) = \frac{1}{(x-3)^2} > \frac{1}{\delta^2} = \frac{1}{\left(\frac{1}{\sqrt{M}}\right)^2} = M$, as desired.

Problem 6: (3 mins + 8 mins + 12 mins)

1. Look them up in your book (or online if you don't have a book). Note that the product, sum, and quotient limit laws in particular only apply if each individual limit exists. E.g., $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ only applies if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Remember, indeterminate forms ($\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$, 0^0, ∞^0, 1^∞) tell you nothing and mean you must take a different approach, or at least try to simplify and cancel before plugging in.

2. Because $\lim_{x \to a} f(x) = L$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| \leq \frac{\epsilon}{6}$. Whenever $0 < |x - a| < \delta$ (the same delta), you then have $|-6f(x) - (-6L)| = |6(f(x) - L)| = 6|f(x) - L| < 6 \cdot \frac{\epsilon}{6} = \epsilon$, so $\lim_{x \to a} -6f(x) = -6L$.

3. Because $\lim_{x\to a^+} f(x) = L$, there exists a δ_1 such that whenever $a < x < a+\delta$, then $|f(x)-L| < \epsilon$. Similarly, because $\lim_{x\to a-} f(x) = L$, there exists a δ_2 such that whenever $a-\delta < x < a$, then $|f(x)-L| < \epsilon$. Pick $\delta = \min(\delta_1, \delta_2)$. Whenever $0 < |x-a| < \delta$, we have either $a < x < a+\delta$ or $a-\delta < x < a$ (that is, x is within δ of a either to the right or to the left). Since $\delta \leq \delta_1$ and $\delta \leq \delta_2$, that means we either have $a < x < a+\delta_1$ or $a-\delta_2 < x < a$. In either case, by above we have $|f(x)-L| < \epsilon$, so $|f(x)-L| < \epsilon$ whenever $0 < |x-a| < \delta$, and thus $\lim_{x\to a} f(x) = L$.

Problem 7: (12 mins)

1. The Squeeze Theorem states: if f, g, and h are real-valued functions and $f(x) \leq g(x) \leq h(x)$ near a (except maybe at a itself) and if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$ as well.

2. Prof. Coward emailed out a nice writeup of this proof. Please read it through and understand it, and be able to replicate it (test yourself on this!) before the exam.

3. Since $-1 \leq \sin(\text{blah}) \leq 1$, we know $-x^2 \leq x^2\sin(e^{1/x}) \leq x^2$. Clearly $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$ (I believe on an exam you can just state something this clear in a Squeeze Theorem question without formal proof; maybe state limit laws if you're concerned), so by the Squeeze Theorem we have that $\lim_{x\to 0} x^2\sin(e^{1/x}) = 0$ as well. Don't forget to state "by the Squeeze Theorem" in your solution.

4. The product limit law only applies when each of the constituent limits exists. Because $\lim_{x\to 0}\sin(e^{\frac{1}{x}})$ doesn't exist (it oscillates faster and faster), the limit law does not apply. Informally, $0 \cdot \text{DNE} \neq 0$. Instead, you must use the Squeeze Theorem.

Problem 8: (1 min + 8 mins + 15 mins)

1. $f(x)$ is continuous at a means that $\lim_{x\to a} f(x) = f(a)$. This also means that $\lim_{x\to a} f(x)$ exists. Frequently one "expands" this out as $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$. f is said to be continuous if it is continuous at all a in its domain (all a at which it is defined).

2. For non-integer a, near a we know $f(x)$ is the constant $\lfloor a \rfloor$, so $\lim_{x\to a} f(x) = \lfloor a \rfloor = f(a)$ and hence f is continuous at a. (try to see how to prove this more formally). At integer a,
$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} \lfloor x \rfloor = \lim_{x\to a^+} a = a$. and

   $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \lfloor x \rfloor = \lim_{x\to a^-} a-1 = a-1$.
   Since $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, overall $\lim_{x\to a} f(x)$ doesn't exist, and so f is not continuous at integers (note that if the limits matched, we'd still have to check if they were equal to $f(a)$, which here would be a).

3.* Pick $\epsilon = 0.3$. We claim that for no $\delta > 0$ is it true that whenever $0 < |x-3| < \delta$, we have $|f(x) - L| = |\lfloor x \rfloor - L| < \epsilon$. Suppose it were. For any $3 < x < 3+\delta$, say $x = 3 + \frac{\delta}{2}$ to be explicit, we have $f(x) = 3$ (note: I'm assuming $\delta < 1$; if not, pick $x = 3.5$, say). Hence $|f(x) - L| < \epsilon$, so $|3 - L| < \epsilon$. Similarly, for any $3 - \delta < x < 3$, say $x = 3 - \frac{\delta}{2}$ (or $x = 2.5$ if $\delta \geq 1$) we have $f(x) = 2$, so $|2 - L| < \epsilon$ also. In other words, L is within less than ϵ of both 2 and 3, which is clearly impossible if $\epsilon \leq 0.5$. To prove this formally, note that $|2-3| = |2-3+L-L| = |(2-L)+(3-L)| \leq |2-L|+|L-3| = |2-L|+|3-L| < \epsilon+\epsilon = 2\epsilon$ by the Triangle Inequality. But $|2-3| = 1$ and $2\epsilon = 0.6 < 1$ for $\epsilon = 0.3$, so we have $1 < 0.6$, which is impossible. Hence no such δ could possibly exist, and so $\lim_{x\to 3} f(x) = \lim_{x \to 3} \lfloor x \rfloor$ doesn't exist (it is not true that for any $\epsilon > 0$ we can find a δ that works, since we just showed that for $\epsilon = 0.3$ there's no δ that works).

Problem 9: (3 mins, most of it for (1))

1. The Intermediate Value Theorem says: if $f(x)$ is a real-valued function defined and continuous on $[a, b]$ and N is strictly between $f(a)$ and $f(b)$ (so either $f(a) < N < f(b)$ or $f(b) < N < f(a)$), then there exists some c in (a, b) such that $f(c) = N$. (Note that c needn't be unique.)

2. Look up pictures about the Intermediate Value Theorem in your book or online. Remember, a continuous function intuitively is one that does not "jump," that you can draw without removing your pencil from the paper (nowhere near formal enough for use on an exam). However, technically a function that "jumps" because it isn't defined at a point can still be continuous, so be careful of that (that's why the function must be defined on $[a, b]$).

3. The Triangle Inequality says: for real numbers a and b, we have $|a + b| \leq |a| + |b|$.

Problem 10: (3 mins + 3 mins + 7 mins)

1. Note that $f(-1) = -9$ and $f(x) = 42$, which are on either side of 0. f is a polynomial and thus continuous, so by the Intermediate Value Theorem there exists $-1 < c < 2$ such that $f(c) = 0$, i.e. a root of f. Remember to state "by the Intermediate Value Theorem" in your solution, as well as that f is continuous and the reason you know f is continuous.

2. f intersects g exactly when $f(x) - g(x) = 0$. Consider $h(x) = f(x) - g(x) = \sin(\pi x) - x^2$. At $x = \frac{1}{2}$, $h(\frac{1}{2}) = \frac{3}{4} > 0$ and at $x = \frac{1}{2}$, $h(1) = -1 < 0$. h is continuous since it's composed of polynomials and trig functions, so by the Intermediate Value Theorem there exists $\frac{1}{2} < c < 1$ such that $h(c) = 0$, which means $f(c) - g(c) = 0$. Thus $f(c) = g(c)$, so f and g intersect between $\frac{1}{2}$ and 1.

3.* Note that $f(0)$ isn't defined. f is continuous (i.e., continuous everywhere it's defined), but the Intermediate Value Theorem requires that f be defined on the entire relevant interval, else it can still "jump" (draw a picture of this). We have $f(-1) < 0$ and $f(1) > 0$. Note that $\lim_{x \to 0} f(x) = -\infty$. So for very small positive or negative x, $f(x)$ is very negative. Hence we'd expect a root between 0 and 1, and maybe not between -1 and 0 (in fact, there isn't one there). For $x = 0.001$, say, $f(x) < 0$ (either plug it in and check, or just say it's obvious). Because f is continuous and defined between 0.001 and 1 and $f(0.001) < 0 < f(1)$, by the Intermediate Value Theorem there exists c between 0.001 and 1 such that $f(c) = 0$. Hence f has a root between 0.001 and 1, which is definitely also between -1 and 1.

Problem 11: (5 mins each, more or less)

1. Take the limits from left and right separately (a good strategy when you have absolute values that are 0 at a):
$\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1^+} x+1 = 2$
$\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \frac{(x+1)(x-1)}{-(x-1)} = \lim_{x\to 1^-} -(x+1) = -2$
These aren't the same, so $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$ doesn't exist.

2. Keep bringing the power of x into the square root:
$\lim_{x\to 0^+} x\sqrt{1+\sqrt{1+\sqrt{1+\frac{1}{x^8}}}} = \lim_{x\to 0^+} \sqrt{x^2+x^2\sqrt{1+\sqrt{1+\frac{1}{x^8}}}} = \lim_{x\to 0^+} \sqrt{x^2+\sqrt{x^4+x^4\sqrt{1+\frac{1}{x^8}}}} =$
$\lim_{x\to 0^+} \sqrt{x^2+\sqrt{x^4+\sqrt{x^8+1}}} = \sqrt{\sqrt{\sqrt{1}}} = 1.$
If it were a 7, we'd get $\lim_{x\to 0^+} \sqrt{x^2+\sqrt{x^4+\sqrt{x^8+x}}} = 0$ instead. If it were a 9, we'd get
$\lim_{x\to 0^+} \sqrt{x^2+\sqrt{x^4+\sqrt{x^8+\frac{1}{x}}}} = \infty.$
Note that the limit from the left doesn't make sense for 7 or 9 - the function isn't defined left of 0 (since the part inside the innermost square root becomes very negative just left of 0). For 8, the limit from the left is actually -1, so the 2-sided limit doesn't exist. Why is it not still 1? Because bringing x inside the first square root assumes that x is positive; you can only bring positive quantities inside square roots like that. If you bring in a negative value, you must also negate the square root (that is, $a\sqrt{b} = \text{sign}(a)\sqrt{a^2b}$, where $\text{sign}(a)$ is 1 if a is positive and -1 if a is negative).

3. Multiply by $\frac{1}{x^3}$ on top and bottom:
$\lim_{x\to\infty} \frac{3000x^2-2x+7}{x^3-x+1} = \lim_{x\to\infty} \frac{3000x^2-2x+7}{x^3-x+1}\cdot\frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{3000\frac{1}{x}-2\frac{1}{x^2}+\frac{7}{x^3}}{1-\frac{1}{x^2}+\frac{1}{x^3}} = \frac{\lim_{x\to\infty}\left(3000\frac{1}{x}-2\frac{1}{x^2}+\frac{7}{x^3}\right)}{\lim_{x\to\infty}\left(1-\frac{1}{x^2}+\frac{1}{x^3}\right)} = \frac{0}{1} = 0$

4. The answer is not at first obvious ($-\infty-(-\infty)$), so put everything over a common denominator:
$\lim_{x\to-\infty} x-\frac{x}{1+\frac{1}{x}} = \lim_{x\to-\infty} x-\frac{x}{\frac{x+1}{x}} = \lim_{x\to-\infty} x-\frac{x^2}{x+1} = \lim_{x\to-\infty} \frac{x(x+1)-x^2}{x+1} = \lim_{x\to-\infty} \frac{x}{x+1}$
$= \lim_{x\to-\infty} \frac{x}{x+1}\cdot\frac{1/x}{1/x} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \frac{\lim_{x\to-\infty} 1}{\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)} = \frac{1}{1} = 1.$
It would also have worked to put the x and the $-\frac{x}{1+\frac{1}{x}}$ over a common denominator of $1+\frac{1}{x}$ first.

5. The answer is not at first obvious (it goes to $\frac{0}{0}$, an indeterminate form), so multiply by the conjugate $3+\sqrt{t+8}$:
$\lim_{t\to 1} \frac{3-\sqrt{t+8}}{t-1} = \lim_{t\to 1} \frac{3-\sqrt{t+8}}{t-1}\cdot\frac{3+\sqrt{t+8}}{3+\sqrt{t+8}} = \lim_{t\to 1} \frac{9-(t+8)}{(t-1)(3+\sqrt{t+8}} = \lim_{t\to 1} \frac{1-t}{(t-1)(3+\sqrt{t+8})} = \lim_{t\to 1} \frac{-1}{(3+\sqrt{t+8})} = \frac{-1}{6}.$

6. Find the limits from the right and left separately:
$\lim_{x\to 3^+} \lceil x\rceil = \lim_{x\to 3^+} 4 = 4$
$\lim_{x\to 3^-} \lceil x\rceil = \lim_{x\to 3^-} 3 = 3$
These aren't equal, so the 2-sided limit $\lim_{x\to 3}\lceil x\rceil$ doesn't exist (just like for Problem 8.2).

Problem 12: (2 mins + 7 mins + 12 mins + 5 mins)

1. The derivative of $f(x)$ at x is defined as:
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, or equivalently as
$f'(x) = \lim_{x_1\to x} \frac{f(x_1)-f(x)}{x_1-x}$.
Personally, I think the first definition is easier to use in practice, so I'll use it for the remaining parts of this problem and Problem 15.

2. If $f(x) = x^2$, we have
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2hx+h^2-x^2}{h} = \lim_{h\to 0} \frac{2hx+h^2}{h} = \lim_{h\to 0} 2x+h = 2x$

3. This one's nastier and involves putting things over a common denominator, but otherwise just like the previous one. If $f(x) = \frac{x}{1-x}$, we have
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(\frac{x+h}{1-(x+h)}\right)-\left(\frac{x}{1-x}\right)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)(1-x)-x(1-x-h)}{(1-x-h)(1-x)}}{h}$
$= \lim_{h\to 0} \frac{h}{h(1-x-h)(1-x)} = \lim_{h\to 0} \frac{1}{(1-x-h)(1-x)} = \frac{1}{(1-x)^2} = \frac{1}{(x-1)^2}$

4. If f and g are differentiable, we know the limits of each of their respective difference quotients exists. Then we have
$\frac{d}{dx}(f+g)(x) = \lim_{h\to 0} \frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h} = \lim_{h\to 0} \frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h}$
$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f'(x) + g'(x)$
Splitting the limit into two limits is justified by the sum limit law and the aforementioned fact that each of the two summand limits exists (because f and g are differentiable). Remember, if you're given an assumption, you probably need to use it somewhere; our logic wouldn't hold if f or g weren't differentiable.

Problem 13: (10 mins)

1. $f(x)$ is clearly continuous for $x \neq 0, 1$. Why? Because on the open intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ we know $f(x)$ is a composition of polynomials, exponentials and trig functions, and hence continuous.
For $x = 0$ and 1 we must check the definition of continuity: $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$ (same thing as $\lim_{x\to a} f(x) = f(a)$):
<u>**x=0:**</u>
   - $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x + 1 = 2$
   - $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 - 2x + 2 = 2$
   - $f(0) = e^0 + 1 = 2$.

   These are all equal, so $f(x)$ is continuous at 0.
<u>**x=1:**</u>
   - $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \cos(\pi x) = -1$
   - $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} e^x + 1 = e + 1$
   - $f(1) = \cos(\pi \cdot 1) = -1$.

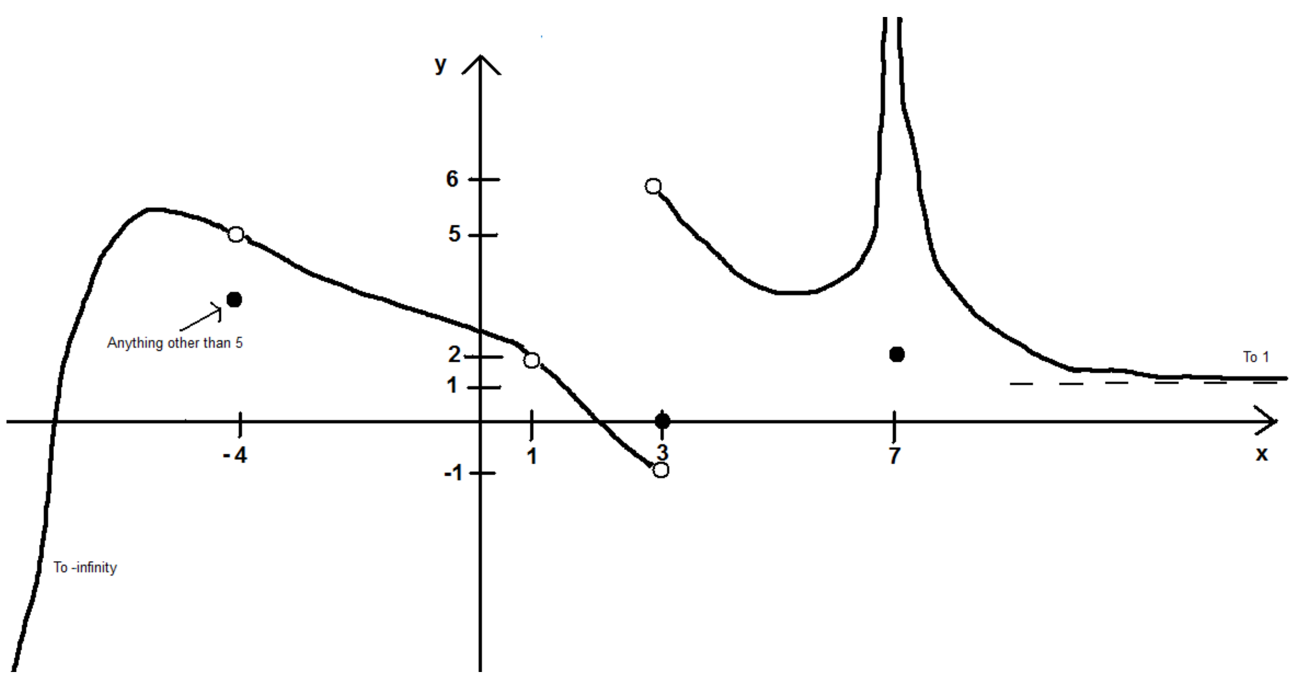
These aren't all equal, so $f(x)$ is not continuous at 1 (in fact, $\lim_{x\to 1} f(x)$ doesn't exist, so we didn't really need to find $f(1)$ in this case; but don't forget to check $f(1)$ when the limit does exist).

Hence f is continuous everywhere except 1; we can write this as $\mathbb{R} - \{1\}$ or as $(\infty, 1) \cup (1, \infty)$. We're essentially just "plugging in" 0 and 1 into the different parts of f to see if they match up. As simple as that sounds, you should write your solution along the same lines as this one in order to not lose points on an exam (don't just say "by plugging in 0 and 1..."), and to demonstrate that you know formally what continuity means ($\lim_{x\to a} f(x) = f(a)$).

Problem 14: (7 mins)

1.
   - At $x = 1$ f isn't defined, but has a limit of 2 there: it has a hole there, at $(1, 2)$.
   - f jumps from -1 to 6 at $x = 3$, with $f(3) = 0$ neither of those. Don't forget to draw the points at $(3, -1)$ and $(3, 6)$ as open circles, and the point at $(3, 0)$ as a filled-in circle.
   - At $x = -4$ f has a limit of 5, but is discontinuous there, so $f(-4) \neq 5$ - there should be a hole at $(-4, 5)$ and a point at $(-4, \_)$ anywhere but at height 5.
   - At $x = 7$ f approaches ∞ from both sides, but has a point at $(7, 2)$.
   - As x gets large f asymptotes to 1 (i.e., a horizontal asymptote to the right of 1).
   - As x gets small (negative) f goes to $-\infty$, so f keeps getting arbitrarily smaller to the left.

See the figure below:



Problem 15: (5 mins)

1. The slope of $f(x)$ at $x = 0$ is given by the derivative of $f(x)$ evaluated at 0, i.e. $\frac{d}{dx}(\sqrt[4]{x})\big|_{x=0}$. We'll compute it, but plug in 0 before we attempt the limit. Note that we can't plug in 0 *before* we write out the limit, since then we'd just be differentiating 0, which isn't what we're asking for. Finally, note that this must be a 1-sided derivative, and hence be defined in terms of a 1-sided limit, since $\sqrt[4]{x}$ isn't defined for $x < 0$:
   $\frac{d}{dx}(\sqrt[4]{x})\big|_{x=0} = \lim_{h\to 0^+} \frac{\sqrt[4]{x+h} - \sqrt[4]{x}}{h}\big|_{x=0} = \lim_{h\to 0^+} \frac{\sqrt[4]{h} - 0}{h} = \lim_{h\to 0^+} \frac{1}{h^{3/4}} = \infty$
   Hence the slope of $f(x)$ is infinite at $x = 0$.

2. One would expect that a slope of ∞ means that the function goes vertically upwards for a ways, and so doesn't pass the vertical line test for functions (namely, any vertical line hits a function in at most one spot). However, infinitesimally past 0 (to the upper-right) the slope of $f(x)$ is no longer ∞, so this isn't actually the case. For the same reason a function that has a slope of 0 somewhere (a horizontal tangent) isn't necessarily flat for any length – think of $y(x) = x^2$.