## MATH 1A QUIZ 1 SOLUTION

Please write your solutions on a separate sheet of paper. Be sure to write your name and section number at the top of each page.

**Problem 1.** (15 points)

(i) State the Squeeze Theorem.
(ii) Prove the Squeeze Theorem.
(iii) Use the Squeeze Theorem to find

$$\lim_{x\to 0} \frac{x^4}{10} \cos \frac{2\pi}{5x} \, .$$

Justify your answer carefully.

*Solution.* (i) If

$$f(x) \le g(x) \le h(x) \tag{1}$$

when $x$ is near $a$ (except possibly at $a$) and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$. The meaning of "near $a$ (except possibly at $a$)" is: there exists some $\delta_1$ such that for all $x$ satisfying $0 < |x-a| < \delta_1$, we have (1).

(ii) Suppose $f, g, h$ satisfy (1) when $0 < |x-a| < \delta_1$ for some fixed $\delta_1$. Also suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|f(x) - L| < \varepsilon$. Also, since $\lim_{x\to a} h(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x-a| < \delta_3$, then $|h(x) - L| < \varepsilon$. Put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Suppose $0 < |x-a| < \delta$. Then (1), $L - \varepsilon < f(x) < L + \varepsilon$ and $L - \varepsilon < h(x) < L + \varepsilon$ hold. Hence $g(x) \le h(x) < L + \varepsilon$. Also $L - \varepsilon < f(x) \le g(x)$. Thus $L - \varepsilon < g(x) < L + \varepsilon$, so $|g(x) - L| < \varepsilon$. Hence $\lim_{x\to a} g(x) = L$.

(iii) Set $f(x) = -\frac{x^4}{10}$, $g(x) = \frac{x^4}{10}\cos\frac{2\pi}{5x}$, and $h(x) = \frac{x^4}{10}$. Note that

$$-1 \le \cos \frac{2\pi}{5x} \le 1 \tag{2}$$

for all $x \ne 0$, so multiplying (2) by $\frac{x^4}{10}$ implies that (1) holds for all $x \ne 0$. (Here we can take $\delta_1$ to be any positive real number.) Note that $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$. Hence, by the Squeeze Theorem, $\lim_{x\to 0} g(x) = 0$.

□

**Problem 2.** (30 points)

(i) State the definition of limit for sequences (i.e. what exactly does $\lim_{n\to\infty} f(n) = L$ mean?).
(ii) Prove that

$$\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0 \, .$$

(iii) Prove that

$$\lim_{n\to\infty} \frac{n^3-1}{n^3} = 1 .$$

*Solution.* (i) The statement "$\lim_{n\to\infty} f(n) = L$" means "For every $\varepsilon > 0$, there exists $N > 0$ such that if $n \geq N$ then $|f(n) - L| < \varepsilon$."

(ii) Let $\varepsilon > 0$. Set $N = \lceil \log_{3/4} \varepsilon \rceil + 1$. Then for $n \geq N$, we have $n > \log_{3/4} \varepsilon$ so $(\frac{3}{4})^n < \varepsilon$ (the direction of the inequality is switched because $\frac{3}{4} < 1$). So $|f(n) - L| = |(\frac{3}{4})^n - 0| = (\frac{3}{4})^n < \varepsilon$.

(iii) Let $\varepsilon > 0$. Set $N = \lceil \frac{1}{\sqrt[3]{\varepsilon}} \rceil + 1$. If $n \geq N$, then $n > \frac{1}{\sqrt[3]{\varepsilon}}$, so $\frac{1}{n^3} < \varepsilon$. Thus $|\frac{n^3-1}{n^3} - 1| = \frac{1}{n^3} < \varepsilon$.

$\square$

**Problem 3.** (35 points)

(i) State the definition of limit for functions (i.e. what exactly does $\lim_{x\to a} f(x) = L$ mean?).

(ii) Let $f(x) = \sqrt{x-3}$. Find a real number $\delta$ such that the following is true: if $x$ is a real number such that $0 < |x - 7| < \delta$, then $|f(x) - 2| < \frac{1}{3}$.

(iii) Prove that

$$\lim_{x\to 0} x^{43} = 0 .$$

(iv) Prove that

$$\lim_{x\to 3} x^2 - 4x = -3 .$$

*Solution.* (i) The statement "$\lim_{x\to a} f(x) = L$" means "For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$."

(ii) Here, $f(x) = \sqrt{x-3}$, $a = 7$, and $L = 2$. We have

$$\begin{aligned} & |f(x) - 2| < \frac{1}{3} \\ \iff\ & 2 - \frac{1}{3} < \sqrt{x-3} < 2 + \frac{1}{3} \\ \iff\ & \frac{25}{9} < x - 3 < \frac{49}{9} \\ \iff\ & -\frac{11}{9} < x - 7 < \frac{13}{9} \end{aligned}$$

so any value of $\delta$ satisfying $\delta < \min\{|-\frac{11}{9}|, |\frac{13}{9}|\}$ works.

(iii) Let $\varepsilon > 0$. Set $\delta = \sqrt[43]{\varepsilon}$. If $0 < |x-0| < \delta = \sqrt[43]{\varepsilon}$, then $|x^{43} - 0| = |x|^{43} < \varepsilon$.

(iv) Let $\varepsilon > 0$. We have $(x^2 - 4x) - (-3) = (x-3)(x-1)$. Set $\delta = \sqrt{\varepsilon+1} - 1$. Suppose $0 < |x-3| < \delta$. Then $3-\delta < x < 3+\delta$ implies $2-\delta < x-1 < 2+\delta$ implies $|x-1| < 2+\delta = \sqrt{\varepsilon+1}+1$. Thus $|(x^2-4x)-(-3)| < |x-3|\cdot|x-1| < \delta(2+\delta) = (\sqrt{\varepsilon+1} - 1)(\sqrt{\varepsilon+1} + 1) = \varepsilon$.

(*How to find* $\delta$: if $|x-3| < \delta$, then $3-\delta < x < 3+\delta$, so $2-\delta < x-1 < 2+\delta$, so $|x-1| < 2+\delta$. If we can find a $\delta$ satisfying $\delta(2+\delta) < \varepsilon$, then $0 < |x-3| < \delta$ implies $|(x^2 - 4x) - (-3)| < |x-3| \cdot |x-1| < \delta(2+\delta) < \varepsilon$, so we will be done. We can complete the square in $\delta(2+\delta) < \varepsilon$ to get $(\delta+1)^2 < \varepsilon + 1$, which is equivalent to $\delta + 1 < \sqrt{\varepsilon+1}$ and $\delta < \sqrt{\varepsilon+1} - 1$.)

$\square$

**Problem 4.** (10 points) Evaluate the following limits and justify each step by indicating the appropriate Limit Laws.

(i)

$$\lim_{x\to -2}\left(\frac{t^2-2}{2t^2-3t+2}\right)^3$$

(ii)

$$\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}}$$

*Solution.* (i) We have

$$\begin{aligned}\lim_{x\to -2}\left(\frac{t^2-2}{2t^2-3t+2}\right)^3 &= \left(\lim_{x\to -2}\frac{t^2-2}{2t^2-3t+2}\right)^3 && \text{(Power Law)}\\ &= \left(\frac{(-2)^2-2}{2(-2)^2-3(-2)+2}\right)^3 && \text{(DSP for rational functions)}\\ &= \frac{1}{512}\,.\end{aligned}$$

(ii) We have

$$\begin{aligned}\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} &= \sqrt{\lim_{x\to 2}\frac{2x^2+1}{3x-2}} && \text{(Root Law)}\\ &= \sqrt{\frac{2(2)^2+1}{3(2)-2}} && \text{(DSP for rational functions)}\\ &= \frac{3}{2}\,.\end{aligned}$$

□

**Problem 5.** (10 points)

(i) What exactly does it mean for a function $f(x)$ to be continuous at the point $x = a$?

(ii) State the Intermediate Value Theorem.

(iii) Use it to show that the polynomial $p(x) = x^2 - \pi x + 2$ has a root between 0 and 1.

*Solution.* (i) The condition "$f(x)$ is continuous at $x = a$" means "$f(a)$ is defined, $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} f(x) = f(a)$".

(ii) Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a, b)$ such that $f(c) = N$. (The expression "between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$" means "in the interval $(f(a), f(b))$ if $f(a) < f(b)$; in the interval $(f(b), f(a))$ if $f(b) < f(a)$".)

(iii) The polynomial $p(x)$ is continuous at every $x = a$. We have $p(0) = 2$ and $p(1) = 3 - \pi < 0$; thus 0 is a number between $p(0)$ and $p(1)$. Hence, by the Intermediate Value Theorem, there exists $c \in (0, 1)$ such that $p(c) = 0$.

□