PROVING LIMITS OF POLYNOMIALS

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Example 1. Prove that

$$\lim_{x \to -1} (2x^4 - 3x^3 - 4x^2 - x - 1) = 1 \; .$$

Solution. Notice that

$$2(-1)^4 - 3(-1)^3 - 4(-1)^2 - (-1) - 1 = 1 \; .$$

So $x - (-1)$ must divide $2x^4 - 3x^3 - 4x^2 - x - 2$. Do polynomial long division to get $\frac{2x^4-3x^3-4x^2-x-2}{x-(-1)} = 2x^3 - 5x^2 + x - 2$.

Let $\varepsilon > 0$. Set $\delta = \min\{\frac{\varepsilon}{40}, 1\}$. Let x be a real number such that $0 < |x - (-1)| < \delta$. Then $|x + 1| < \frac{\varepsilon}{40}$. Also, $|x + 1| < 1$, so $-2 < x < 0$. In particular $|x| < 2$. So

$$\begin{aligned} |2x^3 - 5x^2 + x - 2| &\le |2x^3| + |-5x^2| + |x| + |-2| \\ &= 2|x|^3 + 5|x|^2 + |x| + 2 \\ &< 2(2)^3 + 5(2)^2 + (2) + 2 \\ &= 40 \end{aligned}$$

Thus, $|2x^4 - 3x^3 - 4x^2 - x - 2| = |x + 1| \cdot |2x^3 - 5x^2 + x - 2| < \frac{\varepsilon}{40} \cdot 40 = \varepsilon$.

(Why in the world would you set $\delta = \min\{\frac{\varepsilon}{40}, 1\}$? We want to find $\delta > 0$ such that if $0 < |x-(-1)| < \delta$ then $|2x^4-3x^3-4x^2-x-2| < \varepsilon$. We factor $2x^4-3x^3-4x^2-x-2$ as $(x + 1)(2x^3 - 5x^2 + x - 2)$ and get a bound on $|2x^3 - 5x^2 + x - 2|$ in terms of the bound on $|x + 1|$. It helps to have a bound on $|x|$, because it will give a bound on $|x^n|$ for all positive integers n. Notice that if $|x + 1| < 1$, then $-2 < x < 0$ so $|x| < 2$. So by the above computation, we have $|2x^3 - 5x^2 + x - 2| < 40$. If in addition $|x + 1| < \frac{\varepsilon}{40}$, then we can multiply the two inequalities together to get what we want.) $\square$

Direct Substitution Property for Polynomials

We can generalize the above technique to prove the Direct Substitution Property for polynomials (pg. 101). If you encounter a problem similar to the one above, and you're allowed to use the DSP for polynomials, then cite it and save yourself time. But if you're asked to prove it directly, copy the proof of Theorem 2 below, adapting as necessary to the specific problem (in particular, choosing one of the two cases).

As above, the idea is to manipulate a bound on $|x - a|$ to get a bound on $|x|$, which we can use to bound every term of the polynomial, then bound the polynomial itself.

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Theorem 2. *Let $p(x)$ be a polynomial, and suppose that a is any real number. Prove that*

$$\lim_{x \to a} p(x) = p(a) \; .$$

Proof. Let $p(x)$ be a polynomial. Since the polynomial $p(x) - p(a)$ has $x = a$ as a root, $x - a$ must divide $p(x) - p(a)$. So let

$$\frac{p(x) - p(a)}{x - a} = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0 \; .$$

Let $\varepsilon > 0$. We have two cases, according to whether $a = 0$ or $a \neq 0$.

Case 1: If $a = 0$, set $\delta = \min\{\frac{\varepsilon}{c}, 1\}$ where $c = |b_n| + |b_{n-1}| + \ldots + |b_1| + |b_0|$. Suppose that $|x| < \delta$. Since $|x| < 1$, we have[1]

$$\begin{aligned} |\frac{p(x) - p(a)}{x - a}| &= |b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0| \\ &\leq |b_n x^n| + |b_{n-1} x^{n-1}| + \ldots + |b_1 x| + |b_0| \\ &= |b_n||x|^n + |b_{n-1}||x|^{n-1} + \ldots + |b_1||x| + |b_0| \\ &< |b_n| + |b_{n-1}| + \ldots + |b_1| + |b_0| \\ &= c \; . \end{aligned}$$

Since $|x| < \frac{\varepsilon}{c}$, we have

$$\begin{aligned} |p(x) - p(a)| &= |x| \cdot |b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0| \\ &\leq \frac{\varepsilon}{c} \cdot c \\ &= \varepsilon \; . \end{aligned}$$

Case 2: If $a \neq 0$, set $\delta = \min\{\frac{\varepsilon}{c}, |a|\}$ where $c = |b_n||2a|^n + |b_{n-1}||2a|^{n-1} + \ldots + |b_1||2a| + |b_0|$. Suppose that $|x - a| < \delta$. Since $|x - a| < |a|$, we have $|x| < 2|a|$ so that

$$\begin{aligned} |\frac{p(x) - p(a)}{x - a}| &= |b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0| \\ &\leq |b_n x^n| + |b_{n-1} x^{n-1}| + \ldots + |b_1 x| + |b_0| \\ &= |b_n||x|^n + |b_{n-1}||x|^{n-1} + \ldots + |b_1||x| + |b_0| \\ &< |b_n||2a|^n + |b_{n-1}||2a|^{n-1} + \ldots + |b_1||2a| + |b_0| \\ &= c \; . \end{aligned}$$

Since $|x - a| < \frac{\varepsilon}{c}$, we have

$$\begin{aligned} |p(x) - p(a)| &= |x - a| \cdot |b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0| \\ &\leq \frac{\varepsilon}{c} \cdot c \\ &= \varepsilon \; . \end{aligned}$$

$\square$

[1] In general, for any real numbers $a_1, \ldots, a_n$, we have $|a_1 + \ldots + a_n| \leq |a_1| + \cdots + |a_n|$. This is just a way of saying "in any given path along the number line, the displacement is at most the total distance traveled".