FINAL REVIEW DAY 1

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Problem 1. Prove that

$$\lim_{x\to 1} \frac{x^2+x-3}{x+2} = -\frac{1}{3}$$

using the ϵ, δ definition of limit.

Scratch work. We want to prove the following statement: "for every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x-1| < \delta$ implies $|\frac{x^2+x-3}{x+2} - (-\frac{1}{3})| < \epsilon$". We have

$$\left|\frac{x^2+x-3}{x+2} - \left(-\frac{1}{3}\right)\right| = \left|\frac{3x^2+4x-7}{3x+6}\right| = \left|\frac{(3x+7)(x-1)}{3x+6}\right| \ ;$$

we're going to work with the last expression. If $|x-1|$ is small (i.e. $x \approx 1$), then $3x+7 \approx 10$ and $3x+6 \approx 9$ (this is an inexact statement which needs to be made precise). I can ensure that $9 < 3x+7 < 11$ if $|x-1| < \frac{1}{3}$, which is satisfied whenever $\delta \leq \frac{1}{3}$. Coincidentally, $|x-1| < \frac{1}{3}$ also ensures that $8 < 3x+6 < 10$. Thus $|x-1| < \frac{1}{3}$ implies that $|\frac{3x+7}{3x+6}| < \frac{11}{8}$ (check this). If δ is less than $\frac{\epsilon}{11/8}$ (in addition to being less than or equal to $\frac{1}{3}$), then $|\frac{(3x+7)(x-1)}{3x+6}| < \epsilon$. Notice that the condition "$\delta < \frac{\epsilon}{11/8}$ and $\delta < \frac{1}{3}$" is equivalent to the condition "$\delta < \min\{\frac{\epsilon}{11/8}, \frac{1}{3}\}$". □

Solution. Let $\epsilon > 0$. Set $\delta < \min\{\frac{\epsilon}{11/8}, \frac{1}{3}\}$. Assume $0 < |x-1| < \delta$. Then $|x-1| < \frac{1}{3}$, which implies $9 < 3x+7 < 11$ and $8 < 3x+6 < 10$. Thus $|\frac{3x+7}{3x+6}| < \frac{11}{8}$. Also, we have $|x-1| < \frac{\epsilon}{11/8}$. Thus

$$\left|\frac{x^2+x-3}{x+2} - \left(-\frac{1}{3}\right)\right| = \left|\frac{(3x+7)(x-1)}{3x+6}\right| < \frac{11}{8} \cdot \frac{\epsilon}{11/8} = \epsilon \ .$$

□

Problem 2. Show that the equation $x^4 - 10x^2 + 5 = 0$ has a root in the interval $(0, 2)$.

Solution. Let $f(x) = x^4 - 10x^2 + 5$. Then f is continuous on the closed interval $[0, 2]$. Notice that $f(0) = 5$ and $f(2) = -19$. Then 0 is strictly between $f(0)$ and $f(2)$. Thus, by the Intermediate Value Theorem, there exists $c \in (0, 2)$ such that $f(c) = 0$. □

Problem 3 (pg. 265, #54). Let $f(x) = \frac{1}{2-x}$. Find a general formula for $f^{(n)}(x)$.[1]

Solution. Compute the first few derivatives of $f(x)$: we have $f'(x) = \frac{1}{(2-x)^2}$, $f''(x) = \frac{2}{(2-x)^3}$, and $f'''(x) = \frac{6}{(2-x)^4}$. So I guess that $f^{(n)}(x) = \frac{n!}{(2-x)^{n+1}}$; let P_n be the statement "$f^{(n)}(x) = \frac{n!}{(2-x)^{n+1}}$". We have checked P_0, P_1, P_2, P_3 above. Assume P_{n-1} is true. Then

$$f^{(n)}(x) = (f^{(n-1)}(x))' \overset{(*)}{=} \frac{(n-1)!}{(2-x)^{n+1}}(-n)(-1) = \frac{n!}{(2-x)^{n+1}}$$

where the equality marked $(*)$ is where I use the assumption that P_{n-1} is true. Thus P_{n-1} implies P_n. Since P_0 is true, P_n is true for all $n = 0, 1, 2, \ldots$. So the general formula for $f^{(n)}(x)$ is $f^{(n)}(x) = \frac{n!}{(2-x)^{n+1}}$. □

[1] Recall that $f^{(n)}(x)$ is the nth derivative of $f(x)$.

Problem 4 (pg. 265, #55). Let $f(x) = xe^x$. Prove that $f^{(n)}(x) = (x+n)e^x$.

Solution. We have $f^{(0)}(x) = f(x) = xe^x = (x+0)e^x$. Assume that $f^{(n-1)}(x) = (x+(n-1))e^x$. Then $f^{(n)}(x) = (f^{(n-1)}(x))' = e^x + (x+(n-1))e^x = (x+n)e^x$. Thus $f^{(n)}(x) = (x+n)e^x$ for all nonnegative integers n. □

Problem 5 (pg. 267, #109). Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3}.$$

Solution. This problem is about recognizing indeterminate forms and applying L'Hospital's Rule. I didn't think it would be this messy. Since the power of x on the denominator is a 3, you can expect to have to apply L'Hospital's Rule at most 3 times. Define

$$f(x) = \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3}.$$

Below, if an equality is marked with an $(*)$, then it means that I've used L'Hospital's Rule. Then

$$\begin{aligned}\lim_{x\to 0} f(x) &\overset{(*)}{=} \lim_{x\to 0} \frac{\frac{\frac{1}{2}\sec^2 x}{\sqrt{1+\tan x}} - \frac{\frac{1}{2}\cos x}{\sqrt{1+\sin x}}}{3x^2} \\ &\overset{(*)}{=} \lim_{x\to 0} \frac{\frac{\frac{-1}{\cos^3 x}(-\sin x)\sqrt{1+\tan x} - (\sec^2 x)\frac{\frac{1}{4}\sec^2 x}{\sqrt{1+\tan x}}}{1+\tan x} - \frac{\frac{1}{2}(-\sin x)\sqrt{1+\sin x} - (\frac{1}{2}\cos x)\frac{\frac{1}{2}(\cos x)}{\sqrt{1+\sin x}}}{1+\sin x}}{6x} \\ &= \lim_{x\to 0} g(x) + \lim_{x\to 0} h(x)\end{aligned}$$

where

$$\begin{aligned}g(x) &= \frac{\frac{\frac{-1}{\cos^3 x}(-\sin x)\sqrt{1+\tan x}}{1+\tan x} - \frac{\frac{1}{2}(-\sin x)\sqrt{1+\sin x}}{1+\sin x}}{6x} \\ &= \frac{\sin x}{x} \cdot \frac{\frac{\frac{1}{\cos^3 x}\sqrt{1+\tan x}}{1+\tan x} + \frac{\frac{1}{2}\sqrt{1+\sin x}}{1+\sin x}}{6}\end{aligned}$$

and

$$\begin{aligned}h(x) &= \frac{\frac{-(\sec^2 x)\frac{\frac{1}{4}\sec^2 x}{\sqrt{1+\tan x}}}{1+\tan x} - \frac{-(\frac{1}{2}\cos x)\frac{\frac{1}{2}(\cos x)}{\sqrt{1+\sin x}}}{1+\sin x}}{6x} \\ &= \frac{1}{24}\frac{\frac{-(\sec x)^4}{(1+\tan x)^{3/2}} + \frac{(\cos x)^2}{(1+\sin x)^{3/2}}}{x}.\end{aligned}$$

Then

$$\begin{aligned}\lim_{x\to 0} g(x) &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\frac{\frac{1}{\cos^3 x}\sqrt{1+\tan x}}{1+\tan x} + \frac{\frac{1}{2}\sqrt{1+\sin x}}{1+\sin x}}{6}\right) \\ &\overset{(*)}{=} \left(\lim_{x\to 0}\frac{\cos x}{1}\right)\left(\lim_{x\to 0}\frac{\frac{\frac{1}{\cos^3 0}\sqrt{1+\tan 0}}{1+\tan 0} + \frac{\frac{1}{2}\sqrt{1+\sin 0}}{1+\sin 0}}{6}\right) \\ &= \frac{1}{4}.\end{aligned}$$

Also,

$$\lim_{x\to 0} h(x) \overset{(*)}{=} \lim_{x\to 0} \frac{1}{24}\left(-\frac{4(\sec x)^3(\sec x\tan x)(1+\tan x)^{3/2} - (\sec x)^4(-\frac{3}{2}(1+\tan x)^{-5/2}(\sec x)^2)}{(1+\tan x)^3}\right)$$
$$+ \lim_{x\to 0}\frac{1}{24}\left(\frac{2(\cos x)(-\sin x)(1+\sin x)^{3/2} - (\cos x)^2(-\frac{3}{2}(1+\sin x)^{-5/2}(\cos x))}{(1+\sin x)^3}\right)$$
$$= \frac{1}{24}\left(-\frac{4(\sec 0)^3(\sec 0\tan 0)(1+\tan 0)^{3/2} - (\sec 0)^4(-\frac{3}{2}(1+\tan 0)^{-5/2}(\sec 0)^2)}{(1+\tan 0)^3}\right)$$
$$+ \frac{1}{24}\left(\frac{2(\cos 0)(-\sin 0)(1+\sin 0)^{3/2} - (\cos 0)^2(-\frac{3}{2}(1+\sin 0)^{-5/2}(\cos 0))}{(1+\sin 0)^3}\right)$$
$$= 0\,.$$

Thus

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) + \lim_{x\to 0} h(x) = \frac{1}{4}\,.$$

□

Problem 6 (pg. 267, #110). Suppose f and g are differentiable functions such that $f(g(x)) = x$ and $f'(x) = 1 + (f(x))^2$. Show that $g'(x) = \frac{1}{1+x^2}$.

Solution. The chain rule gives $f'(g(x))g'(x) = 1$. Substituting $g(x)$ into $f'(x) = 1+(f(x))^2$ gives $f'(g(x)) = 1 + (f(g(x)))^2 = 1 + x^2$. Thus $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1+x^2}$. □

Problem 7 (pg. 353, #50). Find two positive integers m, n such that $m + 4n = 1000$ and mn is as large as possible.

Solution. We have $m = 1000 - 4n$, so we want to find a positive integer n such that $1000 - 4n$ is positive and $(1000 - 4n)(n)$ is as large as possible. We have $(1000 - 4n)(n) = 4(250 - n)(n) = 4(125^2 - (n - 125)^2)$, which is maximized when $n = 125$. Thus $(m, n) = (500, 125)$ maximizes mn. [2] □

Problem 8 (pg. 353, #52). Find the point on the hyperbola $xy = 8$ that is closest to the point $(3, 0)$.

Solution. Let $f(x) = \frac{8}{x}$. The distance between the point $(x, f(x)) = (x, \frac{8}{x})$ and $(3, 0)$ is

$$d(x) = \sqrt{(x-3)^2 + (\frac{8}{x} - 0)^2}\,.$$

To minimize $d(x)$, it is equivalent to minimize $e(x) = (d(x))^2 = x^2 - 6x + 9 + \frac{64}{x^2}$, since $d(x_1) < d(x_2)$ if and only if $e(x_1) < e(x_2)$. We have $e'(x) = \frac{2(x^4-3x^3-64)}{x^3}$, which is 0 if and only if $x^4 - 3x^3 - 64 = 0$. We have that 4 is a solution to $x^4 - 3x^3 - 64 = 0$, and $x^4 - 3x^3 - 64 = (x - 4)(x^3 + x^2 + 4x + 16)$ so 4 is the unique nonnegative solution to $x^4 - 3x^3 - 64$. Since $e'(x) < 0$ if $0 < x < 4$ and $e'(x) > 0$ if $x > 4$, we have that $x = 4$ is a local minimum of $e(x)$. Note that $e(4) = (4-3)^2 + \frac{64}{16} = 5$. If $x < 0$, then $e(x) = (x-3)^2 + \frac{64}{x^2} > 3^2 = 9$, so $e(4) < e(x)$ for all negative x. Thus $x = 4$ is the global minimum of $e(x)$, hence of $d(x)$. Thus the point $(4, 2)$ is the point on the hyperbola $xy = 8$ which is closest to $(3, 0)$. □

Problem 9 (pg. 420, #3). If $\int_0^4 e^{(x-2)^4}\, dx = k$, find the value of $\int_0^4 xe^{(x-2)^4}\, dx$.

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[2] This solution doesn't really use calculus since the problem deals with integers (which are discrete) rather than real numbers (which are smooth).

Solution. (I expected this problem to require some integration by parts, but I couldn't figure it out and in the end came up with this.) We have

$$\begin{aligned}\int_0^4 xe^{(x-2)^4}\,dx &= \int_0^4 (x-2)e^{(x-2)^4}\,dx + \int_0^4 2e^{(x-2)^4}\,dx \\ &= \int_0^4 (x-2)e^{(x-2)^4}\,dx + 2k\ .\end{aligned}$$

Let $u = x - 2$. Then $\frac{du}{dx} = 1$, so

$$\int_0^4 (x-2)e^{(x-2)^4}\,dx = \int_{u(0)}^{u(4)} ue^{u^4}\,du = \int_{-2}^{2} ue^{u^4}\,du$$

by the Chain Rule. Notice that ue^{u^4} is an odd function and $-2 = -(2)$. Thus $\int_{-2}^{2} ue^{u^4}\,du = \int_{-2}^{0} ue^{u^4}\,du + \int_0^2 ue^{u^4}\,du = 0$. Thus

$$\int_0^4 xe^{(x-2)^4}\,dx = 2k\ .$$

□

Problem 10 (pg. 420, #6). If $f(x) = \int_0^x x^2\sin(t^2)\,dt$, find $f'(x)$.

Solution. Let $g(x) = \int_0^x \sin(t^2)\,dt$. Then $g'(x) = \sin(x^2)$ by the Fundamental Theorem of Calculus. Thus $f(x) = x^2 g(x)$ implies that $f'(x) = 2xg(x) + x^2 g'(x) = 2xg(x) + x^2\sin(x^2)$. [3] □

Problem 11 (pg. 420, #9). Find the interval $[a, b]$ for which the value of the integral $\int_a^b (2 + x - x^2)\,dx$ is a maximum.

Solution. Let $f(x) = 2 + x - x^2$. We have $f(x) = (2 - x)(x + 1)$ so $f(x) < 0$ on $(-\infty, -1) \cup (2, \infty)$ and $f(x) > 0$ on $(-1, 2)$. The integral $\int_a^b f(x)\,dx$ is maximized when we integrate $f(x)$ over only the intervals where $f(x) > 0$, so it is maximized when $(a, b) = (-1, 2)$. □

Problem 12 (pg. 436, Example 7). A solid has a circular base of radius 1. Each cross-section of the solid by a plane perpendicular to the base is a square. Compute the volume of the solid.[4]

Solution. I assume that each cross-section of solid by a plane perpendicular to the line segment $\overline{(-1,0)(1,0)}$ is a square. The infinitesimal cross-section by the plane $x = a$ is a thin square of side length $2\sqrt{1-a^2}$ and thickness Δx. Thus the volume is

$$\int_{-1}^{1} (2\sqrt{1-x^2})^2\,dx = \int_{-1}^{1} 4 - 4x^2\,dx = 4x - \frac{4}{3}x^3|_{-1}^{1} = \frac{16}{3}\ .$$

□

Problem 13. Let $\{(x, y, z) : x^2 + y^2 = 1\}$ and $\{(x, y, z) : x^2 + z^2 = 1\}$ be two cylinders of radius 1. Find the volume of the solid defined by the intersection of these cylinders.

[3] I don't think the point is to find a nice formula for $g(x)$; in fact it's kind of complicated, as you can check using Wolfram Alpha.

[4] There is a little ambiguity in the statement of the problem: it should have said "each cross-section of the solid by a plane perpendicular to a fixed diameter of the base is a square". I don't know if the solid described in the original problem statement exists.

Solution. The cross section of the solid by the plane $x = a$ is the square $\{(y, z) \;:\; a^2 + y^2 \le 1\,,\; a^2 + z^2 \le 1\}$, which has side length $2\sqrt{1-a^2}$. We integrate along the x-axis. The volume of the solid is then

$$\int_{-1}^{1} (2\sqrt{1-x^2})^2 \; dx = \int_{-1}^{1} 4 - 4x^2 \; dx = 4x - \frac{4}{3}x^3|_{-1}^{1} = \frac{16}{3}\,.$$

Notice that this is the same integral as the one in Problem 12. You can visualize moving each cross section of the solid in Problem 12 so that their centers are lined up; then you'll get the solid in this problem. □

Problem 14 (pg. 445, #12). Let R be the region bounded by the curves $y = 4x^2 - x^3$ and $y = 0$. Use cylindrical shells to find the volume of the solid obtained by rotating R about the y-axis.

Solution. Let $f(x) = 4x^2 - x^3 = x^2(4-x)$. Then $f(x) > 0$ if $x \in (-\infty, 0) \cup (0, 4)$ and $f(x) < 0$ if $x \in (4, \infty)$, and $f(x) = 0$ exactly when $x = 0$ or $x = 4$. Thus the region bounded by $f(x)$ and the x-axis is between $x = 0$ and $x = 4$. The shell with radius x has height $f(x) = 4x^2 - x^3$, so the volume of the solid is

$$\int_{0}^{4} (2\pi x)(4x^2 - x^3) \; dx = 2\pi(x^4 - \frac{x^5}{5})\Big|_0^4 = \frac{512\pi}{5}\,.$$

□