USING INTEGRATION TO DERIVE GEOMETRIC FORMULAS

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Problem 1. Derive the formula for the circumference of a circle of radius r by computing the arclength of the curve $\sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$.

Solution. The general formula for the arclength of the curve defined by the function f defined on an interval $[a, b]$ is $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$. Let $f(x) = \sqrt{r^2 - x^2}$ on $[-r, r]$. (This is the upper-half of the circle.) Its arclength is

$$\begin{aligned}
\int_{-r}^{r} \sqrt{1 + (f'(x))^2}\, dx &= \int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\, dx \\
&= \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\
&= \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx \\
&= r \int_{-r}^{r} \frac{1/r}{\sqrt{1 - (x/r)^2}}\, dx \\
&= r \int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}}\, du \qquad \text{where } u = \frac{x}{r} \\
&= r \arcsin(u)\big|_{-1}^{1} \\
&= r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\
&= \pi r\ .
\end{aligned}$$

Thus the circumference of a circle is $2\pi r$ (twice the arclength of the upper-half of the circle). □

Problem 2. Derive the formula for the area of a circle of radius r by computing the area between the curves $\sqrt{r^2 - x^2}$ and $-\sqrt{r^2 - x^2}$ between $x = -r$ and $x = r$.

Solution. The area of the circle is the area bounded by the curves $\sqrt{r^2 - x^2}$ and $-\sqrt{r^2 - x^2}$ between $x = -r$ and $x = r$, which is the following definite integral:

$$\int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx\ .$$

Let $x = r\sin\theta$; then $\frac{dx}{d\theta} = r\cos\theta$. Thus by the substitution rule we have

$$\begin{aligned}
\int_{-r}^{r} 2\sqrt{r^2 - x^2}\ dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{r^2 - r^2(\sin\theta)^2} \cdot (r\cos\theta)\ d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r^2(\cos\theta)^2\ d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r^2 \frac{\cos(2\theta)+1}{2}\ d\theta \\
&= r^2\left(\frac{\sin(2\theta)}{2} + \theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \pi r^2.
\end{aligned}$$

□

Problem 3. Derive the formula for the volume of a sphere of radius r by computing the volume of "the object obtained by rotating the curve $\sqrt{r^2 - x^2}$ above the x-axis".

Solution. We will integrate vertical disks from $x = -r$ to $x = r$. At the point x, the infinitesimal volume of the disk is $\pi\left(\sqrt{r^2 - x^2}\right)^2 dx$ (the radius of the disk is the value of the function $\sqrt{r^2 - x^2}$). Thus the volume is

$$\begin{aligned}
V &= \int_{-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2\ dx \\
&= \int_{-r}^{r} \pi(r^2 - x^2)\ dx \\
&= \pi\left(r^2 x - \frac{1}{3}x^3\right)\Bigg|_{-r}^{r} \\
&= \frac{4}{3}\pi r^3\ .
\end{aligned}$$

□

Problem 4. Derive the formula for the volume of a cone whose height is h and whose base has area A by "integrating along the height".

Solution. We integrate vertically from top to bottom ($x = 0$ to $x = h$). The horizontal slice at x has area $(\frac{x}{h})^2 A$ because of similar triangles: the horizontal slice is what you get when you shrink the base by a factor of $\frac{x}{h}$. If you scale a figure by a factor of r, then its area gets multiplied by r^2. The infinitesimal volume of the disk at x is $(\frac{x}{h})^2 dx$. Thus the volume is

$$V = \int_0^h \left(\frac{x}{h}\right)^2 A\ dx = \frac{1}{3}\frac{x^3}{h^2}A\Big|_0^h = \frac{1}{3}Ah\ .$$

□