THE SUBSTITUTION RULE

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Theorem 1 (Substitution Rule for Definite Integrals, page 411)**.** *If* g' *is continuous on* $[a, b]$ *and* f *is continuous on the range of* $g(x)$*, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx \; . \tag{1}$$

Exercise 1 (Section 5.5, #27). Find an antiderivative of the function $f(x) = (x^2 + 1)(x^3 + 3x)^4$.

Solution. Set $u(x) = x^3 + 3x$. Then $u'(x) = 3(x^2 + 1)$. Thus

$$\begin{aligned}\int (x^2+1)(x^3+3x)^4 \; dx &= \int \frac{1}{3}(x^3+3x)^4(3x^2+3) \; dx \\ &= \int \frac{1}{3}u^4 \; du \\ &= \frac{1}{15}u^5 + c \\ &= \frac{1}{15}(x^3+3x)^5 + c \; .\end{aligned}$$

□

Exercise 2 (Section 5.5, #32). Find an antiderivative of the function $f(x) = \frac{\sin(\ln x)}{x}$.

Solution. Set $u(x) = \ln x$. Then $u'(x) = \frac{1}{x}$. Thus

$$\begin{aligned}\int \frac{\sin(\ln x)}{x} \; dx &= \int \sin u \; du \\ &= -\cos(u) + c \\ &= -\cos(\ln x) + c \; .\end{aligned}$$

□

Exercise 3 (Section 5.5, #36). Find an antiderivative of the function $f(x) = \frac{2^x}{2^x+3}$.

Solution. Set $u(x) = 2^x + 3$. Then $u'(x) = (\ln 2)2^x$. Thus

$$\begin{aligned}\int \frac{2^x}{2^x+3} \; dx &= \int \frac{1}{\ln 2}\frac{1}{2^x+3}((\ln 2)2^x) \; dx \\ &= \int \frac{1}{\ln 2}\frac{1}{u} \; du \\ &= \frac{1}{\ln 2}\ln u + c \\ &= \frac{1}{\ln 2}\ln(2^x+3) + c \; .\end{aligned}$$

□

Exercise 4 (Section 5.5, #48). Find an antiderivative of the function $f(x) = \frac{x}{1+x^4}$.

Solution. Set $u(x) = x^2$. Then $u'(x) = 2x$. Thus

$$\begin{aligned}\int \frac{x}{1+x^4}\,dx &= \int \frac{1}{2}\frac{1}{1+(x^2)^2}(2x)\,dx \\ &= \int \frac{1}{2}\frac{1}{1+u^2}\,du \\ &= \frac{1}{2}\arctan(u) + c \\ &= \frac{1}{2}\arctan(x^2) + c\,.\end{aligned}$$

□

Exercise 5 (Section 5.5, #60). Evaluate the definite integral

$$\int_0^1 xe^{-x^2}\,dx\,.$$

Solution. Set $f(x)xe^{-x^2}$. We have $f(x) = -\frac{1}{2}e^u u'$ where $u(x) = -x^2$. Thus $-\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$ is an antiderivative of $f(x)$. By the Fundamental Theorem of Calculus, we have

$$\int_0^1 xe^{-x^2}\,dx = \left(-\frac{1}{2}e^{-1^2}\right) - \left(-\frac{1}{2}e^{-0^2}\right) = -\frac{1}{2}e^{-1} + \frac{1}{2}\,.$$

□

Exercise 6 (Section 5.5, #69). Evaluate the definite integral

$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx\,.$$

Solution. Set $f(x) = \frac{1}{x\sqrt{\ln x}}$. We have $f(x) = \frac{1}{\sqrt{u}}u'$ where $u(x) = \ln x$. Thus $2\sqrt{u} = 2\sqrt{\ln x}$ is an antiderivative of $f(x)$. By the Fundamental Theorem of Calculus, we have

$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx = (2\sqrt{\ln e^4}) - (2\sqrt{\ln e}) = (2\cdot 2) - (2\cdot 1) = 2\,.$$

□

Exercise 7 (Section 5.5, #86). If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

Solution. Set $h(x) = xf(x^2)$. Then $h(x) = \frac{1}{2}f(g(x))g'(x)$ where $g(x) = x^2$. Thus

$$\int_0^3 h(x)\,dx = \int_0^3 \frac{1}{2}f(g(x))g'(x)\,dx = \int_{g(0)}^{g(3)} \frac{1}{2}f(x)\,dx = \frac{1}{2}\int_0^9 f(x)\,dx = \frac{1}{2}4 = 2\,.$$

□

Exercise 8 (Section 5.5, #89). If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx\,.$$

Proof. Set $f(x) = x^a(1-x)^b$. Let $g(x) = 1-x$. Then

$$\begin{aligned}\int_0^1 x^a(1-x)^b\,dx &= -\int_{g(0)}^{g(1)} x^a(1-x)^b\,dx \\ &= -\int_0^1 (1-x)^a(1-(1-x))^b(-1)\,dx \\ &= \int_0^1 (1-x)^a x^b\,dx\,.\end{aligned}$$

□

Exercise 9 (Section 5.5, #90,91)*.* If f is continuous on $[0,\pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx\,.$$

Use this to evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx\,.$$

Solution. We have

$$\begin{aligned}\int_0^\pi xf(\sin x)\,dx &= -\int_{u(0)}^{u(\pi)} xf(\sin x)\,dx \\ &= -\int_0^\pi (\pi-x)f(\sin(\pi-x))(-1)\,dx \\ &= \int_0^\pi (\pi-x)f(\sin x)\,dx \qquad \text{since} \qquad \sin(\pi-x)=\sin x \\ &= \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi xf(\sin x)\,dx\end{aligned}$$

so

$$\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx\,.$$

Set $f(x) = \frac{x}{2-x^2}$. Then

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{x\sin x}{2-\sin^2 x}\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx\,.$$

We can compute the last integral by substitution. Set $u(x) = \cos x$. Then

$$\begin{aligned}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx &= \int_0^\pi \frac{-1}{1+(\cos x)^2}(-\sin x)\,dx \\ &= \int_{u(0)}^{u(\pi)} -\arctan u\,du \\ &= \int_{-1}^1 \arctan u\,du \\ &= \frac{\pi}{2}\,.\end{aligned}$$

Thus

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \left(\frac{\pi}{2}\right)^2\,.$$

□