THE SUBSTITUTION RULE

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Theorem 1 (Substitution Rule for Definite Integrals, page 411)**.** *If g' is continuous on $[a, b]$ and f is continuous on the range of $g(x)$, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx\ . \tag{1}$$

Proof. Let F be an antiderivative of f, i.e. we have $F' = f$. By FTC(ii), we have

$$F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(x)dx\ . \tag{2}$$

(You should read the LHS as the net change of $F(x)$ between $g(a)$ and $g(b)$.) Also, notice that

$$(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x)\ .$$

Thus $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. By FTC(ii), we have

$$F(g(b)) - F(g(a)) = \int_a^b f(g(x))g'(x)dx\ . \tag{3}$$

(You should read the LHS as the net change of $F(g(x))$ between a and b.) Then (2) and (3) implies (1).[1] $\square$

The Substitution Rule for Definite Integrals asserts that two numbers are equal. By fixing a and varying b in (1), we obtain an equality of functions

$$G_1(t) = G_2(t)$$

where

$$G_1(t) := \int_a^t f(g(x))g'(x)dx \qquad \text{and} \qquad G_2(t) := \int_{g(a)}^{g(t)} f(x)dx\ .$$

You can use the Substitution Rule backwards and forwards. It's useful to know both methods.

Substitution Method 1. Given a function $f(x)$ of which you want to find an antiderivative, find a function $g(x)$ such that the function $f(g(x))g'(x)$ has a known antiderivative, find an antiderivative $G(x)$ of $f(g(x))g'(x)$, and define $F(x) := G(g^{-1}(x))$. Then $F(x)$ is an antiderivative of $f(x)$. You can check as

[1]Equation (1) is false if we remove the limits of integration. The assertion $\int f(g(x))g'(x)dx = \int f(x)dx$ seems to imply the following: "if $F_1(x)$ is an antiderivative of $f(g(x))g'(x)$ and $F_2(x)$ is an antiderivative of $f(x)$, then there exists a constant C such that $F_1(x) = F_2(x) + C$." But this is false: consider $f(x) = x^2$ and $g(x) = x^2$. Then $f(g(x))g'(x) = 2x^5$. Let's pick the antiderivatives $F_1(x) = \frac{2}{6}x^6$ and $F_2(x) = \frac{1}{3}x^3$ of $f(g(x))g'(x)$ and $f(x)$, respectively. Notice that there is no constant C such that $F_1(x) = F_2(x) + C$ for all x, since this says that the polynomial $\frac{2}{6}x^6 - \frac{1}{3}x^3 - C$ has infinitely many roots. The correct statement is: "if $F_1(x)$ is an antiderivative of $f(g(x))g'(x)$ and $F_2(x)$ is an antiderivative of $f(x)$, then there exists a constant C such that $F_1(x) = F_2(g(x)) + C$." (The only difference is that we have $F_1(x) = F_2(g(x)) + C$ instead of $F_1(x) = F_2(x) + C$.) You can check this for the example $f(x) = x^2$ and $g(x) = x^2$.

follows:

$$\begin{aligned} F'(x) &= G'(g^{-1}(x)){g^{-1}}'(x) \\ &= f(g(g^{-1}(x))) \cdot g'(g^{-1}(x)) \cdot \frac{1}{g'(g^{-1}(x))} \\ &= f(x)\ . \end{aligned}$$

Let's look at an example from class today.

Problem 1. Find an antiderivative of $f(x) = e^{\sqrt{x}}$.

Solution. Let $g(x) = x^2$. Then $f(g(x))g'(x) = e^{\sqrt{x^2}} \cdot 2x = 2xe^x$. An antiderivative of xe^x is $xe^x - e^x$ (which you can find by integration by parts; set $u(x) = x$, $v(x) = e^x$) so set $G(x) = 2xe^x - 2e^x$. Then $G'(x) = f(g(x))g'(x)$. Set $F(x) = G(g^{-1}(x)) = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}}$. Then $F(x)$ is an antiderivative of $f(x)$. □

Substitution Method 2. Given a function $h(x)$ of which you want to find an antiderivative, find functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))g'(x)$ and $f(x)$ has a known antiderivative $F(x)$. Then $h(x) = f(g(x))g'(x) = F'(g(x))g'(x) = (F(g(x)))'$, so $F(g(x))$ is an antiderivative of $h(x)$.

Problem 2. Find an antiderivative of $h(x) = \tan x$.

Solution. Write $h(x) = \tan x = \frac{\sin x}{\cos x}$. Define $f(x) = \frac{-1}{x}$ and $g(x) = \cos x$. Then $h(x) = \frac{-1}{\cos x}(-\sin x) = f(g(x))g'(x)$. Since $-\ln x$ is an antiderivative of $f(x)$, $-\ln(\cos x)$ is an antiderivative of $h(x)$. □

Exercise 1 (Section 5.5, #27). Find an antiderivative of the function $f(x) = (x^2+1)(x^3+3x)^4$.

Exercise 2 (Section 5.5, #32). Find an antiderivative of the function $f(x) = \frac{\sin(\ln x)}{x}$.

Exercise 3 (Section 5.5, #36). Find an antiderivative of the function $f(x) = \frac{2^x}{2^x+3}$.

Exercise 4 (Section 5.5, #48). Find an antiderivative of the function $f(x) = \frac{x}{1+x^4}$.

Exercise 5 (Section 5.5, #60). Evaluate the definite integral

$$\int_0^1 xe^{-x^2}\, dx\ .$$

Exercise 6 (Section 5.5, #69). Evaluate the definite integral

$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx\ .$$

Exercise 7 (Section 5.5, #86). If f is continuous and $\int_0^9 f(x)\, dx = 4$, find $\int_0^3 xf(x^2)\, dx$.

Exercise 8 (Section 5.5, #89). If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\, dx = \int_0^1 x^b(1-x)^a\, dx\ .$$

Exercise 9 (Section 5.5, #90,91). If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi xf(\sin x)\, dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\, dx\ .$$

Use this to evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\, dx\ .$$