# DEFINITE INTEGRALS

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A *definite integral* is generally defined to be the limit of approximations of area. The formal definition is given on page 372, but it will usually suffice to use Theorem 4 on page 374, which says

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \ . \tag{1}$$

In other words, the integral $\int_a^b f(x)dx$ is the limit of the sequence whose $n$th term is equal to the *Riemann sum* $\sum_{i=1}^n f(x_i)\Delta x$, which in turn is the sum of the areas of $n$ rectangles, where the $i$th rectangle has width $\Delta x = \frac{b-a}{n}$ and (possibly negative) height $f(x_i) = f(a + i\Delta x)$.

*Exercise* 1 (Section 5.2, #17). Express the limit $\lim_{n\to\infty} \sum_{i=1}^n x_i \ln(1 + x_i^2)\Delta x$, where $\Delta x = \frac{6-2}{n}$ and $x_i = 2 + i\Delta x$, as a definite integral on the interval $[2, 6]$.

*Solution.* Set $f(x) = x \ln(1 + x^2)$ and $a = 2$ and $b = 6$. Then

$$\lim_{n\to\infty} \sum_{i=1}^n x_i \ln(1 + x_i^2)\Delta x = \int_2^6 x \ln(1 + x^2)\, dx \ .$$

□

*Exercise* 2 (Section 5.2, #24). Use (1) to compute $\int_0^2 (2x - x^3)dx$.

*Solution.* We have

$$\begin{aligned}
\int_0^2 (2x - x^3)\, dx &= \lim_{n\to\infty} \sum_{i=1}^n \left( 2\left(i\frac{2}{n}\right) - \left(i\frac{2}{n}\right)^3 \right) \frac{2}{n} \\
&= \lim_{n\to\infty} \frac{8}{n^2} \sum_{i=1}^n i - \frac{16}{n^4} \sum_{i=1}^n i^3 \\
&= \lim_{n\to\infty} \frac{8}{n^2} \frac{n(n+1)}{2} - \frac{16}{n^4} \left( \frac{n(n+1)}{2} \right)^2 \\
&= \lim_{n\to\infty} \frac{4(n+1)}{n} - 4\frac{(n+1)^2}{n^2} \\
&= 0 \ .
\end{aligned}$$

(You can check that you have the right answer using FTC. Since $F(x) = x^2 - \frac{1}{4}x^4$ is an antiderivative of $2x - x^3$, FTC says $\int_0^2 (2x - x^3)\, dx = F(2) - F(0) = 0$.) □

*Exercise* 3. Use (1) to compute $\int_0^4 e^x dx$.

*Solution.* This turned out to be much harder than I expected. Don't expect anything like this to show up on the midterms or final. For those interested, I'll provide a proof. We have

$$\begin{aligned}\int_0^4 e^x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} e^{i\cdot\frac{4}{n}}\cdot\frac{4}{n}\\ &= \lim_{n\to\infty}\frac{4}{n}\frac{e^{\frac{4}{n}}-e^{(n+1)\frac{4}{n}}}{1-e^{\frac{4}{n}}}\\ &= \lim_{n\to\infty}\frac{4}{n}\frac{e^{\frac{4}{n}}(1-e^4)}{1-e^{\frac{4}{n}}}\\ &= (1-e^4)\lim_{n\to\infty}\frac{\frac{4}{n}e^{\frac{4}{n}}}{1-e^{\frac{4}{n}}}\,.\end{aligned}$$

Let $f$ be the function

$$f(x)=\frac{\frac{4}{x}e^{\frac{4}{x}}}{1-e^{\frac{4}{x}}}$$

with domain of definition $(0,\infty)$. I want to find the limit (of sequences) $\lim_{n\to\infty} f(n)$. I claim that this is equal to the limit (of functions) $\lim_{x\to\infty} f(x)$. This depends crucially on the fact that the function $f(x)$ is monotonically increasing on $(0,\infty)$, which you can check by showing that the derivative is always greater than 0:

$$\begin{aligned}f'(x) &= \frac{\left(-\frac{4}{x^2}e^{\frac{4}{x}}+\frac{4}{x}e^{\frac{4}{x}}(-\frac{4}{x^2})\right)\left(1-e^{\frac{4}{x}}\right)-\left(\frac{4}{x}e^{\frac{4}{x}}\right)\left(-e^{\frac{4}{x}}(-\frac{4}{x^2})\right)}{\left(1-e^{\frac{4}{x}}\right)^2}\\ &= \frac{-\frac{4}{x^2}e^{\frac{4}{x}}\left(\left(1+\frac{4}{x}\right)\left(1-e^{\frac{4}{x}}\right)+\frac{4}{x}e^{\frac{4}{x}}\right)}{\left(1-e^{\frac{4}{x}}\right)^2}\\ &= \frac{\frac{4}{x^2}e^{\frac{4}{x}}}{\left(1-e^{\frac{4}{x}}\right)^2}\left(e^{\frac{4}{x}}-1-\frac{4}{x}\right)\end{aligned}$$

where in the last term, the first factor is always positive. The second factor is always positive since, for any $\alpha>0$, we have $e^\alpha \geq 1+\alpha$ (this is because $e^\alpha = 1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}+\cdots$). This shows that $f$ is monotonically increasing on $(0,\infty)$.

So suppose we know $\lim_{n\to\infty} f(n) = L$ and $L$ is finite. Let $\epsilon>0$; then there exists $M>0$ such that $n\geq M$ implies $|f(n)-L|<\epsilon$. Suppose $x>M$. Then there exists an integer $N$ such that $N\leq x\leq N+1$. Since $|f(N)-L|<\epsilon$ and $|f(N+1)-L|<\epsilon$ and $f(N)\leq f(x)\leq f(N+1)$, we have $|f(x)-L|<\epsilon$. Thus $\lim_{x\to\infty} f(x) = L$.

Conversely, suppose we know $\lim_{x\to\infty} f(x)=L$ and $L$ is finite. Let $\epsilon>0$; then there exists $M>0$ such that $x>M$ implies $|f(x)-L|<\epsilon$. Let $N=\lceil M\rceil$. For any $n>N$, we have $n>M$, so $|f(n)-L|<\epsilon$. Thus $\lim_{n\to\infty} f(n)=L$.

The proofs of the statements "$\lim_{n\to\infty} f(n)=\infty$ if and only if $\lim_{x\to\infty} f(x)=\infty$" and "$\lim_{n\to\infty} f(n) = -\infty$ if and only if $\lim_{x\to\infty} f(x)=-\infty$" are similar.

So let's compute $\lim_{x\to\infty} f(x)$. Since we actually have a function instead of a sequence, we can use L'Hospital's Rule (it turns out that L'Hospital's Rule does include the case when $x$ approaches $\infty$; you can look at Note 2 at the top of page 303). We have

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{-\frac{4}{x^2}e^{\frac{4}{x}} + \frac{4}{x}e^{\frac{4}{x}}(-\frac{4}{x^2})}{-e^{\frac{4}{x}}(-\frac{4}{x^2})} \\ &= \lim_{x\to\infty} \frac{1+\frac{4}{x}}{-1} \\ &= -1\ .\end{aligned}$$

Hence we have

$$\begin{aligned}\int_0^4 e^x\ dx &= (1-e^4)\lim_{n\to\infty}\frac{\frac{4}{n}e^{\frac{4}{n}}}{1-e^{\frac{4}{n}}} \\ &= (1-e^4)\lim_{x\to\infty} f(x) \\ &= (1-e^4)(-1) \\ &= e^4 - 1\ .\end{aligned}$$

□

*Exercise* 4 (Section 5.2, #27). Prove that $\int_a^b x\ dx = \frac{1}{2}(b^2 - a^2)$. Note that this is $f(b) - f(a)$ where $f(x) = \frac{1}{2}x^2$.

*Solution.* We have

$$\begin{aligned}\int_a^b x\ dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} \\ &= \lim_{n\to\infty} a\frac{b-a}{n}\sum_{i=1}^{n} 1 + \left(\frac{b-a}{n}\right)^2\sum_{i=1}^{n} i \\ &= \lim_{n\to\infty} a\frac{b-a}{n}(n) + \left(\frac{b-a}{n}\right)^2\frac{n(n+1)}{2} \\ &= a(b-a) + (b-a)^2\frac{1}{2} \\ &= \frac{1}{2}(b^2-a^2)\end{aligned}$$

□

*Exercise* 5 (Section 5.2, #28). Prove that $\int_a^b x^2\ dx = \frac{1}{3}(b^3 - a^3)$. Note that this is $f(b) - f(a)$ where $f(x) = \frac{1}{3}x^3$.

*Solution.* We have

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^n \left(a+i\frac{b-a}{n}\right)^2\frac{b-a}{n}\\
&= \lim_{n\to\infty}\frac{b-a}{n^3}\sum_{i=1}^n (an+i(b-a))^2\\
&= \lim_{n\to\infty}\frac{b-a}{n^3}\sum_{i=1}^n \left(a^2n^2+2ain(b-a)+i^2(b-a)^2\right)\\
&= \lim_{n\to\infty}\frac{b-a}{n^3}\left(a^2n^2\sum_{i=1}^n 1+2an(b-a)\sum_{i=1}^n i+(b-a)^2\sum_{i=1}^n i^2\right)\\
&= \lim_{n\to\infty}\frac{b-a}{n^3}\left(a^2n^2(n)+2an(b-a)\frac{n(n+1)}{2}+(b-a)^2\frac{n(n+1)(2n+1)}{6}\right)\\
&= (b-a)\left(a^2+2a(b-a)\frac{1}{2}+\frac{(b-a)^2}{3}\right)\\
&= \frac{1}{3}(b^3-a^3)\ .
\end{aligned}$$

□

*Exercise* 6 (Section 5.2, #29). Express the integral $\int_1^{10}(x-4\ln x)\,dx$ as a limit of Riemann sums as in (1).

*Solution.* We have

$$\int_1^{10}(x-4\ln x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n\left(1+i\frac{10-1}{n}-4\ln\left(1+i\frac{10-1}{n}\right)\right)\frac{10-1}{n}\ .$$

□

*Exercise* 7 (Section 5.2, #57). Use the properties of integrals (page 379 to 381) to verify the following inequality without evaluating an integral:

$$2\le\int_{-1}^1\sqrt{1+x^2}\,dx\le 2\sqrt{2}\ .$$

*Solution.* Since $0\le x^2\le 1$ for all $x\in[-1,1]$, we have $1\le\sqrt{1+x^2}\le\sqrt{2}$ for all $x\in[-1,1]$. Thus

$$2=\int_{-1}^1 1\,dx\overset{(*)}{\le}\int_{-1}^1\sqrt{1+x^2}\,dx\overset{(*)}{\le}\int_{-1}^1\sqrt{2}\,dx=2\sqrt{2}$$

where in the inequalities marked by $(*)$ we used property 7 on page 381. Alternatively, you can obtain the result straight from property 8 on the same page. □