MORE FUNCTION GRAPHING; OPTIMIZATION

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Exercise 1. Let n be an arbitrary positive integer. Give an example of a function with exactly n vertical asymptotes. Give an example of a function with infinitely many vertical asymptotes.

Solution. The function $y = \frac{1}{(x-1)(x-2)\cdots(x-n)}$, with domain of definition $\mathbb{R} \setminus \{1, 2, \ldots, n\}$, has exactly n vertical asymptotes, namely at $x = 1, \ldots, n$. In general, given n distinct numbers $a_1, \ldots, a_n$, the function $y = \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_n)}$, with domain of definition $\mathbb{R} \setminus \{a_1, \ldots, a_n\}$, has exactly n vertical asymptotes, namely at $x = a_1, \ldots, a_n$.

The function $y = \tan x$, with domain of definition $\mathbb{R} \setminus \{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}$, has infinitely many vertical asymptotes, exactly at $x \in \{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}$, which is where $\cos x = 0$. □

Exercise 2. Let f be a function which is differentiable everywhere. Suppose that $f'(x) > 1$ for all x. Show that $\lim_{x\to\infty} f(x) = \infty$.

Solution. Let $x_1 < x_2$ be real numbers. By the Mean Value Theorem, there exists some $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$, and since $f'(c) > 1$ we have $f(x_2) - f(x_1) > x_2 - x_1$. In particular, substituting $x_2 = x$ and $x_1 = 0$, we have $f(x) > x + f(0)$ for all $x > 0$. Thus, since $\lim_{x\to\infty}(x + f(0)) = \infty$, we have $\lim_{x\to\infty} f(x) = \infty$.

An alternate solution is to use the Fundamental Theorem of Calculus. We have

$$f(x) - f(0) \overset{(*)}{=} \int_0^x f'(t)\, dt > \int_0^x 1\, dt = x$$

where we used FTC in the equality marked $(*)$ above.

Be careful about using indefinite integrals. It is not necessarily true that $f(x) > x$ for all x; it is only true for all x greater than some number. For example, consider the case $f(x) = 2x$; then $f'(x) = 2$ so $f'(x) > 1$ for all x. But $f(x) > x$ only if $x > 0$. □

Exercise 3. Graph the function

$$f(x) = x^3 + 6x^2 + 9x\ .$$

Indicate domain, critical points, inflection points, regions where the graph is increasing/decreasing, x-intercepts and y-intercepts, regions of concavity (up or down), local maxima and minima, any asymptotes and behavior at infinity.

Solution. The domain is $\mathbb{R}$. A point $(x, f(x))$ on the graph of the function is a critical point if $f'(x) = 0$ or is not defined; $f(x)$ is differentiable everywhere, so the only way a point can be a critical point is if $f'(x) = 0$. We have

$$f'(x) = 3x^2 + 12x + 9 = 3(x + 1)(x + 3)$$

so the only critical points of f are $(-1, f(-1)) = (-1, -4)$ and $(-3, f(-3)) = (-3, 0)$. A point $(x, f(x))$ on the graph of the function is an inflection point if $f''(x) = 0$ or is not defined; again, the only way a point can be an inflection point is if $f''(x) = 0$. We have

$$f''(x) = 6x + 12$$

so the only critical point of f is $(-2, f(-2)) = (-2)$. We can factor $f(x) = x(x + 3)^2$, so the x-intercepts are $x = 0, -3$. The y-intercept is $f(0) = 0$.

The function f is increasing (resp. decreasing) in the interval I if $f(x_1) < f(x_2)$ (resp. $f(x_1) > f(x_2)$) for all $x_1, x_2 \in I$ such that $x_1 < x_2$ (this is the definition on page 19). This is equivalent to saying that $f'(x) > 0$ (or $f'(x) < 0$) for all x in the interval.[1] Since $f'(x) > 0$ on $(-\infty, -3)$ and $(-1, \infty)$ and $f'(x) < 0$ on $(-3, -1)$, f is increasing on $(-\infty, -3)$, decreasing on $(-3, -1)$, and increasing on $(-1, \infty)$.

A function f is concave up (resp. down) on the interval I if $f''(x) > 0$ (resp. $f''(x) < 0$) for all $x \in I$. Since $f''(x) = 6(x + 2)$, $f''(x) > 0$ on the interval $(-2, \infty)$ and $f''(x) < 0$ on the interval $(-\infty, -2)$. So f is concave up on $(-2, \infty)$ and concave down on $(-\infty, -2)$. A function which is differentiable on all of $\mathbb{R}$ has a local maximum (resp. minimum) at $x = a$ if and only if $f'(a) = 0$ and $f''(a) < 0$ (resp. $f''(a) > 0$). We have $f'(-1) = 0$ and $f''(-1) = 6 > 0$, so f has a local minimum at $x = -1$; we have $f'(-3) = 0$ and $f''(-3) = -6 < 0$, so f has a local maximum at $x = -3$.

There are no vertical asymptotes because the function is continuous on the entire real line. Suppose there is a slant or horizontal asymptote; then there exist real numbers m, b such that either

$$\lim_{x\to\infty} (f(x) - (mx + b)) = 0 \quad \text{or} \quad \lim_{x\to-\infty} (f(x) - (mx + b)) = 0 \ .$$

But $f(x)-(mx+b)$ is a polynomial of degree 3, so we have a contradiction (recall the exercise about endpoint behavior of nonconstant polynomials). So there are no slant or horizontal asymptotes.

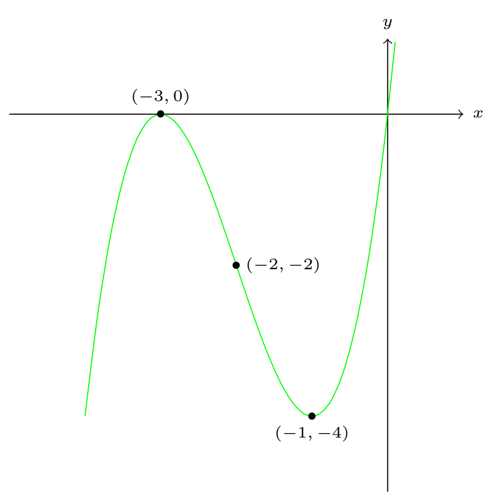

FIGURE 1. Graph of $f(x) = x^3 + 6x^2 + 9x$

□

[1]Notice that the function $f(x) = \frac{1}{x}$ defined on $\mathbb{R} \setminus \{0\}$ has the property that $f'(x) < 0$ for all $x \in \mathbb{R} \setminus \{0\}$, but it is not decreasing on $\mathbb{R} \setminus \{0\}$ (since $f(-1) < f(1)$). This has to do with the fact that f is not defined at $x = 0$.

Exercise 4. Find

$$\lim_{t\to 16} \frac{\sqrt{t}-4}{t-16}$$

in three ways: (i) using methods learned up to and including the first midterm; (ii) by realizing the limit as $f'(c)$ for some function $f(t)$ and some value c; (iii) using L'Hospital's Rule.

Solution. (i) We have

$$\lim_{t\to 16} \frac{\sqrt{t}-4}{t-16} = \lim_{t\to 16} \frac{\sqrt{t}-4}{(\sqrt{t}-4)(\sqrt{t}+4)} = \lim_{t\to 16} \frac{1}{\sqrt{t}+4} = \frac{1}{8}\,.$$

(ii) Set $f(t) = \sqrt{t}$. Then

$$\lim_{t\to 16} \frac{\sqrt{t}-4}{t-16} = \lim_{t\to 16} \frac{f(t)-f(16)}{t-16} = f'(16) = \frac{1/2}{\sqrt{16}} = \frac{1}{8}\,.$$

(iii) We have

$$\lim_{t\to 16} \frac{\sqrt{t}-4}{t-16} = \lim_{t\to 16} \frac{\frac{1/2}{\sqrt{t}}}{1} = \frac{1}{8}\,.$$

□

Exercise 5 (Section 4.7, #19). Find the point on the line $y = 2x + 3$ that is closest to the origin.

Solution. At the point $(x, 2x+3)$, the distance to the origin is

$$d(x) = \sqrt{(x-0)^2 + (2x+3-0)^2} = \sqrt{x^2 + (2x+3)^2} = \sqrt{5x^2 + 12x + 9}$$

by the Pythagorean theorem. The function $d(x)$ is defined for all x and is differentiable everywhere. We want to find a global minimum of $d(x)$. To ease computation, I make the following (perhaps unconventional) argument. Let $p(x) = 5x^2 + 12x + 9$. For any two nonnegative real numbers, the condition $\sqrt{x_1} \le \sqrt{x_2}$ is equivalent to $x_1 \le x_2$. Thus, for any two real numbers x_1, x_2, the condition $d(x_1) \le d(x_2)$ is equivalent to $p(x_1) \le p(x_2)$ (since $d(x) = \sqrt{p(x)}$). Thus $d(a)$ is a global minimum value of the function $d(x)$ if and only if $p(a)$ is a global minimum value of the function $p(x)$. Let's find the global minimum value of $p(x)$. For quadratic polynomials $ax^2 + bx + c$ with $a > 0$, the minimum value takes place at $x = -\frac{b}{2a}$ since that's the vertex of the parabola.[2] So in our case the minimum of $p(x)$ occurs at $x = -\frac{12}{2\cdot 5} = -\frac{6}{5}$. Thus the desired point is $(-\frac{6}{5}, 2(-\frac{6}{5}) + 3) = (-\frac{6}{5}, -\frac{3}{5})$.[3] □

Exercise 6 (Section 4.7, #21). Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

Solution. I'm going to find the point on the upper half of the ellipse which is farthest from the point $(1, 0)$, then note that the ellipse is symmetric with respect to the x-axis, so the reflection will be farthest, too. The equation of the upper part is given by $y = \sqrt{4-4x^2}$, with domain of definition $[-1, 1]$. It is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. The distance from the point $(x, \sqrt{4-4x^2})$ to $(1, 0)$ is given by

$$d(x) = \sqrt{(x-1)^2 + (\sqrt{4-4x^2} - 0)^2} = \sqrt{-3x^2 - 2x + 5}$$

by the Pythagorean theorem. It is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Set $p(x) = -3x^2 - 2x + 5$. By the argument given in the solution to Exercise 5, the problem reduces to finding the maximum

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[2] Exercise: Prove this.

[3] Note that the line joining $(-\frac{6}{5}, -\frac{3}{5})$ to the origin is perpendicular to the line $y = 2x + 3$. This will always be the case for problems of the form "find the point on the line ℓ that is closest to the point P".

of the quadratic $p(x)$ on the interval $[-1,1]$. If the quadratic were defined over all of $\mathbb{R}$, then its global maximum would occur at $x = -\frac{-2}{2\cdot -3} = -\frac{1}{3}$, which is inside the interval $[-1,1]$, so this a global maximum of the quadratic on the interval $[-1,1]$. Thus the point $(-\frac{1}{3}, \sqrt{4-4(-\frac{1}{3})^2}) = (-\frac{1}{3}, \frac{4\sqrt{2}}{3})$ is the point on the upper half of the ellipse which is farthest away from $(1,0)$. On the bottom half, the point $(-\frac{1}{3}, -\frac{4\sqrt{2}}{3})$ is farthest away. So the desired points are $(-\frac{1}{3}, \frac{4\sqrt{2}}{3})$ and $(-\frac{1}{3}, -\frac{4\sqrt{2}}{3})$. □

Exercise 7 (Section 4.7, #24). Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution. Assume without loss of generality that a, b are positive. If $b = a$, then the ellipse is actually a circle and any rectangle inscribed in a circle of radius a is a square of side length $a\sqrt{2}$, so it has area $2a^2$. Now assume $b < a$. I'm going to assume without proof that, for any rectangle inscribed in an ellipse which is not a circle, its sides are parallel to the axes of the ellipse.[4] Since the sides of our rectangles are parallel to the x and y axes, its vertices are $(x,y), (-x,y), (-x,-y), (x,-y)$ for some $x \in [0,a]$ and $y = b\sqrt{1-\frac{x^2}{a^2}}$. Thus the area of the rectangle is

$$A(x) = 2x \cdot 2y = 4bx\sqrt{1-\frac{x^2}{a^2}}$$

where $A(x)$ is a function on the domain $[0,a]$. It is continuous on $[0,a]$ and differentiable on $(0,a)$. We maximize $A(x)$. Let $p(x) = x^2 - \frac{1}{a^2}x^4$. Then $A(x) = 4b\sqrt{p(x)}$. By the argument given in the solution to Exercise 5, the problem reduces to finding the maximum of $p(x)$ on the interval $[0,a]$. Considering $p(x)$ to be a quadratic polynomial in the variable x^2, we have that it takes a maximum when $x^2 = -\frac{1}{2\cdot -\frac{1}{a^2}} = \frac{a^2}{2}$, or when $x = \frac{a}{\sqrt{2}}$, which is inside the interval $[0,a]$. For this value of x, we have $y = b\sqrt{1-\frac{(a/\sqrt{2})^2}{a^2}} = \frac{b}{\sqrt{2}}$, so the rectangle has area $2\frac{a}{\sqrt{2}} \cdot 2\frac{b}{\sqrt{2}} = 2ab$.[5] □

Exercise 8 (Section 4.7, #54). At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

Solution. Let $f(x) = 1 + 40x^3 - 3x^5$. At $(x, f(x))$, the tangent line to the curve $y = f(x)$ has slope $f'(x) = 120x^2 - 15x^4$. So our problem reduces to maximizing $f'(x)$. We have $f''(x) = 240x - 60x^3 = 60x(x-2)(x+2)$, which is 0 only if $x = 0, \pm 2$. We have $f'''(x) = 240 - 180x^2$ and $f'''(0) = 240 > 0, f'''(2) = -480 < 0, f'''(-2) = -480 < 0$, so 0 is a local minimum while ± 2 are local maxima. Since $f''(x) > 0$ on the intervals $(-\infty, -2)$ and $(0,2)$ and $f''(x) < 0$ on the intervals $(-2,0)$ and $(2,\infty)$, we have that $f'(x)$ is increasing on the intervals $(-\infty,-2)$ and $(0,2)$ and $f'(x)$ is decreasing on the intervals $(-2,0)$ and $(2,\infty)$. Thus $(-2, f(-2)) = (-2,-223)$ and $(2, f(2)) = (2, 225)$ are the global maxima of $f'(x)$.

Alternatively, you can complete the square in $f'(x)$. We have $f'(x) = -15(x^4 - 8x^2)$. Completing the square gives $f'(x) = -15(x^4 - 8x^2 + 16) + 240 = -15(x^2-4)^2 + 240$, so $f'(x)$ takes its maximum whenever

[4]This is a minor technical point which is less relevant to Math 1A but is still necessary in a complete solution: if a rectangle is inscribed in an ellipse which is not a circle, how do I know that its sides are going to be parallel to the x and y axes? (For example, if the ellipse is a circle, then inscribed rectangles are squares, whose sides don't have to be parallel to the axes.) I hope this following argument will satisfy you. Assume without loss of generality that $b < a$ and a rectangle is inscribed in the ellipse such that its sides aren't parallel to the axes of the ellipse. Apply the linear transformation which shrinks everything along the x-axis by the factor $k = \frac{b}{a}$. So the ellipse now becomes a circle of radius b. Under this transformation, the rectangle that was inscribed in the ellipse becomes a parallelogram. The four sides of the original rectangle have slope $m, -\frac{1}{m}, m, -\frac{1}{m}$ for some nonzero m, so the parallelogram has four sides whose slopes are $mk, -\frac{k}{m}, mk, -\frac{k}{m}$. Since $k > 1$, adjacent sides of the parallelogram aren't perpendicular, and the parallelogram is not a rectangle. Thus the parallelogram has an acute angle, and such parallelograms cannot be inscribed in circles, contradiction.

[5]Note that this gives the same answer as above for the special case $b = a$.

$(x^2-4)^2$ is minimized, i.e. when $x^2-4=0$, or $x=\pm 2$. Thus our points are $(-2, f(-2)) = (-2, -223)$ and $(2, f(2)) = (2, 225)$. $\square$