## GRAPHS OF FUNCTIONS; L'HOSPITAL'S RULE

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The line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L\ . \tag{1}$$

The line $x = a$ is called a *vertical asymptote* of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty\ . \tag{2}$$

The line $y = mx + b$, for $m \neq 0$, is called a *slant asymptote* of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} \big(f(x) - (mx + b)\big) = 0 \quad \text{or} \quad \lim_{x\to-\infty} \big(f(x) - (mx + b)\big) = 0\ . \tag{3}$$

*Exercise* 1. Give an example of a function $f$ which is continuous everywhere but has a horizontal asymptote. Give an example of a function $f$ which is continuous everywhere but has a slant asymptote.

*Solution.* The simplest example of a function with a horizontal asymptote is the function $y = 0$. More interesting examples are $y = \frac{1}{x^2+1}$ and $y = e^x$. The simplest example of a function with a horizontal asymptote is the function $y = mx + b$ for $m \neq 0$. More interesting examples are $y = x(e^{-x} + 1)$ and $y = x + \frac{1}{x}\sin x$.

Note that, if the curve $y = f(x)$ has a slant asymptote $mx+b$, then, for any $c$, the curve $y = f(x)-(mx+c)$ has a horizontal asymptote at $y = b - c$.

A function can have two distinct horizontal asymptotes; consider the function $f(x) = \frac{e^x+2}{e^x+1}$. What are the asymptotes for this function? No curve defined as the graph of a function has more than two horizontal asymptotes. (Why?) □

*Exercise* 2. Give a proof, using $N$ and $\delta$, of the following fact: If a function $f(x)$ is continuous at $x = a$, then $f(x)$ cannot have a vertical asymptote at $x = a$.

*Solution.* (Note: I added "at $x = a$" to the end, since otherwise the statement is false.) Suppose that $f$ is continuous at $x = a$. Then $\lim_{x\to a} f(x) = f(a)$. Suppose that $f$ has a vertical asymptote at $x = a$. Then at least one of (2) is true. By shifting the function by $a$, we can assume that $a = 0$. Also, by reflecting $f$ over the line $x = 0$ and/or $y = 0$, we can assume that $\lim_{x\to 0^+} f(x) = \infty$. While one might be tempted to conclude here that, since $f(0) \neq \infty$, we have a contradiction, there is more that needs to be said because limits involving infinity have a different format: the statement "$\lim_{x\to 0} f(x) = f(0)$" means "for every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x| < \delta$ implies $|f(x) - f(0)| < \epsilon$" and the statement "$\lim_{x\to 0^+} f(x) = \infty$" means "for every $N > 0$ there exists $\delta > 0$ such that $0 < x < \delta$ implies $f(x) > N$". In particular, there exists $\delta_1 > 0$ such that $0 < |x| < \delta_1$ implies $|f(x) - f(0)| < 1$ (we're taking $\epsilon = 1$). There exists $\delta_2 > 0$ such that $0 < x < \delta_2$ implies $f(x) > f(0) + 1$ (we're taking $N = f(0) + 1$). Consider any $x$ such that $0 < |x - 0| < \delta_1$ and $0 < x < \delta_2$ (i.e. $0 < x < \min\{\delta_1, \delta_2\}$). Then $|f(x) - f(0)| < 1$ and $f(x) > f(0) + 1$, contradiction.

(Regarding the Note: Let $f(x) = \frac{1}{x}$ on $(-\infty, 0) \cup (0, \infty)$ and $a = 1$. Then $f$ is continuous at $x = a$ and still has a vertical asymptote at $x = 0$.) □

*Exercise* 3. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with $n \geq 1$ and $a_n \neq 0$. Prove that $\lim_{x\to\infty} f(x) = \infty$ if $a_n > 0$ and $-\infty$ if $a_n < 0$. Use this to show that $\lim_{x\to-\infty} f(x) = \infty$ if either (i) $n$ is even and $a_n > 0$ or (ii) $n$ is odd and $a_n < 0$; show that $\lim_{x\to-\infty} f(x) = -\infty$ if either (iii) $n$ is odd and $a_n > 0$ or (iv) $n$ is even and $a_n < 0$.

*Solution.* Suppose $a_n > 0$. We prove the statement "$\lim_{x\to\infty} f(x) = \infty$", which means "for every $N > 0$, there exists $M$ such that $x > M$ implies $f(x) > N$". Let $N > 0$. We're done if we can find $M$ such that $x > M$ implies $f(x) > N$. We're done if we can find $M$ such that $x > M$ implies $a_n x^n - |a_{n-1}x^{n-1}| - \cdots - |a_1 x| - |a_0| > N$, because $f(x)$ is even bigger than $a_n x^n - |a_{n-1}x^{n-1}| - \cdots - |a_1 x| - |a_0|$. If $x \geq 1$, then $a_n x^n - |a_{n-1}x^{n-1}| - \cdots - |a_1 x| - |a_0| \geq a_n x^n - (|a_{n-1}| + \cdots + |a_1| + |a_0|)x^{n-1}$, so it suffices to find $M \geq 1$ such that $x > M$ implies $a_n x^n - (|a_{n-1}| + \cdots + |a_1| + |a_0|)x^{n-1} > N$. Choose $M = \max\{\frac{|a_{n-1}|+\cdots+|a_1|+|a_0|+N}{a_n}, 1\}$. Suppose $x > M$. Then $x > \frac{|a_{n-1}|+\cdots+|a_1|+|a_0|+N}{a_n}$, so $a_n x - (|a_{n-1}| + \cdots + |a_1| + |a_0|) > N$. Also, $x > 1$, so $x^{n-1} > 1$, and thus $a_n x^n - (|a_{n-1}| + \cdots + |a_1| + |a_0|)x^{n-1} > N$, which implies $f(x) > N$ as above.

If $a_n < 0$, then note that the leading coefficient of the polynomial $-f(x)$ is positive and apply the above proof to $-f(x)$, observing that $\lim_{x\to\infty} f(x) = \infty$ if and only if $\lim_{x\to\infty}(-f(x)) = -\infty$.

If $n$ is even and $a_n > 0$, then the leading coefficient of $f(-x) = a_n x^n - a_{n-1}x^{n-1} + \ldots - a_1 x + a_0$ is positive and $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x)$, so we have $\lim_{x\to-\infty} f(x) = \infty$ as above. The other cases are analogous; consider either $-f(x)$, $f(-x)$, or $-f(-x)$. □

*Exercise* 4 (Section 4.4, #53). Find the limit $\lim_{x\to\infty}(x - \ln x)$.

*Solution.* Note that $x - \ln x = \ln(\frac{e^x}{x})$. We have $\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$ by L'Hospital. Thus $\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} \ln(\frac{e^x}{x}) = \infty$, where the last step follows from the fact that if $f, g$ are two functions such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} f(g(x)) = \infty$. □

*Exercise* 5 (Section 4.4, #61). Find the limit $\lim_{x\to\infty} x^{1/x}$.

*Solution.* Note that $x^{1/x} = e^{\frac{1}{x}\ln x}$. We have $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$ by L'Hospital. Thus $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^0 = 1$, where the last step follows from the fact that if $f, g$ are two functions such that $f$ is continuous and $\lim_{x\to\infty} g(x) = a$, then $\lim_{x\to\infty} f(g(x)) = f(a)$. □

*Exercise* 6 (Section 4.4, #67). Find the limit $\lim_{x\to\infty}(1+\frac{2}{x})^x$. More generally, find the limit $\lim_{x\to\infty}(b+\frac{a}{x})^x$ where $a$ is a real number and $b$ is a positive real number.

*Solution.* Suppose that $b = 1$. Then $(1 + \frac{a}{x})^x = e^{x\ln(1+\frac{a}{x})}$, and $\lim_{x\to\infty} x\ln(1 + \frac{a}{x}) = \lim_{x\to\infty} \frac{\ln(1+\frac{a}{x})}{1/x} = \lim_{x\to\infty} \frac{1/(1+\frac{a}{x})(-a/x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{a}{1+\frac{a}{x}} = a$. Thus $\lim_{x\to\infty}(1 + \frac{a}{x})^x = \lim_{x\to\infty} e^{x\ln(1+\frac{a}{x})} = e^a$, where the last step follows from the fact that if $f, g$ are two functions such that $f$ is continuous and $\lim_{x\to\infty} g(x) = a$, then $\lim_{x\to\infty} f(g(x)) = f(a)$.

Suppose that $b > 1$. Then $\lim_{x\to\infty}(b + \frac{a}{x}) = b > 1$, so $\lim_{x\to\infty}(b + \frac{a}{x})^x = \infty$ (not an indeterminate form). Suppose that $b > 1$. Then $\lim_{x\to\infty}(b + \frac{a}{x}) = b < 1$, so $\lim_{x\to\infty}(b + \frac{a}{x})^x = 0$ (not an indeterminate form). □

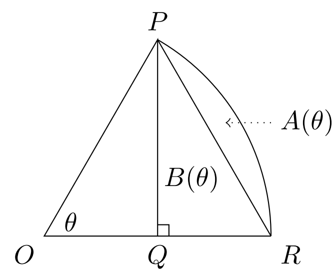
*Exercise* 7 (Section 4.4, #71). Prove that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for any positive integer $n$. (Intuition: The exponential function grows faster than any large power of $x$.)

*Solution.* We have $\lim_{x\to\infty} \frac{e^x}{1} = \infty$. If we assume that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ and prove $\lim_{x\to\infty} \frac{e^x}{x^{n+1}} = \infty$, we will be done by induction. We have $\lim_{x\to\infty} \frac{e^x}{x^{n+1}} = \lim_{x\to\infty} \frac{e^x}{(n+1)x^n} = \infty$, where the first step follows by L'Hospital and the second by assumption. □

*Exercise* 8 (Section 4.4, #72). Prove that $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ for any $p > 0$. (Intuition: The natural logarithm grows slower than any small power of $x$. Application to computer science: Many sorting algorithms have a worst-case running time of $O(n \ln n)$, which is better than $O(n^{1+p})$ for any $p > 0$. But it's worse than $O(n)$. Wait, what?)

*Solution.* We have $\lim_{x\to\infty} \frac{1}{x^p} = 0$ for any $p > 0$. Thus $\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \frac{1}{p} \lim_{x\to\infty} \frac{1}{x^p} = 0$. □

*Exercise* 9 (Section 4.4, #82). Let $A(\theta)$ be the area of the region between the chord $PR$ and the arc $PR$. Let $B(\theta)$ be the area of the triangle $PQR$. Find $\lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)}$.




*Solution.* The problem doesn't specify lengths because it doesn't matter; scaling everything by a constant preserves ratios of areas. Let $r = PO = OR$. Then $A(\theta) = \frac{\theta}{2\pi}(\pi r^2) - \frac{1}{2} \cdot r \cdot (r \sin\theta)$, and $B(\theta) = \frac{1}{2}(r - r\cos\theta)(r\sin\theta)$. Then

$$\frac{A(\theta)}{B(\theta)} = \frac{\frac{\theta}{2\pi}(\pi r^2) - \frac{1}{2} \cdot r \cdot (r\sin\theta)}{\frac{1}{2}(r - r\cos\theta)(r\sin\theta)} = \frac{\theta - \sin\theta}{(1-\cos\theta)(\sin\theta)}$$

so

$$\begin{aligned}
\lim_{\theta\to 0} \frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0} \frac{\theta - \sin\theta}{(1-\cos\theta)(\sin\theta)} \\
&\overset{(1)}{=} \lim_{\theta\to 0} \frac{1-\cos\theta}{\sin^2\theta + (1-\cos\theta)(\cos\theta)} \\
&\overset{(2)}{=} \lim_{\theta\to 0} \frac{\sin\theta}{2\sin\theta\cos\theta + \sin\theta\cos\theta + (1-\cos\theta)(-\sin\theta)} \\
&= \lim_{\theta\to 0} \frac{1}{2\cos\theta + \cos\theta - (1-\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{1}{4\cos\theta - 1} \\
&= \frac{1}{3}
\end{aligned}$$

where we used L'Hospital in the steps marked (1) and (2). □

*Exercise* 10 (Section 4.4, #84). Let $f$ and $g$ be functions such that $f(x) > 0$ for all $x$. Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$. Show that $\lim_{x\to a}(f(x))^{g(x)} = 0$. (Hint: This is not an indeterminate form. Intuition: You are multiplying increasingly many copies of increasingly small numbers.)

*Solution 1.* Since $\lim_{x\to a} f(x) = 0$, we have $\lim_{x\to a} \ln(f(x)) = -\infty$. Thus $\lim_{x\to a} \ln(f(x))\cdot g(x) = -\infty$. Thus $\lim_{x\to a}(f(x))^{g(x)} = \lim_{x\to a} e^{\ln(f(x))\cdot g(x)} = 0$. (Here, we're using the following facts: (1) if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = -\infty$ then $\lim_{x\to a} f(x)g(x) = -\infty$; (2) if $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to -\infty} g(x) = 0$, then $\lim_{x\to a} g(f(x)) = 0$.) □

*Solution 2.* Let $\epsilon > 0$. Since $\lim_{x\to a} f(x) = 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies $|f(x)| < \min\{\epsilon, 1\}$. Since $\lim_{x\to a} g(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ implies $g(x) > 1$. Set $\delta = \min\{\delta_1, \delta_2\}$ and suppose that $0 < |x-a| < \delta$. Then $|f(x)| < \epsilon$ and $|f(x)| < 1$ and $g(x) > 1$. Thus $|(f(x))^{g(x)} - 0| = (f(x))^{g(x)} < f(x) < \epsilon$. Hence $\lim_{x\to a}(f(x))^{g(x)} = 0$. $\square$