GRAPHS OF FUNCTIONS; L'HOSPITAL'S RULE

WED, OCT 23, 2013

Recall (or look up) the definitions of *horizontal asymptote* (pg. 131), *vertical asymptote* (pg. 94), and *slant asymptote* (pg. 315).

Exercise 1. Give an example of a function f which is continuous everywhere but has a horizontal asymptote. Give an example of a function f which is continuous everywhere but has a slant asymptote.

Exercise 2. Give a proof, using N and δ, of the following fact: If a function $f(x)$ is continuous at $x = a$, then $f(x)$ cannot have a vertical asymptote.

Exercise 3. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with $n \geq 1$ and $a_n \neq 0$. Prove that $\lim_{x\to\infty} f(x) = \infty$ if $a_n > 0$ and $-\infty$ if $a_n < 0$. Use this to show that $\lim_{x\to-\infty} f(x) = \infty$ if either (i) n is even and $a_n > 0$ or (ii) n is odd and $a_n < 0$; show that $\lim_{x\to-\infty} f(x) = -\infty$ if either (iii) n is odd and $a_n > 0$ or (iv) n is even and $a_n < 0$.

Exercise 4 (Section 4.4, #53). Find the limit $\lim_{x\to\infty}(x - \ln x)$.

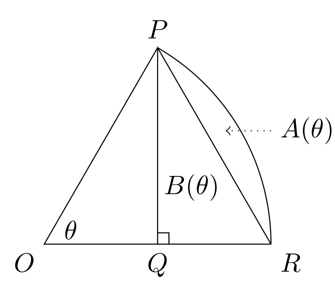
Exercise 5 (Section 4.4, #61). Find the limit $\lim_{x\to\infty} x^{1/x}$.

Exercise 6 (Section 4.4, #67). Find the limit $\lim_{x\to\infty}(1+\frac{2}{x})^x$. More generally, find the limit $\lim_{x\to\infty}(b+\frac{a}{x})^x$ where a is a real number and b is a positive real number.

Exercise 7 (Section 4.4, #71). Prove that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for any positive integer n. (Intuition: The exponential function grows faster than any large power of x.)

Exercise 8 (Section 4.4, #72). Prove that $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ for any $p > 0$. (Intuition: The natural logarithm grows slower than any small power of x. Application to computer science: Many sorting algorithms have a worst-case running time of $O(n \ln n)$, which is better than $O(n^{1+p})$ for any $p > 0$. But it's worse than $O(n)$. Wait, what?)

Exercise 9 (Section 4.4, #82). Let $A(\theta)$ be the area of the region between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)}$.



Exercise 10 (Section 4.4, #84). Let f and g be functions such that $f(x) > 0$ for all x. Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$. Show that $\lim_{x\to a}(f(x))^{g(x)} = 0$. (Hint: This is not an indeterminate form. Intuition: You are multiplying increasingly many copies of increasingly small numbers.)