ROLLE'S THEOREM; MEAN VALUE THEOREM

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Theorem 1 (Rolle's)**.** *Let f be a function such that:*

(i) *f is continuous on $[a, b]$;*
(ii) *f is differentiable on (a, b);*
(iii) *$f(a) = f(b)$.*

Then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 2 (Mean Value)**.** *Let f be a function such that:*

(i) *f is continuous on $[a, b]$;*
(ii) *f is differentiable on (a, b).*

Then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Exercise 3. Let f be a function. Show that if f is differentiable at $x = a$, then f is continuous at $x = a$.

Solution. By the definition of differentiability, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0 \ .$$

We have

$$\lim_{x \to a} x - a = 0 \ .$$

Then

$$\begin{aligned} \lim_{x \to a} f(x) - f(a) &= \lim_{x \to a} \left(\left(\frac{f(x) - f(a)}{x - a} \right) (x - a) \right) \\ &\overset{(*)}{=} \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \cdot \left(\lim_{x \to a} x - a \right) \\ &= 0 \end{aligned}$$

where the step marked $(*)$ follows by the product rule. Thus

$$\lim_{x \to a} f(x) = f(a) \ ,$$

which is what it means for f to be continuous at $x = a$. □

Exercise 4. Show that Rolle's Theorem is a special case of the Mean Value Theorem.

Solution. Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b) and such that $f(a) = f(b)$. By the Mean Value Theorem, there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, but $f(a) = f(b)$ implies $\frac{f(b)-f(a)}{b-a} = 0$, so $f'(c) = 0$. □

(In lecture, the Mean Value Theorem was proved using Rolle's Theorem. Since each can be used to prove the other, it follows that Rolle's Theorem and the Mean Value Theorem are equivalent.)

Exercise 5. Let $f(x) = |x|$. Show that $f(-1) = f(1)$, but there does not exist any $c \in (-1, 1)$ such that f is differentiable at c and $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

Solution. Rolle's Theorem does not apply to f because there exists a point in the interval $(-1, 1)$ such that f is not differentiable at that point, namely at $x = 0$. □

Exercise 6. Let $f(x) = \frac{1}{x}$ defined on $(-\infty, 0) \cup (0, \infty)$. Show that the line through $(1, f(1))$ and $(-1, f(-1))$ has slope 1, but there does not exist any $c \in (-\infty, 0) \cup (0, \infty)$ such that f is differentiable at c and $f'(c) = 1$. Why does this not contradict the Mean Value Theorem?

Solution. Notice that if $x \in (-\infty, 0) \cup (0, \infty)$, then $f'(x) = -\frac{1}{x^2}$, which is always negative, hence cannot equal 1. This does not contradict MVT because the function f is not defined on the whole interval $[-1, 1]$. Perhaps a more interesting fact is that we cannot make f continuous by giving f a value at $x = 0$; this is because the two one-sided limits at $x = 0$ are different: $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. □

Exercise 7. Let f be the function $f(x) = e^x$. Let $a_1 < a_2$ be two real numbers and set $P_1 = (a_1, f(a_1))$ and $P_2 = (a_2, f(a_2))$. Let $L = \overline{P_1 P_2}$ be the line containing P_1 and P_2. Show that there exists a unique real number c such that the tangent line to f at $x = c$ is parallel to the line L.

Solution 1. Let ℓ_c denote the tangent line to f at $x = c$. Then L is parallel to ℓ_c if and only if their slopes are equal. The slope of ℓ_c is $f'(c) = e^c$ and the slope of L is $\frac{f(a_2)-f(a_1)}{a_2-a_1}$. Thus the slope of L is equal to the slope of ℓ_c if and only if $c = \ln\left(\frac{f(a_2)-f(a_1)}{a_2-a_1}\right)$. □

Solution 2. (May be more useful if you don't know the inverse of $f'(x)$, such as in the case $f(x) = x^6 + x^4 + x^2 + 1$) As before, we want to show that there exists a unique c such that $f'(c) = \frac{f(a_2)-f(a_1)}{a_2-a_1}$. By the MVT, there exists $c \in (a_1, a_2)$ such that $f'(c) = \frac{f(a_2)-f(a_1)}{a_2-a_1}$. Notice that $f'(x)$ is strictly increasing. So $f'(x) > f'(c)$ if $x > c$ and $f'(x) < f'(c)$ if $x < c$; thus such a c is unique. □

Exercise 8 (Section 4.2, Exercise #18). Show that the equation $x^3 + e^x = 0$ has exactly one real root.

Solution. Set $f(x) = x^3 + e^x$. Notice that f is differentiable (in particular, continuous) everywhere. Notice that $f(-1) = -1 + \frac{1}{e} < 0$ and $f(1) = 1 + e > 0$. Thus, by the Intermediate Value Theorem, there exists $c \in (-1, 1)$ such that $f(c) = 0$. Let's show that f has no other roots. Suppose that β is another root, i.e. $\beta \neq c$ and $f(\beta) = 0$. By Rolle's Theorem, there exists some α strictly between c and β such that $f'(\alpha) = 0$. But $f'(x) = 3x^2 + e^x > 0$ for all x. This is a contradiction; thus f has exactly one root. □

Exercise 9 (Section 4.2, Exercise #26). Suppose that f is an odd function and is differentiable everywhere. Prove that, for every positive real number b, there exists $c \in (-b, b)$ such that $f'(c) = \frac{f(b)}{b}$.

Solution. Recall that a function f is *odd* if $f(x) = -f(-x)$ for all x. This is just saying that the graph of f is symmetric about the origin. Plugging in $x = 0$, we have $f(0) = 0$. Since f is differentiable everywhere (in particular, continuous everywhere), by the MVT there exists $c_1 \in (0, b)$ such that $f'(c_1) = \frac{f(b)-f(0)}{b-0} = \frac{f(b)}{b}$. Similarly, there exists $c_2 \in (-b, 0)$ such that $f'(c_2) = \frac{f(0)-f(-b)}{0-(-b)} = \frac{f(b)}{b}$. So there actually exists at least two values of c such that $f'(c) = \frac{f(b)}{b}$ ($c_1 \neq c_2$ because $c_1 > 0$ and $c_2 < 0$).

By the way, notice that $f(x) = -f(-x)$ implies $f'(x) = f'(-x)$ for all x, by the chain rule. So if $f'(c) = \frac{f(b)}{b}$ then $f'(-c) = \frac{f(b)}{b}$. □