ROLLE'S THEOREM; MEAN VALUE THEOREM

MON, OCT 21, 2013

Theorem 1 (Rolle's)**.** *Let f be a function such that:*

(i) *f is continuous on $[a, b]$;*
(ii) *f is differentiable on (a, b);*
(iii) *$f(a) = f(b)$.*

Then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 2 (Mean Value)**.** *Let f be a function such that:*

(i) *f is continuous on $[a, b]$;*
(ii) *f is differentiable on (a, b).*

Then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Exercise 3. Let f be a function. Show that if f is differentiable at $x = a$, then f is continuous at $x = a$.

Exercise 4. Show that Rolle's Theorem is a special case of the Mean Value Theorem.

(In lecture, the Mean Value Theorem was proved using Rolle's Theorem. Since each can be used to prove the other, it follows that Rolle's Theorem and the Mean Value Theorem are equivalent.)

Exercise 5. Let $f(x) = |x|$. Show that $f(-1) = f(1)$, but there does not exist any $c \in (-1, 1)$ such that f is differentiable at c and $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

Exercise 6. Let $f(x) = \frac{1}{x}$ defined on $(-\infty, 0) \cup (0, \infty)$. Show that the line through $(1, f(1))$ and $(-1, f(-1))$ has slope 1, but there does not exist any $c \in (-\infty, 0) \cup (0, \infty)$ such that f is differentiable at c and $f'(c) = 1$. Why does this not contradict the Mean Value Theorem?

Exercise 7. Let f be the function $f(x) = e^x$. Let $a_1 < a_2$ be two real numbers and set $P_1 = (a_1, f(a_1))$ and $P_2 = (a_2, f(a_2))$. Let $L = \overline{P_1P_2}$ be the line containing P_1 and P_2. Show that there exists a unique real number c such that the tangent line to f at $x = c$ is parallel to the line L.

Exercise 8 (Section 4.2, Exercise #18). Show that the equation $x^3 + e^x = 0$ has exactly one real root.

Exercise 9 (Section 4.2, Exercise #26). Suppose that f is an odd function and is differentiable everywhere. Prove that, for every positive real number b, there exists $c \in (-b, b)$ such that $f'(c) = \frac{f(b)}{b}$.