CHAPTER 3 REVIEW

(*Last edited October 23, 2013 at 1:59pm.*)

Problem 1 (Section 3.1 Exercise #51). *Find the points on the curve* $y = 2x^3 + 3x^2 - 12x + 1$ *where the tangent line is horizontal.*

Solution. The tangent line is horizontal if and only if the slope is 0. The slope of the tangent line to the curve at (x, y) is $y'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$, and this is 0 exactly when $x = -2$ or $x = 1$. $\square$

Problem 2 (Section 3.1 Exercise #53). *Show that the curve* $y = 2e^x + 3x + 5x^3$ *has no tangent line with slope* 2.

Solution. The slope of the tangent line at (x, y) is $y'(x) = 2e^x + 3 + 15x^2$. So we want to show that $y'(x) \neq 2$ for all x. In fact, we can show that the slope is always greater than 3: since $2e^x > 0$ for all x and $15x^2 \geq 0$ for all x, we have $y'(x) = 2e^x + 3 + 15x^2 > 0 + 3 + 0 = 3$ for all x. $\square$

Problem 3 (Section 3.1 Exercise #67). *Let*

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}.$$

Is f *differentiable at* $x = 1$*? Sketch the graphs of* f *and* f'*.*

Solution. Recall that f is differentiable at $x = 1$ if and only if the limit $\lim_{x\to1} \frac{f(x)-f(1)}{x-1}$ exists. The function $\frac{f(x)-f(1)}{x-1}$ is equal to $x + 1$ if $x < 1$ and 1 if $x > 1$, hence $\lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} x + 1 = 2$ and $\lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} 1 = 1$. Since the two one-sided limits do not agree, the (two-sided) limit does not exist, and f is not differentiable at $x = 1$. For the graph, refer to figure 1. The green solid line is the graph of f, and the red dashed line is the graph of f'. Note that while f is defined on all of $\mathbb{R}$, f' is not defined at $x = 0$ (precisely because f is not differentiable at $x = 0$).

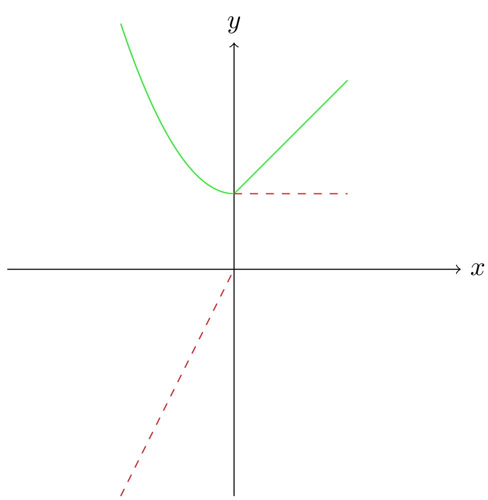

FIGURE 1. Graphs of f and f'

□

Problem 4 (Section 3.2 Exercise #33). *Find equations of the tangent line and normal line to the curve $y = 2xe^x$ at the point $(0,0)$.*

Solution. The slope of the tangent line to the curve at $(0,0)$ is $y'(0) = 2e^0 + 2(0)e^0 = 2$. So the tangent line at $(0,0)$ is $y - 0 = 2(x-0)$, or just $y = 2x$. By definition, the normal line is perpendicular to the tangent line and passes through the tangent point, so its equation is $y = 0 = -\frac{1}{2}(x-0)$, or just $y = -\frac{1}{2}x$. □

Problem 5 (Section 3.2 Exercise #55). *Find $R'(0)$, where*

$$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9} .$$

Solution. Set $f(x) = x - 3x^3 + 5x^5$ and $g(x) = 1 + 3x^3 + 6x^6 + 9x^9$. Then $f'(x) = 1 - 9x^2 + 25x^4$ and $g'(x) = 9x^2 + 36x^5 + 81x^8$. Thus we have

$$\begin{aligned} R'(0) &= \frac{f'(0) \cdot g(0) + f(0) \cdot g'(0)}{(g(0))^2} \\ &= \frac{(1)(1) + (0)(0)}{(1)^2} \\ &= 1 . \end{aligned}$$

Notice that I didn't have to compute the general formula for $R'(x)$. □

Problem 6 (Section 3.4 Exercise #72). *If g is a twice-differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', g''.*

Solution. We have

$$\begin{aligned} f'(x) &= (x)'g(x^2) + x(g(x^2))' \\ &= g(x^2) + x \cdot g'(x^2)(2x) \\ &= g(x^2) + 2x^2 g'(x^2) \end{aligned}$$

and

$$\begin{aligned} f''(x) &= (g(x^2))' + (2x^2)'g'(x^2) + 2x^2(g'(x^2))' \\ &= g'(x^2)2x + (4x)g'(x^2) + 2x^2 g''(x^2)2x \\ &= 6xg'(x^2) + 4x^3 g''(x^2) . \end{aligned}$$

□

Problem 7 (Section 3.5 Exercise #21). *If $f(x) + x^2(f(x))^3 = 10$ and $f(1) = 2$, find $f'(1)$.*

Solution. Differentiate implicitly:

$$f'(x) + 2x(f(x))^3 + x^2 \cdot 3(f(x))^2 f'(x) = 0 .$$

At $x = 1$, we have

$$f'(1) + 2 \cdot 1(f(1))^3 + 1^2 \cdot 3(f(1))^2 f'(1) = 0 ,$$

or

$$f'(1) + 16 + 12f'(1) = 0$$

which implies $f'(1) = -\frac{16}{13}$. □

Problem 8 (Section 3.6 Exercise #51). *Find y' if $y = \ln(x^2 + y^2)$.*

Solution. We have $e^y = x^2+y^2$; differentiate implicitly to get $e^y y' = 2x+2yy'$. We thus have $y' = \frac{2x}{e^y-2y}$. □

Problem 9 (Section 3.6 Exercise #54). *Find $\frac{d^9}{dx^9}(x^8 \ln x)$.*

Solution. For any integer n, define

$$f_n(x) = \frac{d^{n+1}}{dx^{n+1}}(x^n \ln x) \ .$$

Then we have

$$\begin{aligned} f_n(x) &= \frac{d^n}{dx^n}\left(\frac{d}{dx}(x^n \ln x)\right) \\ &= \frac{d^n}{dx^n}(nx^{n-1}\ln x + x^{n-1}) \\ &= \frac{d^n}{dx^n}(nx^{n-1}\ln x) + \frac{d^n}{dx^n}(x^{n-1}) \\ &\overset{(*)}{=} \frac{d^n}{dx^n}(nx^{n-1}\ln x) \\ &= nf_{n-1}(x) \end{aligned}$$

where, in the step labeled $(*)$, I used the fact that the nth derivative of any polynomial of degree less than n is 0 (this is just a repeated application of the Power Rule). We're asked to find $f_8(x)$. We have

$$\begin{aligned} f_8(x) &= 8 \cdot f_7(x) \\ &= 8 \cdot 7 \cdot f_6(x) \\ &= 8 \cdot 7 \cdot 6 \cdot f_5(x) \\ &\vdots \\ &= 8 \cdot 7 \cdots 2 \cdot 1 \cdot f_0(x) \\ &= 8! \frac{d^1}{dx^1}(x^0 \ln x) \\ &= \frac{8!}{x} \ . \end{aligned}$$

□

Problem 10 (Section 3.8 Exercise #10). *A sample of tritium-3 decayed to* 94.5% *of its original amount after a year. What is the half-life of tritium-3? How long would it take the sample to decay to* 20% *of its original amount?*

Solution. Let $y(t)$ be the quantity left at time t, in years. Then $y(t) = y(0)e^{Ct}$ for some C. Let's determine C. We are given that $y(1) = 0.925y(0)$, so $0.945y(0) = y(0)e^{C\cdot 1}$ implies $0.945 = e^C$ implies $C = \ln(0.945)$.

Let λ be the half-life of tritium-3. For any t, we have $y(t+\lambda) = \frac{1}{2}y(t)$. In particular, $y(\lambda) = \frac{1}{2}y(0)$ so $\frac{1}{2}y(0) = y(0)e^{\ln(0.945)\lambda}$, which implies $\frac{1}{2} = e^{\ln(0.945)\lambda}$, or $\lambda = \frac{\ln\frac{1}{2}}{\ln(0.945)}$. Notice that both $\frac{1}{2}$ and 0.945 are less than 1, so $\ln\frac{1}{2}$ and $\ln(0.945)$ are both negative, so λ is positive.

Let t_0 be the time required for the sample to decay to 20% of its original amount. Then $y(t_0) = 0.20y(0)$. Thus $0.20y(0) = y(0)e^{\ln(0.945)t_0}$. Thus $0.20 = e^{\ln(0.945)t_0}$, or $t_0 = \frac{\ln(0.20)}{\ln(0.945)}$. □

Problem 11 (Section 3.9 Exercise #7(a))**.** *Suppose* $y = \sqrt{2x+1}$*, where* x *and* y *are functions of* t*. If* $\frac{dx}{dt} = 3$*, find* $\frac{dy}{dt}$ *when* $x = 4$*.*

Solution. Differentiate both sides of $y = \sqrt{2x+1}$ with respect to t to get $\frac{dy}{dt} = \frac{1/2}{\sqrt{2x+1}} \cdot 2 \cdot \frac{dx}{dt}$. Suppose that $\frac{dx}{dt}(t_0) = 3$ and $x(t_0) = 4$. Then $\frac{dy}{dt}(t_0) = \frac{1}{\sqrt{2x(t_0)+1}} \cdot \frac{dx}{dt}(t_0) = \frac{1}{\sqrt{2 \cdot 4+1}} \cdot 3 = 1$. □

Problem 12 (Section 3.9 Exercise #31)**.** *The top of a ladder slides down a vertical wall at a rate of* 0.15 *m/s. At the moment when the bottom of the ladder is* 3 *m from the wall, it slides away from the wall at a rate of* 0.2 *m/s. How long is the ladder?*

Solution. Let ℓ be the length of the ladder. (Using common sense, we're going to assume that it stays constant over time.) Let $y(t)$ be the distance from the top of the ladder to the floor, and $x(t)$ the distance from the bottom of the ladder to the base of the wall. We have $\ell^2 = (x(t))^2 + (y(t))^2$ for all t. By differentiating implicitly, we have $0 = 2x(t)x'(t) + 2y(t)y'(t)$. We're also given that $y'(t) = -0.15$ for all t. Suppose that $x(t_0) = 3$ and $x'(t_0) = 0.2$. Thus $0 = 2(3)(0.2) + 2y(t_0)(-0.15)$, which implies $y(t_0) = 4$. This implies $\ell^2 = (3)^2 + (4)^2 = 25$, or that $\ell = 5$. □