EXPONENTIAL GROWTH AND DECAY; RELATED RATES

(*Last edited October 16, 2013 at 3:32pm.*)

Problem 1 (Section 3.8 Exercise #9)**.** *The half-life of cesium-137 is 30 years. Suppose we have a* 100*-mg sample.*

(a) *Find the mass that remains after* t *years.*
(b) *How much of the sample remains after* 100 *years?*
(c) *After how long will only* 1 *mg remain?*

Solution. Let $y(t)$ be the quantity of the sample at time t. Since cesium undergoes radioactive decay, we know that $y(t)$ is of the form

$$y(t) = y(0)e^{Ct}$$

for some constant C. We know that $y(t+30) = \frac{1}{2}y(t)$, so $y(0)e^{C(t+30)} = \frac{1}{2}y(0)e^{Ct}$, which implies $e^{30C} = \frac{1}{2}$. Thus $C = \frac{\ln\frac{1}{2}}{30}$. We also know that $y(0) = 100$. Thus

$$y(t) = 100e^{\frac{\ln\frac{1}{2}}{30}t} = 100(\tfrac{1}{2})^{\frac{t}{30}} .$$

The quantity of cesium remaining after 100 years is $y(100) = 100(\frac{1}{2})^{\frac{100}{30}} \approx 9.92$ mg. (Intuitively, more than 3 but less than 4 half-lives have passed, so there should be less than $\frac{1}{8}$ and more than $\frac{1}{16}$ of the original amount remaining.) To find out at what time only 1 mg will remain, we set $y(t) = 1$ and solve for t: $1 = 100(\frac{1}{2})^{\frac{t}{30}} \iff \frac{1}{100} = (\frac{1}{2})^{\frac{t}{30}} \iff 100 = 2^{\frac{t}{30}} \iff \ln 100 = \frac{t}{30}\ln 2 \iff t = \frac{30\ln 100}{\ln 2}$. □

Problem 2 (Section 3.8 Exercise #12)**.** *A curve passes through the point* $(0,5)$ *and has the property that the slope of the curve at every point* P *is twice the* y*-coordinate of* P*. What is the equation of the curve?*

Solution. Let $y(t)$ be the equation of the curve. Since the curve passes through the point $(0,5)$, we have $y(0) = 5$. The condition "the slope of the curve at every point P is twice the y-coordinate at P" is just saying that $\frac{dy}{dt} = 2y(t)$. We proved in lecture that $y(t) = y(0)e^{2t}$ is the unique solution to this equation. Thus $y(t) = 5e^{2t}$. □

Problem 3 (Section 3.9 Exercise #6)**.** *The radius of a sphere is increasing at the rate of* 4 *mm/s. How fast is the volume increasing when the diameter is* 80 *mm?*

Solution. Let $V(t)$ and $r(t)$ be the volume and radius of the sphere at time t, respectively. Then we have

$$V(t) = \frac{4\pi}{3}(r(t))^3 .$$

Taking the derivative with respect to t gives

$$V'(t) = 4\pi(r(t))^2 r'(t) .$$

We are given that $r'(t) = 4$ for all t and that for some t_0 we have $2r(t_0) = 80$, so $r(t_0) = 40$. Thus $V'(t_0) = 4\pi(r(t_0))^2 r'(t_0) = 4\pi(40)^2(4) = 25600\pi$. □

Problem 4 (Section 3.9 Exercise #9)**.** *If* $x^2 + y^2 + z^2 = 9$*,* $\frac{dx}{dt} = 5$*, and* $\frac{dy}{dt} = 4$*, find* $\frac{dz}{dt}$ *when* $(x,y,z) = (2,2,1)$.

Solution. Differentiating with respect to t, we have

$$2x(t)\frac{dx}{dt}(t) + 2y(t)\frac{dy}{dt}(t) + 2z(t)\frac{dz}{dt}(t) = 0 \ .$$

At some time t_0, we have $\frac{dx}{dt}(t_0) = 5$, $\frac{dy}{dt}(t_0) = 4$, $x(t_0) = 2$, $y(t_0) = 2$, and $z(t_0) = 1$. Thus

$$2(2)(5) + 2(2)(4) + 2(1)\frac{dz}{dt}(t_0) = 0 \ ,$$

which implies $\frac{dz}{dt}(t_0) = -18$. □

Problem 5 (Section 3.9 Exercise #22)**.** *A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $(\frac{1}{3}, 1)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?*

Solution. Let $(x(t), y(t))$ be the position of the particle at time t. Let $d(t)$ be the distance from the particle to the origin at time t. We have

$$d(t)^2 = x(t)^2 + y(t)^2 \qquad 2d(t)d'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

$$y(t) = 2\sin(\tfrac{\pi}{2}x(t)) \qquad y'(t) = 2\cos(\tfrac{\pi}{2}x(t))\tfrac{\pi}{2}x'(t)$$

Let "this instant" refer to time $t = t_0$. Then $(x(t_0), y(t_0)) = (\frac{1}{3}, 1)$ and $x'(t_0) = \sqrt{10}$. Thus

$$\begin{aligned}
d(t_0)d'(t_0) &= x(t_0)x'(t_0) + y(t_0)y'(t_0) \\
\sqrt{(1/3)^2 + 1^2} \cdot d'(t_0) &= \tfrac{1}{3} \cdot \sqrt{10} + 1 \cdot \left(2\cos(\tfrac{\pi}{2} \cdot \tfrac{1}{3}) \cdot \tfrac{\pi}{2} \cdot \sqrt{10}\right) \\
\tfrac{\sqrt{10}}{3} d'(t_0) &= \tfrac{\sqrt{10}}{3} + (2\tfrac{\sqrt{3}}{2} \cdot \tfrac{\pi}{2} \cdot \sqrt{10}) \\
d'(t_0) &= 1 + \tfrac{3\pi\sqrt{3}}{2}
\end{aligned}$$

□

Problem 6 (Section 3.9 Exercise #23)**.** *Water is leaking out of an inverted conical tank at a rate of* 10,000 *cm*^{3}*/min at the same time that water is being pumped into the tank at a constant rate. The tank has height* 6 *m and the diameter at the top is* 4 *m. If the water level is rising at a rate of* 20 *cm/min when the height of the water is* 2 *m, find the rate at which water is being pumped into the tank.*

Solution. I'll convert everything into meters. Let $h(t), r(t), V(t)$ be the height of the water level, the radius at the top of the water, and the volume of the water at time t. Let $t = t_0$ be the instant of time we are interested in. We have $h(t_0) = 2$ and $h'(t_0) = 0.2$. By similar triangles, we have $\frac{r(t)}{h(t)} = \frac{2}{6}$ so $r(t) = \frac{1}{3}h(t)$. Since $V(t) = \frac{1}{3}(\pi(r(t)^2)h(t) = \frac{\pi}{27} \cdot h(t)^3$, thus $V'(t) = \frac{\pi}{9}h(t)^2h'(t)$. Substituting $t = t_0$, we have $V'(t_0) = \frac{\pi}{9}(2)^2(0.2) = \frac{4\pi}{45}$. Let C be the rate (in m^3/min) at which water is being pumped into the tank. Then $V'(t) = -\frac{10{,}000}{(100)^3} + C$ (we divide by 100^3 since the units changed from cm^3/min to m^3/min). Thus $C = V'(t_0) + \frac{1}{100} = \frac{4\pi}{45} + \frac{1}{100}$. □