MATH 1A WORKSHEET

MON, OCT 14, 2013

Exponential Growth and Decay

Consider the equation

$$\frac{dy}{dt} = Cy \tag{1}$$

where C is a constant. The question we are interested in is: given a real number k, does there exist a function $y(t)$ such that $y(0) = k$ and y satisfies equation (1)? The answer is yes: the unique solution to (1) is

$$y(t) = ke^{Ct}\ . \tag{2}$$

In exponential growth, the constant C is positive. In exponential decay, the constant C is negative.

A "half-life" is a length of time, and different radioactive substances have different half-lives. Let's say the half-life of substance A is 20 years. Then $y(t+20)$ is half of $y(t)$ for all t: $y(23.5) = \frac{1}{2} \cdot y(3.5)$, $y(10029) = \frac{1}{2} \cdot y(10009)$, etc. In general, if the half-life of a substance is λ, then $C = \frac{\ln \frac{1}{2}}{\lambda}$. Be careful to use the same units for λ and t.

Problem 1 (Section 3.8 Exercise #9)**.** *The half-life of cesium-137 is 30 years. Suppose we have a* 100*-mg sample.*

(a) *Find the mass that remains after t years.*
(b) *How much of the sample remains after* 100 *years?*
(c) *After how long will only* 1 *mg remain?*

Problem 2 (Section 3.8 Exercise #12)**.** *A curve passes through the point* $(0, 5)$ *and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?*

Related Rates

In related rates problems, you're usually given at least two quantities (say x and y) which are functions of a third variable (say t), and an equation

$$F(x(t), y(t)) = 0 \tag{3}$$

relating the two quantities. Differentiate (3) with respect to t, and you get an equation involving $x(t)$, $y(t)$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$.

Problem 3 (Section 3.9 Exercise #6)**.** *The radius of a sphere is increasing at the rate of* 4 *mm/s. How fast is the volume increasing when the diameter is* 80 *mm?*

Problem 4 (Section 3.9 Exercise #9)**.** *If* $x^2 + y^2 + z^2 = 9$*,* $\frac{dx}{dt} = 5$*, and* $\frac{dy}{dt} = 4$*, find* $\frac{dz}{dt}$ *when* $(x, y, z) = (2, 2, 1)$.

Problem 5 (Section 3.9 Exercise #22)**.** *A particle moves along the curve* $y = 2\sin(\pi x/2)$*. As the particle passes through the point* $(\frac{1}{3}, 1)$*, its x-coordinate increases at a rate of* $\sqrt{10}$ *cm/s. How fast is the distance from the particle to the origin changing at this instant?*

Problem 6 (Section 3.9 Exercise #23)**.** *Water is leaking out of an inverted conical tank at a rate of* $10,000$ *cm*^{3}*/min at the same time that water is being pumped into the tank at a constant rate. The tank has height* 6 *m and the diameter at the top is* 4 *m. If the water level is rising at a rate of* 20 *cm/min when the height of the water is* 2 *m, find the rate at which water is being pumped into the tank.*