MATH 1A WORKSHEET SOLUTIONS

(*Last edited October 12, 2013 at 12:22am.*)

Notation: "$f'(x)$" is the same thing as "$\frac{df}{dx}$", and "$f^{(n)}(x)$" is the same thing as "$\frac{d^n f}{dx^n}$". Whenever I say "the curve $F(x, y) = 0$", what I really mean is "the set of points (x, y) in the plane satisfying the condition $F(x, y) = 0$".

Problem 1. *Find the derivative of the following functions.*

(a) $f(x) = x^{\cos x}$, *defined on the domain* $(0, \infty)$
(b) $f(x) = \log_5(3x^2 - 2)$, *defined on the domain* $(-\infty, -\sqrt{2/3}) \cup (\sqrt{2/3}, \infty)$
(c) $f(x) = (\sqrt{x})^x$, *defined on the domain* $(0, \infty)$

Solution. (a) Take the natural log of both sides to get

$$\ln(f(x)) = \cos x \cdot \ln x \ .$$

Take the derivative with respect to x of both sides (using the chain rule on the left hand side and the product rule on the right hand side) to get

$$\tfrac{1}{f(x)} f'(x) = (-\sin x)(\ln x) + (\cos x)(\tfrac{1}{x})$$

so that

$$\begin{aligned} f'(x) &= f(x)\left((-\sin x)(\ln x) + (\cos x)(\tfrac{1}{x})\right) \\ &= x^{\cos x}\left((-\sin x)(\ln x) + (\cos x)(\tfrac{1}{x})\right) \ . \end{aligned}$$

(b) Write $f(x)$ as

$$f(x) = x^{\frac{1}{2}x} = e^{\frac{1}{2}x \ln x}$$

so that we have, by the chain rule,

$$f'(x) = e^{\frac{1}{2}x \ln x} \cdot \left(\frac{1}{2} x \ln x\right)' = x^{\frac{1}{2}x} \cdot \left(\frac{1}{2}\ln x + \frac{1}{2}\right) \ .$$

Alternatively, take the natural log of both sides to get

$$\ln(f(x)) = \frac{1}{2} x \ln x$$

and differentiate both sides (using the chain rule on the left hand side) to get

$$\frac{1}{f(x)} f'(x) = \frac{1}{2}\ln x + \frac{1}{2} x \cdot \frac{1}{x}$$

so that

$$f'(x) = x^{\frac{1}{2}x}\left(\frac{1}{2}\ln x + \frac{1}{2}\right) \ .$$

(c) Remember that, in general, $\log_x y = \frac{\ln y}{\ln x}$ (but these expressions are not equivalent to $\ln(\frac{x}{y})$, which equals $\ln(x) - \ln(y)$). So we can write $f(x)$ as

$$f(x) = \frac{\ln(3x^2 - 2)}{\ln 5} = \frac{1}{\ln 5} \cdot \ln(3x^2 - 2)$$

so that

$$f'(x) = \frac{1}{\ln 5} \cdot \frac{1}{3x^2 - 2} \cdot 6x \ .$$

□

Problem 2. *Let f be the function $f(x) = \ln(x-1)$, defined on the domain $(1, \infty)$. Find $\frac{d^n f}{dx^n}$ for any positive integer n.*

Solution. We have

$$\begin{aligned} f^{(1)}(x) &= (x-1)^{-1} \\ f^{(2)}(x) &= (-1) \cdot (x-1)^{-2} \\ f^{(3)}(x) &= (-1)(-2) \cdot (x-1)^{-3} \\ f^{(4)}(x) &= (-1)(-2)(-3) \cdot (x-1)^{-4} \\ &\vdots \\ f^{(n)}(x) &= (-1)(-2)\cdots(-(n-1)) \cdot (x-1)^{-n} \end{aligned}$$

so that

$$f^{(n)}(x) = (-1)^{n-1} \cdot (n-1)! \cdot (x-1)^{-n}$$

for all positive integers n. □

Problem 3. *Find $\frac{dy}{dx}$ by implicit differentiation.*

(a) $x^3 + y^3 = 6xy$
(b) $x \sin y + y \sin x = 1$

Solution. (a) We differentiate implicitly to get

$$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx} \ .$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} \ .$$

(b) We differentiate implicitly to get

$$\sin y + x(\cos y)\frac{dy}{dx} + (\sin x)\frac{dy}{dx} + y \cos x = 0 \ .$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{-y \cos x - \sin y}{x \cos y + \sin x} \ .$$

□

Problem 4. *Find all points on the curve C defined by*

$$x^2y^2 + xy = 2 \tag{1}$$

where the slope of the tangent line is -1.

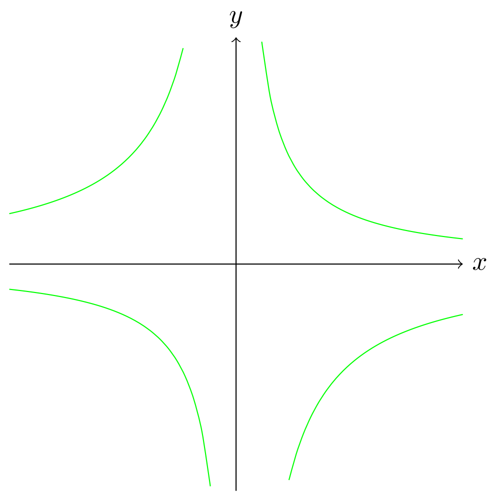

FIGURE 1. The curve $x^2y^2 + xy = 2$

Solution 1. We differentiate (1) implicitly to get

$$2xy^2 + 2x^2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \; ;$$

solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{2xy^2 + y}{2x^2y + x} \; . \tag{2}$$

Thus if the slope of the tangent line to the curve at a point (x_0, y_0) is -1, then we must have

$$-\frac{2x_0y_0^2 + y_0}{2x_0^2y_0 + x_0} = -1 \; ,$$

or

$$2x_0y_0^2 + y_0 = 2x_0^2y_0 + x_0$$

which rearranges to

$$(x_0 - y_0)(2x_0y_0 + 1) = 0 \; .$$

This means that either $x_0 - y_0 = 0$ or $2x_0y_0 + 1 = 0$.

Suppose that $2x_0y_0 + 1 = 0$; thus $x_0y_0 = -\frac{1}{2}$. Substituting into (1), we get

$$x_0^2y_0^2 + x_0y_0 - 2 = \left(-\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}\right) - 2 \neq 0 \; ,$$

which is a contradiction.

Thus we must have $x_0 - y_0 = 0$, which implies that $x_0 = y_0$. Thus $x_0^4 + x_0^2 - 2 = 0$. This factors as $(x_0^2 + 2)(x_0^2 - 1) = 0$. Since $x_0^2 + 2$ is always nonzero, we must have $x_0^2 - 1 = 0$, or $x_0 = \pm 1$. So $(-1, -1)$ and $(1, 1)$ are the two desired points. We can check this, by substituting $(x, y) = (-1, -1)$ and $(x, y) = (1, 1)$

into (2), obtaining

$$\frac{dy}{dx} = -\frac{2(-1)(-1)^2 + (-1)}{2(-1)^2(-1) + (-1)} = -1 \quad \text{and} \quad \frac{dy}{dx} = -\frac{2(1)(1)^2 + (1)}{2(1)^2(1) + (1)} = -1 ,$$

respectively. □

Solution 2. The following is true in general: when a curve is given implicitly by an equation of the form $(y - f(x))(y - g(x)) = 0$, then it is the union of the graphs of $y = f(x)$ and $y = g(x)$. This is because, if x is fixed, then y must equal either $f(x)$ or $g(x)$ in order to satisfy $(y - f(x))(y - g(x)) = 0$.

How does this help us? Let (x_0, y_0) be a point on the curve such that the tangent line at (x_0, y_0) has slope -1. The given equation is equivalent to $(xy)^2 + (xy) - 2 = 0$, which factors as

$$(xy + 2)(xy - 1) = 0 . \tag{3}$$

If $x = 0$, then $x^2y^2 + xy - 2 \neq 0$, so C does not contain any points of the form $(0, y)$. Thus we can divide by x^2 in (3) to get

$$\left(y + \frac{2}{x}\right)\left(y - \frac{1}{x}\right) = 0 .$$

This is the union of the two hyperbolas $y = -\frac{2}{x}$ and $y = \frac{1}{x}$. Their derivatives are $y' = \frac{2}{x^2}$ and $y' = \frac{-1}{x^2}$, respectively. We can never have $\frac{2}{x_0^2} = -1$ since x_0^2 is always nonnegative. So the point (x_0, y_0) must be on the hyperbola $y = \frac{1}{x}$, and we must have $\frac{-1}{x_0^2} = -1$. This implies $x_0 = \pm 1$. Since $y_0 = \frac{1}{x_0}$, the two desired points are $(-1, -1)$ and $(1, 1)$. □

Problem 5. (a) *Draw a graph of the curve $x^2 - y^2 = 1$. Use implicit differentiation to find the tangent line at all points except $(-1, 0)$ and $(1, 0)$. What goes wrong for these points? What is the tangent line to the curve at the point $(1, 0)$?*

(b) *Draw a graph of the curve $x^2 - y^2 = 0$. Argue that there isn't a good way to define the "tangent line" to the curve at the point $(0, 0)$.*

(c) *Draw a graph of the curve $y^2 - x^4 = 0$. Use implicit differentiation to find the tangent line at all points except $(0, 0)$. What goes wrong for $(0, 0)$? Find the tangent line to the curve at $(0, 0)$.*

Solution. (a) Let C be the curve $x^2 - y^2 = 1$.

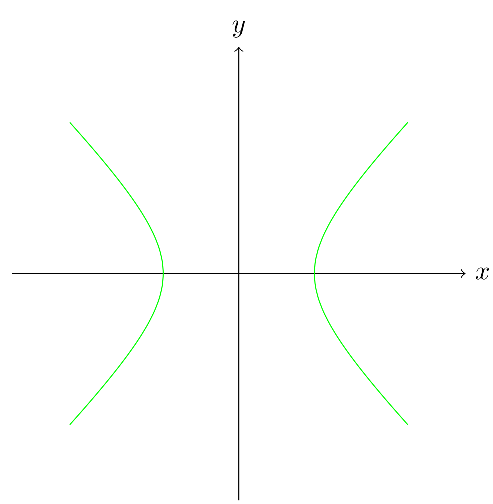

FIGURE 2. The curve $x^2 - y^2 = 1$

Differentiate the given equation $x^2 - y^2 = 1$ implicitly to get

$$2x - 2y\frac{dy}{dx} = 0$$

and solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = \frac{x}{y}\,. \tag{4}$$

This is valid for all points (x_0, y_0) on C where $y_0 \neq 0$ (otherwise you end up dividing by 0 in (4)), and the only points of the form $(x_0, 0)$ on C are $(-1, 0)$ and $(1, 0)$. Thus the tangent line to C at (x_0, y_0), where $(x_0, y_0) \notin \{(-1, 0), (1, 0)\}$, is given by $\frac{y-y_0}{x-x_0} = \frac{x_0}{y_0}$, or $y - y_0 = \frac{x_0}{y_0}(x - x_0)$.

In order to find the tangent lines at $(-1, 0)$ and $(1, 0)$, view x as a function of y and differentiate implicitly:

$$2x\frac{dx}{dy} - 2y = 0$$

so that

$$\frac{dx}{dy} = \frac{y}{x}\,.$$

At $(-1, 0)$, we have $\frac{dx}{dy} = \frac{0}{-1} = 0$ so the tangent line is $\frac{x-(-1)}{y-(0)} = 0$, which is just $x = -1$. Similarly, at $(1, 0)$, we have $\frac{dx}{dy} = \frac{0}{1} = 0$ so the tangent line is $\frac{x-(1)}{y-(0)} = 0$, which is just $x = 1$.

(b) Let C be the curve $x^2 - y^2 = 0$. Since the condition $x^2 - y^2 = 0$ is equivalent to $(x + y)(x - y) = 0$, C is just the union of the two lines $x = y$ and $x = -y$.

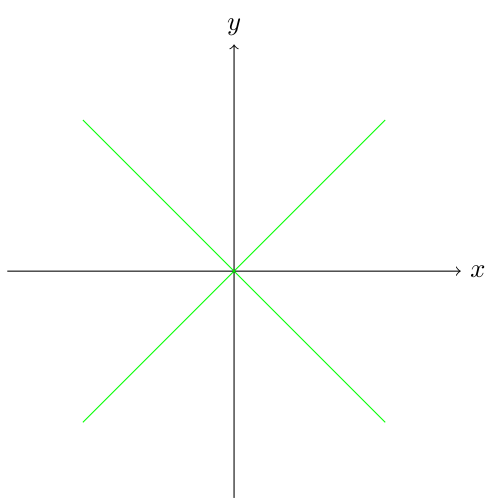

FIGURE 3. The curve $x^2 - y^2 = 0$

Informal argument: At $(0, 0)$, two lines of slope 1 and -1 cross each other. The tangent line to each point on the line $y = x$ has slope 1, and the tangent line to each point on the line $y = -x$ has slope -1; but a line cannot have slope equal to both 1 and -1.

Attempt to formalize the informal argument: For any $\epsilon > 0$, there exist points $P_1, P_2 \in C$ (with $P_i = (x_i, y_i)$) such that

(i) $P_1 \neq (0, 0)$ and $P_2 \neq (0, 0)$;

(ii) for each i, the distance between P_i and $(0, 0)$ is less than ϵ (i.e. $\sqrt{(x_i - 0)^2 + (y_i - 0)^2} < \epsilon$);

(iii) P_1 is on the line $y = x$ and P_2 is on the line $y = -x$.
Then the slope of the line containing $(0, 0)$ and P_1 has slope 1, and the slope of the line containing $(0, 0)$ and P_2 has slope -1.

(c) Let C be the curve $y^2 - x^4 = 0$. Since the condition $y^2 - x^4 = 0$ is equivalent to $(y - x^2)(y + x^2) = 0$, C is just the union of the two parabolas $y = x^2$ and $y = -x^2$.

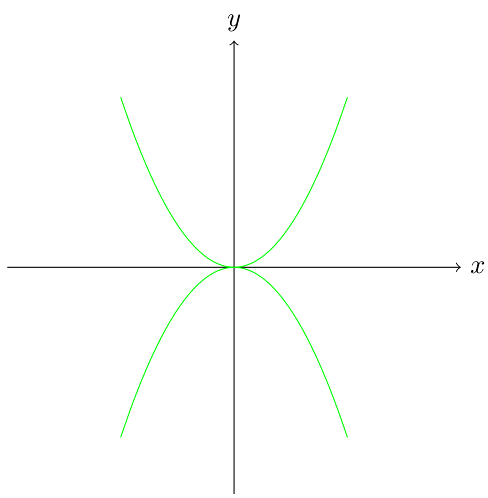

FIGURE 4. The curve $y^2 - x^4 = 0$

Differentiate implicitly to get

$$2y\frac{dy}{dx} - 4x^3 = 0$$

so that

$$\frac{dy}{dx} = \frac{4x^3}{y} \, .$$

So that tangent line to C at the point $(x_0, y_0) \in C$ has slope $\frac{4x_0^3}{y_0}$ (as long as $y_0 \neq 0$), and its equation is $y - y_0 = \frac{4x_0^3}{y_0}(x - x_0)$.

You can use any method you like to check that the tangent line to $y = x^2$ at $(0, 0)$ has slope 0 and that the tangent line to $y = -x^2$ at $(0, 0)$ has slope 0. Thus the tangent line to C at $(0, 0)$ is $y - 0 = 0(x - 0)$, which is just $y = 0$.

□