PRACTICE PROBLEMS FOR MATH 1A

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Limits of sequences

Problem 1. *Define the sequence* $a_n = \frac{n}{2^n}$ *for all* $n \geq 1$. *Prove that* $\lim_{n\to\infty} a_n = 0$.

Solution. We first prove that $0 < a_n < 2(\frac{2}{3})^n$ for $n \geq 4$. First, $a_4 = \frac{4}{16}$ and $2(\frac{2}{3})^4 = \frac{32}{81}$ so we have $a_4 < 2(\frac{2}{3})^4$. Assume $n \geq 4$ and that $0 < a_n < 2(\frac{2}{3})^n$. We have $\frac{n+1}{2n} < \frac{2}{3}$, so $a_{n+1} = \frac{n+1}{2n} a_n < \frac{2}{3} \cdot 2(\frac{2}{3})^n = 2(\frac{2}{3})^{n+1}$. Thus $0 < a_{n+1} < 2(\frac{2}{3})^{n+1}$. Now we proceed to the main argument. Let $\varepsilon > 0$. Set

$$N = \max\{\lceil \log_{3/2} 2/\varepsilon \rceil + 1, 4\} .$$

Then if $n \geq N$, we have $n > \log_{3/2} 2/\varepsilon$, which is equivalent to $(\frac{3}{2})^n > \frac{2}{\varepsilon}$ and to $2(\frac{2}{3})^n < \varepsilon$. Since $n \geq 4$, we have $0 < a_n < 2(\frac{2}{3})^n$ by the argument above. Thus $|a_n - 0| < \varepsilon$.

(Intuition/scratch work: The numerator of a_n is n and that of a_{n+1} is $n+1$, so the numerator increases by a factor of $\frac{n+1}{n}$. Meanwhile, the denominator increases by a factor of 2. Since $\frac{n+1}{n}$ approaches 1, we have $a_{n+1} \approx \frac{1}{2} a_n$ for large values of n. In particular, for $n > 3$, we have $a_n < \frac{2}{3} a_n$. So the sequence a_n is "squeezed" between 0 and $(\frac{2}{3})^n c_0$ for some fixed constant c_0.) ☐

Limits and continuity of functions

Problem 2. *Let* $f(x) = x^5$ *for all* x. *Prove that* $\lim_{x\to 0} f(x) = 0$.

Solution. Let $\varepsilon > 0$. Set $\delta = \sqrt[5]{\varepsilon}$. If x is a real number such that $0 < |x - 0| < \delta = \sqrt[5]{\varepsilon}$, then $|x^5 - 0| = |x|^5 < (\sqrt[5]{\varepsilon})^5 = \varepsilon$. Hence $\lim_{x\to 0} f(x) = 0$.

(How you might find δ: the condition $|x^5 - 0| < \epsilon$ is equivalent to $|x - 0| < \sqrt[5]{\varepsilon}$ by taking the 5th root of both sides.) ☐

Note: We can replace every instance of 5 in the above problem with any other positive integer and obtain a valid solution.

Problem 3. *Let* $f(x) = x^2 - 1$ *for all* x. *Prove that* $\lim_{x\to 0} f(x) = -1$.

Solution. Let $\varepsilon > 0$. Set $\delta = \sqrt[2]{\varepsilon}$. If x is a real number such that $0 < |x - 0| < \delta = \sqrt[2]{\varepsilon}$, then $|(x^2 - 1) - (-1)| = |x|^2 < (\sqrt[2]{\varepsilon})^2 = \varepsilon$. Hence $\lim_{x\to 0} f(x) = -1$.

(How you might find δ: the condition $|(x^2 - 1) - (-1)| < \epsilon$ is equivalent to $|x| < \sqrt[2]{\varepsilon}$ by canceling the $(-1) - (-1)$ and taking the square root of both sides.) ☐

Problem 4. *Let* $f(x) = \frac{1}{x}$ *for all* $x > 0$. *Prove that* $\lim_{x\to 1} f(x) = 1$.

Date: Sat, Sep 14, 2013.

Solution. Let $\varepsilon > 0$. Choose $\delta > 0$ so that $\delta < \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$. If x is a real number such that $0 < |x-1| < \delta$, then in particular $|x-1| < \frac{1}{2}$ so $\frac{1}{2} < x < \frac{3}{2}$, which means $|x| > \frac{1}{2}$, or $\frac{1}{|x|} < 2$. Also, $|x-1| < \frac{\varepsilon}{2}$. Then multiplying $|x-1| < \frac{\varepsilon}{2}$ and $\frac{1}{|x|} < 2$ gives $\frac{|x-1|}{|x|} < \frac{\varepsilon}{2} \cdot 2$, which is equivalent to $|\frac{1}{x} - 1| < \varepsilon$.

(Alternate solution) Let $\varepsilon > 0$. Choose $\delta > 0$ so that $\delta < \frac{\varepsilon}{1+\varepsilon}$. If x is a real number such that $0 < |x-1| < \delta$, then $|x-1| < \frac{\varepsilon}{1+\varepsilon} \iff -\frac{\varepsilon}{1+\varepsilon} < x - 1 < \frac{\varepsilon}{1+\varepsilon} \iff \frac{1}{1+\varepsilon} < x < \frac{1+2\varepsilon}{1+\varepsilon} \iff \frac{1+\varepsilon}{1+2\varepsilon} < \frac{1}{x} < 1+\varepsilon \iff \frac{-\varepsilon}{1+2\varepsilon} < \frac{1}{x} - 1 < \varepsilon$. The last condition implies that $|\frac{1}{x} - 1| < \max\{|\frac{-\varepsilon}{1+2\varepsilon}|, |\varepsilon|\} = \varepsilon$.

(How you might find δ: First, deal with the case $\varepsilon \geq 1$ separately. Now suppose $\varepsilon < 1$. Find the points where the lines $y = 1+\varepsilon$ and $y = 1-\varepsilon$ intersect the graph of $f(x)$; they are $(\frac{1}{1+\varepsilon}, 1+\varepsilon)$ and $(\frac{1}{1-\varepsilon}, 1-\varepsilon)$, respectively. Since $f(x)$ is strictly decreasing where we defined it, we have that if $\frac{1}{1+\varepsilon} < x < \frac{1}{1-\varepsilon}$ then $1 - \varepsilon < f(x) < 1 + \varepsilon$. In other words, if we want $f(x)$ to be within ε of 1, x better be between $\frac{1}{1+\varepsilon}$ and $\frac{1}{1-\varepsilon}$. Since we want our interval centered around $x = 1$, we can take $\delta = \min\{|\frac{1}{1+\varepsilon} - 1|, |\frac{1}{1-\varepsilon} - 1|\} = \min\{\frac{\varepsilon}{1+\varepsilon}, \frac{\varepsilon}{1-\varepsilon}\} = \frac{\varepsilon}{1+\varepsilon}$.) □

In order to understand objects with property P, it helps to see some examples of objects which do not have property P. We now see two examples of functions which are not continuous at a particular point.

Problem 5. *Let*

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

Prove that $\lim_{x\to 0} f(x)$ *does not exist (i.e.* $f(x)$ *is not continuous at* $x = 0$*).*

Solution. We argue by contradiction. Suppose that $\lim_{x\to 0} f(x)$ does exist and is equal to some L. Thus, taking $\varepsilon = 1$ in the definition, there exists some $\delta > 0$ such that $|f(x) - L| < 1$ for every real number x satisfying $0 < |x - 0| < \delta$. Let $x_0 = \min\{\frac{1}{|L|+1}, \frac{\delta}{2}\}$. Since $|x_0| \leq \frac{\delta}{2} < \delta$, we have $|f(x_0) - L| < 1$, and we substitute $f(x_0) = \frac{1}{x_0}$ to get $|\frac{1}{x_0} - L| < 1$. Since $x_0 \leq \frac{1}{|L|+1}$, we have $|L| + 1 \leq \frac{1}{x_0}$ and $|L| - L + 1 \leq \frac{1}{x_0} - L$. But this contradicts $|\frac{1}{x_0} - L| < 1$ since $1 \leq |L| - L + 1$.

(Intuition/scratch work: we can see easily that $f(x) \to +\infty$ as $x \to 0^+$ and $f(x) \to -\infty$ as $x \to 0^-$, and since the left/right limits are different the limit should not exist at 0.) □

Recall that a real number x is *rational* if there exist integers p and q such that $x = \frac{p}{q}$.

Problem 6. *Let*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ -1 & \text{if } x \text{ is an irrational number} \end{cases}.$$

Prove that $f(x)$ *is not continuous at* $x = 0$.

Solution. Fact 1: For any $\delta > 0$, there exists a rational number x_1 such that $0 < x_1 < \delta$. For example, we can take $x_1 = \frac{1}{\lceil \frac{1}{\delta} \rceil + 1}$: we have $\frac{1}{x_1} = \lceil \frac{1}{\delta} \rceil + 1 > \frac{1}{\delta}$

which means $x_1 < \delta$. Also, x_1 is nonzero, and it is a rational number since we can set $p = 1$ and $q = \lceil \frac{1}{\delta} \rceil + 1$.

Fact 2: For any $\delta > 0$, there exists an irrational number x_2 such that $0 < x_2 < \delta$. For example, we can take $x_2 = \frac{\sqrt{2}}{2} x_1$: since $\frac{\sqrt{2}}{2} < 1$, we have $x_2 < x_1$ which means $x_2 < \delta$. Also, x_2 is nonzero, and it is an irrational number since $\sqrt{2}$ is irrational.

Now we proceed to the main argument. We show that $f(x)$ is not continuous at 0 by contradiction. So suppose that $f(x)$ is continuous at 0. By definition, this means that there exists $\delta > 0$ such that $0 < |x - 0| < \delta$ implies $|f(x) - f(0)| < 1$ (we took $\varepsilon = 1$ in the definition). By Fact 1, there exists a rational number x_1 between 0 and δ. By Fact 2, there exists an irrational number x_2 between 0 and δ. So $f(x_1) = 1$ and $f(x_2) = -1$. Since $0 < |x_1 - 0| < \delta$ and $0 < |x_2 - 0| < \delta$, we have $|f(x_1) - f(0)| < 1$ and $|f(x_2) - f(0)| < 1$. By the Triangle Inequality[1], we have

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(0)| + |f(x_2) - f(0)| \,. \tag{1}$$

But the left hand side in (1) is 2, and each of the summands in the right hand side is less than 1, so (1) implies $2 < 2$, which is a contradiction. Hence $f(x)$ is not continuous at $x = 0$. □

(By the way, this is a perfectly good function. The graph of $f(x)$ looks like the union of two horizontal lines $y = 1$ and $y = -1$, except that the line $y = 1$ is punctured at $(x, 1)$ for every irrational x and the line $y = 2$ is punctured at $(x, -1)$ for every rational x. It's really messed up in the sense that it's not continuous at any point of its domain. As an exercise, adapt the solution above to prove that the function is not continuous at any other x, either.)

[1] The Triangle Inequality says that for any real numbers a, b, c, we have $|a - c| \leq |a - b| + |c - b|$.