PRACTICE MIDTERM 2 SOLUTIONS

(*Last edited November 12, 2013 at 6:56pm.*)

Problem 1. (i) Complete the sentence: "A function f is continuous at $x = a$ if ..." Give an example of a function f and some a in the domain of f such that f is not continuous at $x = a$.

(ii) Complete the sentence: "A function f is differentiable at $x = a$ if ..." Give an example of a function f and some a in the domain of f such that f is not differentiable at $x = a$.

Solution. (i) A function f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. (For the question of continuity to make sense, the domain of definition of f needs to contain an open interval around a.) The function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is not continuous at $x = 0$, since $\lim_{x\to 0^+} f(x) = 1 \neq f(0)$ and $\lim_{x\to 0^-} f(x) = 1 \neq f(0)$.

(ii) A function f is differentiable at $x = a$ if the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and is finite. (For the question of differentiability to make sense, the domain of definition of f needs to contain an open interval around a.) The function $f(x) = |x|$ is not differentiable at $x = 0$, since $\frac{f(x+h)-f(x)}{h}$ is 1 if $h > 0$ and -1 if $h < 0$, so the two one-sided limits $\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$ and $\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$ do not agree.

□

Problem 2. (i) Find the equation of the tangent line to the curve $y = \frac{2\ln x}{x^3} - (\sin^{-1}(x))^5$ at the point where $x = \frac{1}{2}$.

(ii) Find the tangent line to the graph $x^2 + 2y^4 = 3$ at the point $(1, 1)$.

Solution. (i) We have

$$y\left(\frac{1}{2}\right) = \frac{2\ln\frac{1}{2}}{\left(\frac{1}{2}\right)^3} - \left(\sin^{-1}\frac{1}{2}\right)^5 = -16\ln 2 - \left(\frac{\pi}{6}\right)^5$$

and

$$y'(x) = \frac{2}{x^4} - \frac{6\ln x}{x^4} - 5\left(\sin^{-1}(x)\right)^4 \frac{1}{\sqrt{1-x^2}}$$

so

$$y'\left(\frac{1}{2}\right) = 32 + 96\ln 2 - 5\left(\frac{\pi}{6}\right)^4 \frac{2}{\sqrt{3}}\,.$$

Thus the equation of the tangent line at $x = \frac{1}{2}$ is

$$\boxed{y - \left(-16\ln 2 - \left(\frac{\pi}{6}\right)^5\right) = \left(32 + 96\ln 2 - 5\left(\frac{\pi}{6}\right)^4 \frac{2}{\sqrt{3}}\right)\left(x - \frac{1}{2}\right)}\,.$$

(ii) Use implicit differentiation:

$$2x + 8y^3y' = 0 \qquad \Longrightarrow \qquad y'(x) = \frac{-2x}{8y^3}$$

so $y'(1) = \frac{-2(1)}{8(1)^3} = -\frac{1}{4}$. Thus the equation of the tangent line is

$$\boxed{y - 1 = -\frac{1}{4}(x-1)}\,.$$

□

Problem 3. (i) Let $f(x)$ be a function of x such that $(f(x))^x = 1$. Find $f'(x)$ in terms of f.

(ii) Let f, g be differentiable functions such that $g(x) > 0$ for all x. Set $h(x) = f(x^{g(x)})$. Find h' in terms of f, g, f', g'.

(iii) Let $f(x) = 3x$. Find a function F whose derivative is f.

Solution. (i) Taking natural logarithm of both sides, we have

$$x\ln(f(x)) = 0\,.$$

Taking the derivative with respect to x gives

$$\ln(f(x)) + x\frac{1}{f(x)}f'(x) = 0\,.$$

Thus

$$\boxed{f'(x) = -\frac{f(x)\ln(f(x))}{x}}\,.$$

(ii) Define $k(x) = x^{g(x)}$. Then $h(x) = f(k(x))$. We have

$$\ln(k(x)) = g(x)\ln x$$

so

$$\frac{1}{k(x)}k'(x) = g'(x)\ln x + g(x)\frac{1}{x}$$

and

$$k'(x) = x^{g(x)}\left(g'(x)\ln x + g(x)\frac{1}{x}\right)$$

thus

$$\begin{aligned} h'(x) &= f'(k(x))\cdot k'(x) \\ &= \boxed{f'(x^{g(x)})\cdot x^{g(x)}\left(g'(x)\ln x + g(x)\frac{1}{x}\right)}\,. \end{aligned}$$

(iii) An example is $\boxed{F(x) = \frac{3}{2}x^2}$.

□

Problem 4. Strontium-90 has a half-life of 28 days. A sample has a mass of 50 mg initially.

(a) How much of the sample remains after t days?

(b) How long does it take the sample to decay to a mass of 2 mg?

Solution. (a) Let $y(t)$ be the quantity which remains after t days. Then there exists a constant C such that $y(t) = y(0)e^{Ct}$. We know $y(0) = 50$ so $y(t) = 50e^{Ct}$. Since the half-life is 28 days, we have $y(28) = \frac{1}{2}y(0) = 25$. Thus $y(28) = y(0)e^{28C}$ implies $e^{28C} = \frac{1}{2}$, or $C = \frac{1}{28}\ln\frac{1}{2}$. We have $e^{Ct} = (e^{\frac{1}{28}\ln\frac{1}{2}})^t = (\frac{1}{2})^{\frac{t}{28}}$ so $\boxed{y(t) = 50(\frac{1}{2})^{\frac{t}{28}}}$.

(b) Let t_0 be the amount of time it takes for the sample to decay to 2 mg. We have $2 = y(t_0) = 50(\frac{1}{2})^{\frac{t_0}{28}}$ so $(\frac{1}{2})^{\frac{t_0}{28}} = \frac{1}{25}$, which means $\frac{t_0}{28} = \frac{\ln\frac{1}{25}}{\ln\frac{1}{2}} = \frac{\ln 25}{\ln 2}$, and $\boxed{t_0 = \frac{28\ln 25}{\ln 2}}$.

□

Problem 5. (i) State Rolle's Theorem.

(ii) Use Rolle's Theorem to prove the Mean Value Theorem.

(iii) Let $f(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x$. Prove that $f(\pi) > f(1)$.

Solution. (i) The statement of Rolle's Theorem is:

Theorem 1. Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

(ii) We use Rolle's Theorem to prove the Mean Value Theorem:

Theorem 2. Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Define the function $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$. Then g is a function which is continuous on $[a, b]$ and differentiable on (a, b). We also have $g(a) = g(b)$. By Rolle's Theorem, there exists some $c \in (a, b)$ such that $g'(c) = 0$. We have $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Thus $f'(c) = \frac{f(b)-f(a)}{b-a}$. □

(iii) We have

$$f'(x) = x^4 - 2x^2 + 1 = (x^2 - 1)^2 .$$

Since f is a polynomial, it is continuous on $[1, \pi]$ and differentiable on $(1, \pi)$. By the Mean Value Theorem, there exists some $c \in (1, \pi)$ such that $f'(c) = \frac{f(\pi)-f(1)}{\pi-1}$. But $f'(c) = (c^2 - 1)^2 > 0$ since $c > 1$ (it's important that $(c^2 - 1)^2 > 0$ is a strict inequality). Thus $\frac{f(\pi)-f(1)}{\pi-1} > 0$ and $f(\pi) - f(1) > 0$ and $f(\pi) > f(1)$.

□

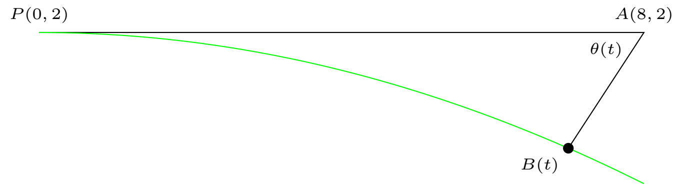
Problem 6. Find $\lim_{x\to 0^+} x\ln x$.

Solution. Rewrite the limit as $\lim_{x\to 0^+} \frac{\ln x}{1/x}$. As $x \to 0^+$, we have $\ln x \to -\infty$ and $\frac{1}{x} \to \infty$ so $\lim_{x\to 0^+} \frac{\ln x}{1/x}$ is an indeterminate form. Thus we have

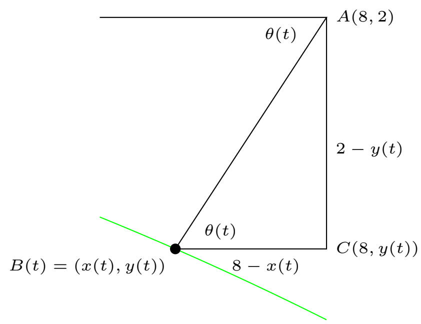
$$\begin{aligned} \lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \quad \text{by L'Hospital's Rule} \\ &= \lim_{x\to 0^+} -x \\ &= \boxed{0} . \end{aligned}$$

□

Problem 7. A pitcher at $P = (0, 2)$ throws a ball whose position at time t is $B(t) = (x(t), y(t))$ where $x(t) = t$ and $y(t) = -\frac{1}{32}(x(t))^2 + 2$. You are watching the ball from the point $A = (8, 2)$. Let $\theta(t)$ be the angle $\angle PAB(t)$. How fast (in radians) is $\theta(t)$ changing when the ball hits home plate, i.e. when $B(t) = (8, 0)$?



Solution. We have to express $\theta(t)$ in terms of $x(t)$ and $y(t)$ somehow. Consider the following diagram:



where $C = (8, y(t))$. Then we have

$$\cot(\theta(t)) = \frac{8 - x(t)}{2 - y(t)} = \frac{8 - t}{2 - (-\frac{1}{32}t^2 + 2)} = \frac{32(8 - t)}{t^2} .$$

We have $B(t) = (8, 0)$ if $t = 8$. Differentiating with respect to t gives

$$-\csc^2(\theta(t))\theta'(t) = \frac{32(-1)(t^2) - 32(8 - t)(2t)}{(t^2)^2}$$

and at $t = 8$ we have $\theta(8) = \frac{\pi}{2}$ and $\csc(\frac{\pi}{2}) = 1$ so

$$-\csc^2(\theta(8))\theta'(8) = \frac{32(-1)(8^2) - 32(8 - 8)(2 \cdot 8)}{(8^2)^2}$$

implies

$$\boxed{\theta'(8) = \frac{1}{2}} .$$

□

Problem 8. Consider the graph of the function $f(x) = x^3 - x$.

(a) Find the regions of the graph where $f(x)$ is increasing. Find the regions of the graph where $f(x)$ is decreasing. Find all local extrema of f. Which of these are global extrema?

(b) Find the regions of the graph where $f(x)$ is concave up. Find the regions of the graph where $f(x)$ is concave down. Find all inflection points.

Solution. (a) We have

$$f'(x) = 3x^2 - 1 .$$

Thus $f'(x) > 0$ on $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$ and $f'(x) < 0$ on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. So f is increasing on $(-\infty, -\frac{1}{\sqrt{3}})$, decreasing on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and increasing on $(\frac{1}{\sqrt{3}}, \infty)$. [1] The critical points of f are at $x = \pm\frac{1}{\sqrt{3}}$. The point $(-\frac{1}{\sqrt{3}}, f(-\frac{1}{\sqrt{3}}))$ is a local maximum and $(\frac{1}{\sqrt{3}}, f(\frac{1}{\sqrt{3}}))$ is a local minimum by the First Derivative Test, since f' changes from positive to negative at $x = -\frac{1}{\sqrt{3}}$ and from negative to positive at $x = \frac{1}{\sqrt{3}}$.

(b) We have

$$f''(x) = 6x$$

so f is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$. The unique inflection point is at $x = 0$.

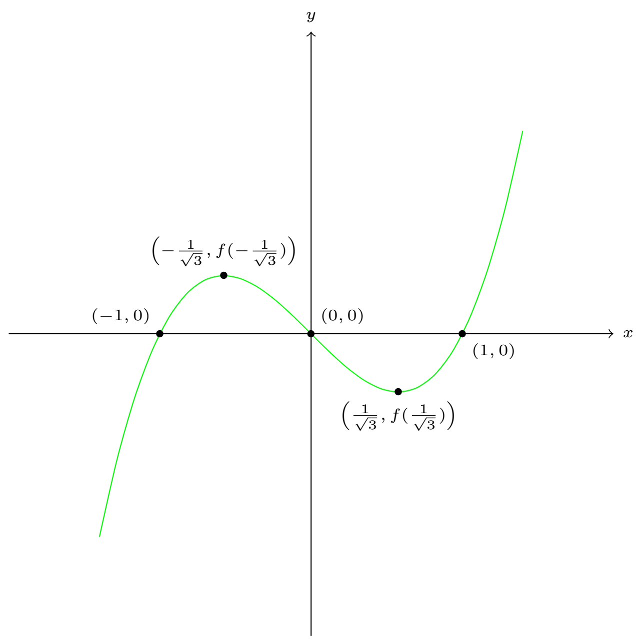


FIGURE 1. Graph of $f(x) = x^3 - x$

□

[1]But f is not increasing on $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$, because $f(-\frac{2}{3}) > f(\frac{2}{3})$. See the definition on page 19.