PRACTICE FINAL/STUDY GUIDE SOLUTIONS

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Feel free to send me any feedback, including comments, typos, and mathematical errors.

Problem 1. Give the precise meaning of the following statements.

(i) "$\lim_{x\to a} f(x) = L$"
(ii) "$\lim_{x\to a^+} f(x) = L$"
(iii) "$\lim_{x\to +\infty} f(x) = L$"
(iv) "$\lim_{x\to +\infty} f(x) = -\infty$"
(v) "$\lim_{x\to a^-} f(x) = -\infty$"

Solution. (i) For every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$.
(ii) For every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < x - a < \delta$ then $|f(x) - L| < \epsilon$.
(iii) For every $\epsilon > 0$, there exists $N > 0$ such that if $x > N$ then $|f(x) - L| < \epsilon$.
(iv) For every $M < 0$, there exists $N > 0$ such that if $x > N$ then $f(x) < M$.
(v) For every $M < 0$, there exists $\delta > 0$ such that if $0 < a - x < \delta$ then $f(x) < M$.

□

Problem 2. Prove the following statements using the limit definitions.

(i) "$\lim_{x\to 0} \frac{1}{x^2+1} = 1$"
(ii) "$\lim_{x\to 1} \frac{x^2-4x+5}{x+4} = \frac{2}{5}$" [1]
(iii) "$\lim_{x\to +\infty} \frac{e^x}{e^x+x} = 1$"

Scratch work. (i) We want to prove the following statement: "for every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x - 0| < \delta$ implies $|\frac{1}{x^2+1} - 1| < \epsilon$". We have $|\frac{1}{x^2+1} - 1| = \frac{x^2}{x^2+1} \le x^2$.

(ii) We want to prove the following statement: "for every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x-1| < \delta$ implies $|\frac{x^2-4x+5}{x+4} - \frac{2}{5}| < \epsilon$". We have

$$\left|\frac{x^2-4x+5}{x+4} - \frac{2}{5}\right| = \left|\frac{5x^2-22x+17}{5x+20}\right| = \left|\frac{(5x-17)(x-1)}{5x+20}\right| ;$$

we're going to work with the last expression. If $|x-1|$ is small (i.e. $x \approx 1$), then $5x - 17 \approx -12$ and $5x + 20 \approx 25$ (this is an inexact statement which needs to be made precise). I can ensure that $-13 < 5x - 12 < -11$ if $|x - 1| < \frac{1}{5}$, which is satisfied whenever $\delta \le \frac{1}{5}$. Coincidentally, $|x - 1| < \frac{1}{5}$ also ensures that $19 < 5x + 20 < 21$. Thus $|x - 1| < \frac{1}{5}$ implies that $|\frac{5x-12}{5x+20}| < \frac{13}{19}$ (check this). If δ is less than $\frac{\epsilon}{13/19}$ (in addition to being less than or equal to $\frac{1}{3}$), then $|\frac{(5x-12)(x-1)}{5x+20}| < \epsilon$. Notice that the condition "$\delta < \frac{\epsilon}{13/19}$ and $\delta < \frac{1}{5}$" is equivalent to the condition "$\delta < \min\{\frac{\epsilon}{13/19}, \frac{1}{5}\}$".

(iii) (This turned out to be harder than I expected.) We want to prove the following statement: "for every $\epsilon > 0$, there exists $N > 0$ such that $x > N$ implies $|\frac{e^x}{e^x+x} - 1| < \epsilon$". We have $|\frac{e^x}{e^x+x} - 1| = \frac{x}{e^x+x} < \frac{x}{e^x}$ (as long as $x > 0$, which we can force by taking $N \ge 0$). If $\frac{x}{e^x} < \epsilon$, then $\frac{x}{e^x+x} < \epsilon$, so we're going to look for an N such that $x > N$ implies $\frac{x}{e^x} < \epsilon$. We will show that $\lim_{n\to\infty} \frac{n}{e^n} = 0$. We will also show that the function $f(x) = \frac{x}{e^x}$ is decreasing on the interval $(1, \infty)$. This will show that $\lim_{x\to\infty} \frac{x}{e^x} = 0$.

[1] Typo: the printed version said $\frac{1}{5}$.

□

Solution. (i) Let $\epsilon > 0$. Set $\delta > \sqrt{\epsilon}$. Assume $0 < |x - 0| < \delta$. Then $x^2 = |x|^2 = |x-0|^2 < \delta^2 = \epsilon$. Since $x^2 + 1 \geq 1$, we have $\frac{x^2}{x^2+1} < \epsilon$. Thus $|\frac{1}{x^2+1} - 1| = |\frac{x^2}{x^2+1}| < \epsilon$.

(ii) Let $\epsilon > 0$. Set $\delta < \min\{\frac{\epsilon}{13/19}, \frac{1}{5}\}$. Assume $0 < |x - 1| < \delta$. Then $|x - 1| < \frac{1}{5}$, which implies $-13 < 5x - 12 < -11$ and $19 < 5x + 20 < 21$. Thus $|\frac{5x-12}{5x+20}| < \frac{13}{19}$. Also, we have $|x - 1| < \frac{\epsilon}{13/19}$. Thus

$$\left|\frac{x^2 - 4x + 5}{x + 4} - \frac{1}{5}\right| = \left|\frac{(5x-12)(x-1)}{5x+20}\right| < \frac{13}{19} \cdot \frac{\epsilon}{13/19} = \epsilon\ .$$

(iii) [2] Let's first show that $\lim_{n\to\infty} \frac{n}{e^n} = 0$. We have $\lim_{n\to\infty} \frac{n}{2^n} = 0$ by Problem 1 in `http://math.berkeley.edu/~shinms/FA13-1A/practice-problems-01.pdf`. We have $\frac{n}{e^n} < \frac{n}{2^n}$ for all n since $2 < e$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} \frac{n}{2^n} = 0$, there exists $N > 0$ such that $x > N$ implies $|\frac{n}{2^n} - 0| < \epsilon$. Assume $x > N$. Then $|\frac{n}{e^n} - 0| = \frac{n}{e^n} < \frac{n}{2^n} = |\frac{n}{2^n} - 0| < \epsilon$. Thus $\lim_{n\to\infty} \frac{n}{e^n} = 0$.

Let's show that if $1 \leq x < y$, then $f(x) > f(y)$. We have

$$e^{y-x} \overset{(*)}{>} 1 + (y - x) \geq 1 + \frac{y-x}{x} = \frac{y}{x}$$

where the inequality marked $(*)$ follows from the fact that $e^x > 1 + x$ for all positive x (since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$). Then $e^{y-x} > \frac{y}{x}$ implies $f(x) > f(y)$.

We now show that $\lim_{x\to\infty} \frac{x}{e^x} = 0$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} \frac{n}{e^n} = 0$, there exists a positive integer $N > 0$ such that if n is a positive integer such that $n \geq N$, then $\frac{n}{e^n} = |\frac{n}{e^n} - 0| < \epsilon$. Assume that x is a real number such that $x > N$. Let n be an integer such that $x > n \geq N$. Then $|\frac{x}{e^x} - 0| = \frac{x}{e^x} < \frac{n}{e^n} < \epsilon$. Thus $\lim_{x\to\infty} \frac{x}{e^x} = 0$.

We now prove that $\lim_{x\to\infty} \frac{e^x}{e^x+x} = 1$. Let $\epsilon > 0$. Since $\lim_{x\to\infty} \frac{x}{e^x} = 0$, there exists $N > 0$ such that $x > N$ implies $\frac{x}{e^x} = |\frac{x}{e^x} - 0| < \epsilon$. Then $|\frac{e^x}{e^x+x} - 1| = \frac{x}{e^x+x} < \frac{x}{e^x} < \epsilon$. Thus $\lim_{x\to\infty} \frac{e^x}{e^x+x} = 1$.

□

Problem 3. (i) State and prove the Squeeze Theorem.

(ii) Use the Squeeze Theorem to compute

$$\lim_{x\to 0} x^2(\sin x)^4(\cos x)^3\ .$$

Justify your answer carefully.

Solution. (i) If f, g, h are real-valued functions such that

$$f(x) \leq g(x) \leq h(x) \tag{1}$$

when x is near a (except possibly at a) and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

The precise meaning of "near a (except possibly at a)" is: "there exists some $\delta_1 > 0$ such that (1) holds for all x satisfying $0 < |x - a| < \delta_1$".

Proof: Suppose f, g, h satisfy (1) when $0 < |x - a| < \delta_1$ for some fixed $\delta_1 > 0$. Also suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$.

Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|f(x) - L| < \varepsilon$. Also, since $\lim_{x\to a} h(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then $|h(x) - L| < \varepsilon$.

[2] We can also use L'Hospital's rule, but the point was to compute the limit using the definitions.

Put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Suppose $0 < |x - a| < \delta$. Then (1), $L - \varepsilon < f(x) < L + \varepsilon$ and $L - \varepsilon < h(x) < L + \varepsilon$ hold. Hence $g(x) \le h(x) < L + \varepsilon$. Also $L - \varepsilon < f(x) \le g(x)$. Thus $L - \varepsilon < g(x) < L + \varepsilon$, so $|g(x) - L| < \varepsilon$. Hence $\lim_{x\to a} g(x) = L$.

(ii) Let $g(x) = x^2(\sin x)^4(\cos x)^3$, $f(x) = -x^2$, $h(x) = x^2$. Since $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$ for all x, we have

$$-1 \le (\sin x)^4(\cos x)^3 \le 1$$

for all x. Thus (1) holds. In addition, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$. Thus $\lim_{x\to 0} g(x) = 0$.

□

Problem 4. Let f and g be functions, and suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$. Prove that $\lim_{x\to a}(f(x) + g(x)) = L + K$.

Solution. Let f and g be functions, and suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$. Let $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2}$. Also, since $\lim_{x\to a} g(x) = K$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|g(x)-K| < \frac{\epsilon}{2}$. Put $\delta = \min\{\delta_1, \delta_2\}$. Suppose $0 < |x - a| < \delta$. Then $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - K| < \frac{\epsilon}{2}$. Hence

$$\begin{aligned}
|(f(x) + g(x)) - (L + K)| &= |(f(x) - L) + (g(x) - K)| \\
&\le |f(x) - L| + |g(x) - K| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon\ .
\end{aligned}$$

Hence $\lim_{x\to a}(f(x) + g(x)) = L + K$. □

Problem 5. Evaluate

$$\lim_{x\to 0}\left(\frac{1}{\sqrt{1+x}} - \frac{1}{1+x}\right)^2$$

You should show your reasoning carefully; however you may use any of the limit laws without explanation or proof.

Solution. We have

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sqrt{1+x}} - \frac{1}{1+x}\right)^2 &= \left(\lim_{x\to 0}\left(\frac{1}{\sqrt{1+x}} - \frac{1}{1+x}\right)\right)^2 \\
&= \left(\lim_{x\to 0}\left(\frac{1}{\sqrt{1+x}}\right) - \lim_{x\to 0}\left(\frac{1}{1+x}\right)\right)^2 \\
&= \left(\sqrt{\lim_{x\to 0}\left(\frac{1}{1+x}\right)} - \lim_{x\to 0}\left(\frac{1}{1+x}\right)\right)^2 \\
&= \left(\sqrt{\frac{1}{1+0}} - \frac{1}{1+0}\right)^2 \\
&= 0\ .
\end{aligned}$$

□

Problem 6. Indicate “true” if the statement is always true; indicate “false” if there exists a counterexample.

(i) "If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a^+} f(x) = L$."
(ii) "If $\lim_{x\to a^+} f(x) = L$, then $\lim_{x\to a} f(x) = L$."
(iii) "If $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f(x)e^x = 0$."
(iv) "If $\lim_{x\to a}(f(x))^2 = 1$, then $\lim_{x\to a} f(x) = 1$."

Solution. (i) True. (Intuition: If the two-sided limit is L, then a one-sided limit is also L.) Proof: Suppose $\lim_{x\to a} f(x) = L$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$. Thus if $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$. Thus $\lim_{x\to a^+} f(x) = L$.

(ii) False. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Then $\lim_{x\to 0^+} f(x) = 1$ but $\lim_{x\to 0} f(x)$ does not exist.

(iii) False. We have $\lim_{x\to\infty} \frac{1}{e^x} = 0$ but $\lim_{x\to\infty} \frac{1}{e^x} e^x = \lim_{x\to\infty} 1 = 1 \neq 0$.

(iv) False. Consider the function f in the solution to (ii) again; then $(f(x))^2$ is identically 1 on $(-\infty, 0) \cup (0, \infty)$ so $\lim_{x\to 0}(f(x))^2 = 1$, but, as noted above, $\lim_{x\to 0} f(x)$ does not exist.

□

Problem 7. (i) Give the precise meaning of the statement "f is continuous at $x = a$".
(ii) Using the definition in (i), show that $f(x) = x$ is continuous at $x = 1$.

Solution. (i) The statement "f is continuous at $x = a$" means "f is defined in some interval containing a and $\lim_{x\to a} f(x) = f(a)$".

(ii) The function $f(x) = x$ is defined everywhere, so it is defined in an interval containing $a = 1$. We have $\lim_{x\to 1} x = 1$, so $f(x)$ is continuous at $x = 1$.

□

Problem 8. (i) State the Intermediate Value Theorem.[3]
(ii) Prove that $e^x \sin x = 40$ has a solution in $(0, \infty)$.

Solution. (i) Let f be a continuous function on the interval $[a, b]$. Suppose $f(a) \neq f(b)$ and let N be a value strictly between $f(a)$ and $f(b)$. Then there exists some $c \in (a, b)$ such that $f(c) = N$.

(Remark: f needs to be continuous on the closed interval $[a, b]$, not just on the open interval (a, b), and we can guarantee that such c exists in the open interval (a, b), not just the closed interval $[a, b]$.)

(ii) Let $f(x) = e^x \sin x$. Note that $f(x)$ is continuous on $\mathbb{R}$. We have $f(0) = e^0 \sin 0 = 0 < 40$. We have $f(\frac{2013\pi}{2}) = e^{\frac{2013\pi}{2}} \sin \frac{2013\pi}{2} = e^{\frac{2013\pi}{2}} \sin(1006\pi + \frac{\pi}{2}) = e^{\frac{2013\pi}{2}} > 2^{\frac{2013\pi}{2}} > 2^{2013} > 40$. Since $f(x)$ is continuous on the closed interval $[0, \frac{2013\pi}{2}]$, the IVT implies that there exists $c \in (0, \frac{2013\pi}{2})$ such that $f(c) = 40$. Thus $f(x) = 40$ has a solution in $(0, \frac{2013\pi}{2})$. Thus $f(x) = 40$ has a solution in $(0, \infty)$.

□

Problem 9. (i) Give the precise meaning of the statement "f is differentiable at $x = a$".
(ii) Using the definition in (i), show that $f(x) = x$ is differentiable at $x = 1$.

Solution. (i) The statement "f is differentiable at $x = a$" means "f is defined in some interval containing a and the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists (which we denote '$f'(a)$')".[4]

[3]The paper version asked you to prove the IVT, too; it was a typo.
[4]It's equivalent to say that the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

(ii) Let $f(x) = x$. Then f is defined on all of $\mathbb{R}$, so it is defined on an interval containing 1. We have $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(a+h)-(a)}{h} = \lim_{h\to 0} 1$, which exists (and is equal to 1), so f is differentiable at $x = 1$.

□

Problem 10. (i) State Rolle's Theorem.
(ii) State the Mean Value Theorem.
(iii) Prove the Mean Value Theorem using Rolle's Theorem.

Solution. (i) Let f be a function continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.
(ii) Let f be a function continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
(iii) Let f be a real-valued function that is continuous on $[a, b]$ and differentiable on (a, b). Define $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$. Note that g is a real-valued function that is continuous on $[a, b]$ and differentiable on (a, b). Note that $g(a) = g(b)$. Hence, by Rolle's Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. But $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Hence $f'(c) = \frac{f(b)-f(a)}{b-a}$.

□

Problem 11. In each of the following cases, evaluate $\frac{dy}{dx}$.

(i) $y = \frac{2x}{x^2+1}$
(ii) $y = \arctan((\sin x)^2)$
(iii) $y^2 + 3xy + x^2 = e^x \cos x$
(iv) $y = x^{x^x}$

Solution. (i) Use the quotient rule:

$$\frac{dy}{dx} = \frac{2(x^2 + 1) - (2x)(2x)}{(x^2 + 1)^2} = \frac{-2x^2 + 2}{(x^2 + 1)^2} .$$

(ii) Use the chain rule:

$$\frac{dy}{dx} = \frac{1}{1 + ((\sin x)^2)^2}((\sin x)^2)' = \frac{2(\sin x)(\cos x)}{1 + ((\sin x)^2)^2} = \frac{2 \sin x \cos x}{1 + (\sin x)^4} .$$

(iii) Differentiate implicitly:

$$2y\frac{dy}{dx} + \left(3y + 3x\frac{dy}{dx}\right) + 2x = e^x \cos x - e^x \sin x .$$

Thus

$$(2y + 3x)\frac{dy}{dx} = e^x \cos x - e^x \sin x - 3y - 2x$$

so

$$\frac{dy}{dx} = \frac{e^x \cos x - e^x \sin x - 3y - 2x}{2y + 3x} .$$

(iv) Let $g(x) = x^x = e^{x \ln x}$. Then $g'(x) = e^{x \ln x}(\ln x + x\frac{1}{x}) = x^x(\ln x + 1)$. Then

$$\begin{aligned} \frac{dy}{dx} &= (x^{g(x)})' \\ &= (e^{g(x) \ln x})' \\ &= (e^{g(x) \ln x})(g(x) \ln x)' \\ &= (e^{g(x) \ln x}) \left(g'(x) \ln x + g(x) \frac{1}{x} \right) \\ &= x^{x^x} \left(x^x(\ln x + 1)(\ln x) + x^x \frac{1}{x} \right) . \end{aligned}$$

□

Problem 12. Alexander Coward's youtube channel has 21 subscribers at time $t = 0$, and the number of subscribers grows exponentially with respect to time. At time $t = 4$, he has 103 subscribers. After how long will Alexander have 10^6 subscribers?

Solution. Let $y(t)$ be the number of subscribers at time t. Since $y(t)$ grows exponentially, we have $y(t) = y(0)e^{Ct}$ for some constant C. We're given that $y(0) = 21$ and $y(4) = 103$. Thus $103 = 21e^{4C}$, so $C = \frac{\ln \frac{103}{21}}{4}$. Let t_0 be the time at which $y(t_0) = 10^6$. Then $10^6 = 21e^{Ct_0}$ implies $t_0 = \frac{\ln \frac{10^6}{21}}{C} = \frac{4 \ln \frac{10^6}{21}}{\ln \frac{103}{21}}$. □

Problem 13. Which point on the graph of $y = x^2$ is closest to the point $(5, -1)$?

Solution. An arbitrary point on the graph $y = x^2$ is of the form (x, x^2). The distance between (x, x^2) and $(5, -1)$ is

$$d(x) = \sqrt{(x-5)^2 + (x^2+1)^2} = \sqrt{x^4 + 3x^2 - 10x + 26} \,.$$

We have

$$d'(x) = \frac{1/2}{\sqrt{x^4 + 3x^2 - 10x + 26}}(4x^3 + 6x - 10) = \frac{2x^3 + 3x - 5}{\sqrt{x^4 + 3x^2 - 10x + 26}} \,.$$

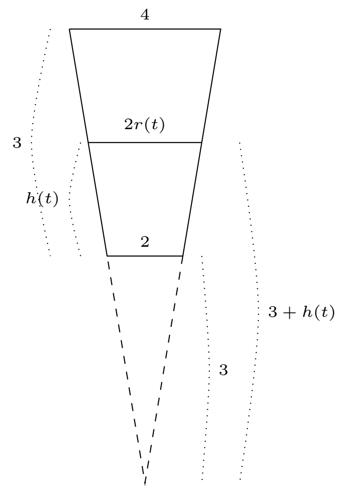
Thus $d'(x) = 0$ if and only if $2x^3 + 3x - 5 = 0$. Since the coefficients of $2x^3 + 3x - 5$ sum to 0, a root of $2x^3 + 3x - 5$ is 1. Thus $x - 1$ divides $2x^3 + 3x - 5$, and

$$2x^3 + 3x - 5 = (x-1)(2x^2 + 2x + 5) \,. \tag{2}$$

Since $2x^2 + 2x + 5 = x^2 + (x+1)^2 + 4$, it is always positive. Thus $2x^3 + 3x - 5$ has exactly one root, namely $x = 1$. Furthermore, by (2), we have $2x^3 + 3x - 5 > 0$ if $x > 1$ and $2x^3 + 3x - 5 < 0$ if $x < 1$. Thus $d'(x) > 0$ if $x > 1$ and $d'(x) < 0$ if $x < 1$. Thus $x = 1$ is a global minimum of $d(x)$. Thus $(1, 1)$ is the point on the graph of $y = x^2$ which is closest to $(5, -1)$. □

Problem 14. The interior of a bowl is a "conic frustum", where the top surface is a disk of radius 2 and the bottom surface is a disk of radius 1 and the height of the cup is 3. A liquid is being poured into the bowl at a constant rate of 4. How fast is the height of the water increasing when the bowl is full?

Solution. Let $h(t)$, $r(t)$, and $V(t)$ be the height, radius (of the surface), and volume of the water in the bowl at time t, respectively. The following is the side view of the bowl:



By similar triangles, we have

$$\frac{2}{3} = \frac{2r(t)}{3 + h(t)} \ .$$

Thus

$$r(t) = \frac{h(t) + 3}{3} \ .$$

We have

$$\begin{aligned} V(t) &= \frac{1}{3}(\pi \cdot (r(t))^2)(3 + h(t)) - \frac{1}{3}(\pi \cdot (1/2)^2)(3) \\ &= \frac{1}{3}\left(\pi \cdot \left(\frac{h(t)+3}{3}\right)^2\right)(3 + h(t)) - \frac{\pi}{4} \\ &= \frac{\pi}{27}(h(t) + 3)^3 - \frac{\pi}{4} \ . \end{aligned}$$

Thus

$$V'(t) = \frac{\pi}{9}(h(t) + 3)^2(h'(t)) \ .$$

Let t_0 be the time at which the bowl is full. Then $h(t_0) = 3$. Since $V'(t) = 4$ for all t by assumption, we have

$$h'(t_0) = \frac{36}{\pi(h(t_0) + 3)^2} = \frac{36}{\pi \cdot 6^2} = \frac{1}{\pi} \ .$$

□

Problem 15. Showing your work carefully, evaluate the limit

$$\lim_{x \to 0} \frac{(1 + \sin x)^2 - (\cos x)^2}{x^2} \ .$$

Solution. We have

$$\lim_{x \to 0} \frac{(1 + \sin x)^2 - (\cos x)^2}{x^2} = \lim_{x \to 0} \frac{2(1 + \sin x)(\cos x) - 2(\cos x)(-\sin x)}{2x}$$

by L'Hospital's Rule. The second limit does not exist since

$$\lim_{x \to 0^+} \frac{2(1 + \sin x)(\cos x) - 2(\cos x)(-\sin x)}{2x} = +\infty \tag{3}$$

and

$$\lim_{x\to 0^-} \frac{2(1+\sin x)(\cos x) - 2(\cos x)(-\sin x)}{2x} = -\infty\ .$$

(Notice that we cannot apply L'Hospital's Rule to (3) since it is not an indeterminate form: $\lim_{x\to 0^+} 2(1+\sin x)(\cos x) - 2(\cos x)(-\sin x) = 2$ while $\lim_{x\to 0^+} 2x = 0$.) Thus the desired limit does not exist. □

Problem 16. (i) Give the precise definition of the definite integral using Riemann sums.
(ii) What's the difference between a definite integral and an indefinite integral?
(iii) Using the definition in (i), compute $\int_0^2 x^2\, dx$.

Solution. (i) Let $f : [a,b] \to \mathbb{R}$ be a function. For each $n \in \mathbb{N}$ (recall that $\mathbb{N}$ is the set of positive integers), pick a collection of sample points $x_1^*, \dots, x_n^*$ so that x_i^* lies in $[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}]$. The definite integral of f from a to b is defined as

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*) \left(\frac{b-a}{n}\right)\ ,$$

provided this limit exists and gives the same value for all possible choices of sample points.
(ii) A definite integral is a number, while an indefinite integral (i.e. antiderivative) is a function.
(iii) Let $f(x) = x^2$, $a = 0$, and $b = 2$. For convenience, let's choose $x_i^* = a + i\frac{b-a}{n}$ for all i, n (we're computing the right-hand Riemann sums). We have

$$\begin{aligned}\int_0^2 x^2\, dx &= \lim_{n\to\infty} \sum_{i=1}^n \left(i\frac{2}{n}\right)^2 \frac{2}{n} \\ &= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^n i^2 \\ &= \lim_{n\to\infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= \lim_{n\to\infty} \frac{8}{3} \cdot \frac{n+1}{n} \cdot \frac{n+1/2}{n} \\ &= \frac{8}{3}\ .\end{aligned}$$

□

Problem 17. (i) State the Fundamental Theorem of Calculus.
(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that if g is an antiderivative of f', then there exists a constant C such that $f(x) = g(x) + C$ for all x.
(iii) Are all continuous functions differentiable?
(iv) Do all continuous functions have antiderivatives?

Solution. (i) Let f be a continuous real-valued function on $[a,b]$. Let $g : [a,b] \to \mathbb{R}$ be defined by

$$g(x) = \int_a^x f(t)\, dt\ .$$

Then $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$. Furthermore, if F is any antiderivative of f, then

$$F(b) - F(a) = \int_a^b f(t)\, dt\ .$$

(By letting b vary, this implies, in particular, that $F(x) - F(a) = \int_a^x f(t)\, dt$ for all $x \in [a,b]$.)

(ii) Notice that f and g are functions defined on $\mathbb{R}$. Choose an interval $[a,b]$. Notice that both f and g are antiderivatives of f'. The second part of the FTC states that $f(x)-f(a)=g(x)-g(a)$ for all $x \in [a,b]$. This implies that $f(x)-g(x)=f(a)-g(a)$ for all $x \in [a,b]$. Let $C=f(a)-g(a)$. If $x \notin [a,b]$, then we can repeat the above argument to any closed interval $I=[a',b']$ which contains $[a,b]$ and x (which will yield the conclusion that $f(x)-g(x)=f(a')-g(a')$ for all $x\in[a',b']$). Thus $f(x)-g(x)=C$ for all $x\in\mathbb{R}$.

(iii) Not all continuous functions are differentiable. For example, $f(x)=|x|$ is not differentiable at $x=0$.

(iv) All continuous functions have antiderivatives, by the first part of FTC: if f is continuous on $[a,b]$, then the function $g(x)=\int_a^x f(t)\,dt$ is an antiderivative of f.

□

Problem 18. Compute an antiderivative of the following functions.

(i) $f(x)=8x^3+3x^2$

(ii) $f(x)=(\sqrt[5]{x}+1)^2$

(iii) $f(x)=x\sqrt{1+x^2}$

(iv) $f(x)=\tan(\arcsin(x))$

(v) $f(x)=\frac{x^3}{\sqrt{x^2+1}}$

Solution. (i) We have

$$\int (8x^3+3x^2)\,dx = \frac{8}{4}x^4+\frac{3}{3}x^3+C = 2x^4+x^3+C\;.$$

(ii) We have

$$\int (\sqrt[5]{x}+1)^2\,dx = \int (x^{2/5}+2x^{1/5}+1)\,dx = \frac{5}{7}x^{7/5}+\frac{2\cdot 5}{6}x^{6/5}+x+C$$

(iii) Use the substitution $u=1+x^2$. Then $\frac{du}{dx}=2x$. We have

$$\int x\sqrt{1+x^2}\,dx = \int \frac{1}{2}\sqrt{u}\frac{du}{dx}\,dx \overset{(*)}{=} \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{3}u^{3/2}+C = \frac{1}{3}(1+x^2)^{3/2}+C$$

where we have used the Substitution Rule in the step marked $(*)$.

(iv) Write $\tan(\arcsin(x))=\frac{x}{\sqrt{1-x^2}}$ and use the substitution $u=1-x^2$. Thus

$$\begin{aligned}\int \tan(\arcsin(x))\,dx &= \int \frac{x}{\sqrt{1-x^2}}\,dx \\ &= \int \frac{-\frac{1}{2}}{\sqrt{u}}\frac{du}{dx}\,dx \\ &\overset{(*)}{=} \int \frac{-\frac{1}{2}}{\sqrt{u}}\,du \\ &= -\sqrt{u}+C \\ &= -\sqrt{1-x^2}+C\end{aligned}$$

(v) Write

$$\frac{x^3}{\sqrt{x^2+1}} = \frac{x^3+x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}} = x\sqrt{x^2+1} - \frac{x}{\sqrt{x^2+1}}\;.$$

We compute antiderivatives of $x\sqrt{x^2+1}$ and $\frac{x}{\sqrt{x^2+1}}$ separately. An antiderivative of $x\sqrt{x^2+1}$ is $\frac{1}{3}(1+x^2)^{3/2}+C_1$ by (iii). We use the same substitution $u=1+x^2$ to compute an antiderivative of

$\frac{x}{\sqrt{x^2+1}}$. Then $\frac{du}{dx} = 2x$. We have

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{2}\frac{1}{\sqrt{u}}\frac{du}{dx}\,dx \overset{(*)}{=} \int \frac{1}{2}\frac{1}{\sqrt{u}}\,du = \sqrt{u} + C_1 = \sqrt{1+x^2} + C_2$$

where we have used the Substitution Rule in the step marked $(*)$. Hence

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C\ .$$

□

Problem 19. (i) Find the volume of the solid obtained by rotating the region $\{(x,y) : 0 \le x \le e^y, 1 \le y \le 2\}$ about the y-axis.

(ii) Find the volume of the solid obtained by rotating about the y-axis the region between $y = \sqrt{x}$ and $y = x^2$.

Solution. (i) The intersection of the solid with the plane $y = y_0$ is a circle of radius e^{y_0}. Thus the infinitesimal cross-section of the solid has volume $\pi(e^{y_0})^2 dy$ where dy is the thickness of the cross-section. Thus the volume of the solid is

$$\int_1^2 \pi(e^y)^2\ dy = \int_1^2 \pi e^{2y}\ dy = \frac{\pi}{2}e^{2y}\Big|_1^2 = \frac{\pi}{2}(e^4 - e^2)\ .$$

(ii) The curves $y = \sqrt{x}$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$ and they are strictly increasing, so the region they bound is contained in the box $\{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. The intersection of the solid with the plane $y = y_0$ is an annulus (a disk with a smaller concentric disk removed from it) of inner radius y_0^2 and outer radius $\sqrt{y_0}$, which has area $\pi(\sqrt{y_0})^2 - \pi(y_0^2)^2$. Thus the volume of the solid is

$$\int_0^1 \pi(\sqrt{y})^2 - \pi(y^2)^2\ dy = \int_0^1 \pi(y - y^4)\ dy = \pi\left(\frac{1}{2}y^2 - \frac{1}{5}y^5\right)\Big|_0^1 = \frac{3\pi}{10}\ .$$

□

Problem 20. Simplify $\log_{\log_3 9}(\log_4 2)$.

Solution. We have

$$\log_{\log_3 9}(\log_4 2) = \log_2(1/2) = -1\ .$$

□