PRACTICE FINAL/STUDY GUIDE

Write in complete sentences and show all work.

Problem 1. Give the precise meaning of the following statements.

(i) "$\lim_{x\to a} f(x) = L$"
(ii) "$\lim_{x\to a^+} f(x) = L$"
(iii) "$\lim_{x\to +\infty} f(x) = L$"
(iv) "$\lim_{x\to +\infty} f(x) = -\infty$"
(v) "$\lim_{x\to a^-} f(x) = -\infty$"

Problem 2. Prove the following statements using the limit definitions.

(i) "$\lim_{x\to 0} \frac{1}{x^2+1} = 1$"
(ii) "$\lim_{x\to 1} \frac{x^2-4x+5}{x+4} = \frac{2}{5}$"
(iii) "$\lim_{x\to +\infty} \frac{e^x}{e^x+x} = 1$"

Problem 3. (i) State and prove the Squeeze Theorem.
(ii) Use the Squeeze Theorem to compute

$$\lim_{x\to 0} x^2(\sin x)^4(\cos x)^3 \ .$$

Justify your answer carefully.

Problem 4. Let f and g be functions, and suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$. Prove that $\lim_{x\to a}(f(x) + g(x)) = L + K$.

Problem 5. Evaluate

$$\lim_{x\to 0} \left(\frac{1}{\sqrt{1+x}} - \frac{1}{1+x} \right)^2$$

You should show your reasoning carefully; however you may use any of the limit laws without explanation or proof.

Problem 6. Indicate "true" if the statement is always true; indicate "false" if there exists a counterexample.

(i) "If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a^+} f(x) = L$."
(ii) "If $\lim_{x\to a^+} f(x) = L$, then $\lim_{x\to a} f(x) = L$."
(iii) "If $\lim_{x\to \infty} f(x) = 0$, then $\lim_{x\to \infty} f(x)e^x = 0$."
(iv) "If $\lim_{x\to a}(f(x))^2 = 1$, then $\lim_{x\to a} f(x) = 1$."

Problem 7. (i) Give the precise meaning of the statement "f is continuous at $x = a$".
(ii) Using the definition in (i), show that $f(x) = x$ is continuous at $x = 1$.

Problem 8. (i) State and prove the Intermediate Value Theorem.
(ii) Prove that $e^x \sin x = 40$ has a solution in $(0, \infty)$.

Problem 9. (i) Give the precise meaning of the statement "f is differentiable at $x = a$".
(ii) Using the definition in (i), show that $f(x) = x$ is differentiable at $x = 1$.

Problem 10. (i) State Rolle's Theorem.
(ii) State the Mean Value Theorem.
(iii) Prove the Mean Value Theorem using Rolle's Theorem.

Problem 11. In each of the following cases, evaluate $\frac{dy}{dx}$.
(i) $y = \frac{2x}{x^2+1}$
(ii) $y = \arctan((\sin x)^2)$
(iii) $y^2 + 3xy + x^2 = e^x \cos x$
(iv) $y = x^{x^x}$

Problem 12. Alexander Coward's youtube channel has 21 subscribers at time $t = 0$, and the number of subscribers grows exponentially with respect to time. At time $t = 4$, he has 103 subscribers. After how long will Alexander have 10^6 subscribers?

Problem 13. Which point on the graph of $y = x^2$ is closest to the point $(5, -1)$?

Problem 14. The interior of a bowl is a "conic frustum", where the top surface is a disk of radius 2 and the bottom surface is a disk of radius 1 and the height of the cup is 3. A liquid is being poured into the bowl at a constant rate of 4. How fast is the height of the water increasing when the bowl is full?

Problem 15. Showing your work carefully, evaluate the limit

$$\lim_{x \to 0} \frac{(1 + \sin x)^2 - (\cos x)^2}{x^2}.$$

Problem 16. (i) Give the precise definition of the definite integral using Riemann sums.
(ii) What's the difference between a definite integral and an indefinite integral?
(iii) Using the definition in (i), compute $\int_0^2 x^2 \, dx$.

Problem 17. (i) State the Fundamental Theorem of Calculus.
(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that if g is an antiderivative of f', then there exists a constant C such that $f(x) = g(x) + C$ for all x.
(iii) Are all continuous functions differentiable?
(iv) Do all continuous functions have antiderivatives?

Problem 18. Compute an antiderivative of the following functions.
(i) $f(x) = 8x^3 + 3x^2$
(ii) $f(x) = (\sqrt[5]{x} + 1)^2$
(iii) $f(x) = x\sqrt{1 + x^2}$
(iv) $f(x) = \tan(\arcsin(x))$
(v) $f(x) = \frac{x^3}{\sqrt{x^2+1}}$

Problem 19. (i) Find the volume of the solid obtained by rotating the region $\{(x, y) : 0 \leq x \leq e^y, 1 \leq y \leq 2\}$ about the y-axis.
(ii) Find the volume of the solid obtained by rotating about the y-axis the region between $y = \sqrt{x}$ and $y = x^2$.

Problem 20. Simplify $\log_{\log_3 9}(\log_4 2)$.